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Packing the Cartesian Product of Two Complete Graphs with Hexagons*

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Abstract

In this paper, we completely solve the problem of finding a maximum packing of $K_m \times K_n$, the cartesian product of two complete graphs with edge-disjoint 6-cycles, and minimum leaves are explicitly given. Subsequently, we also find a minimum covering of $K_m \times K_n$.

1 Introduction and Preliminaries

A *k-cycle packing* of a graph G is a set of edge-disjoint k-cycles in G. A k-cycle packing C of G is *maximum* if $|C| \geq |C'|$ for all other k-cycle packings C' of G. The *leave* L of a packing C is the subgraph induced by the set of edges of G that do not occur in any k-cycle of the packing C. The leave L of a maximum packing is referred to as a minimum leave, a leave with

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minimum number of edges. A packing with empty leave is known as a *k-cycle system* of G. (In terms of graph decomposition, we say $C_k|G$.) And a k-cycle system of K_v is referred to as a k-cycle system of order v.

Clearly, if $C_k|K_v$ then v is odd and $k|\binom{v}{2}$. To determine whether the above necessary condition is also sufficient is commonly referred to as the existence problem of k-cycle system.

The existence problem for *k-cycle system* of order v has been studied for more than 35 years. Recently, it has been completely solved by Alspach et. al, see [1,2]. But, the packing of K_v with k-cycles is not that lucky, only partial results are obtained so far, see [12]. Mainly, $k \in \{3,4,5,6\}$ is considered.

The cartesian product of two graphs, $G = G_1 \times G_2$, is the graph which has vertex set $V(G_1) \times V(G_2)$ and in which vertex (u_1, u_2) is adjacent to (v_1, v_2) if and only if either $u_1 = v_1$ in G_1 and u_2 is adjacent to v_2 in G_2, or $u_2 = v_2$ in G_2 and u_1 is adjacent to v_1 in G_1. For convenience, we also define the union of two graphs G_1 and G_2. The union $G = G_1 \cup G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Since we are dealing with even cycle decompositions, the following result of Sotteau deserves to be mentioned first.

Theorem 1.1. *[13] The complete bipartite graph $K_{m,n}$ can be decomposed into $2k$-cycles if and only if (i) $m, n \geq k$, (ii) m and n are even, and (iii)*

$2k|mn$.

Now, consider the packing of K_v with hexagons. The following two results are obtained by Kennedy, and Ashe et. al. respectively.

Theorem 1.2. *[10] The minimum leaves of the maximum packings of K_v with hexagons are as follows: v is considered to be the number modulo 12.*

v	0	1	2	3	4	5	6	7	8	9	10	11
L	F	ϕ	F	C_3	F_4	C_4	F	C_3	F	ϕ	F_4	E_7

F is a 1-factor, C_i is a cycle of length i, F_4 is an odd graph with $v/2+4$ edges and E_7 is a simple even graph with 7 edges.

Theorem 1.3. *[3] Let R be a 2-regular graph in the complete graph K_m. Then there exists a 6-cycle system of $G = K_m - E(R)$ if and only if (i) $|E(K_m - E(R))|$ is divisible by 6, (ii) m is odd and (iii) $m \neq 5$.*

The following terminology was introduced by Billington and Cavenagh[7]. A graph G is said to be k-sufficient if (i) each vertex in G has even degree and (ii) $k||E(G)|$. Recently, Farrell and Pike proved the following:

Theorem 1.4. *[8] All 6-sufficient $K_m \times K_n$ are decomposable into 6-cycles.*

By Theorem 1.4, we shall consider only the case when $K_m \times K_n$ is not 6-sufficient in the next section.

2 The maximum packing

First, we consider the packing of some small cases for m and n. If the proofs are direct, we omit the details.

Lemma 2.1. *$C_3 \times C_3$ can be decomposed into a 6-cycle system.*

Lemma 2.2. *Let $S = \{((i,0),(i,1),(i,2),(i,3)) \mid i \in Z_n\}$, $T = \{((0,j),(1,j),\cdots,(n-1,j)) \mid j = 0,3\}$, then $S \cup T$ can be decomposed into a 6-cycle system.*

Proof. $S \cup T$ can be packed with $((i,0),(i,1),(i,2),(i,3),(i+1,3),(i+1,0))$, $\forall i \in Z_n$.

□

For convenience, we use tH to denote t disjoint copies of H.

Lemma 2.3. *Let the vertex set of $3C_4$ be V. Then there exists a 6-cycle C defined on V such that $3C_4 \cup C$ can be decomposed into 6-cycles.*

Lemma 2.4. *Let $4C_3$ be $\{((i,0),(i,1),(i,2)) \mid i \in Z_4\}$, and $H = \{((0,j),(1,j),(a_j,b_j),(2,j),(3,j),(c_j,d_j)) \mid j = 0,1\}$. Then $4C_3 \cup H$ can be decomposed into four 6-cycles.*

Proof. $4C_3 \cup H$ can be decomposed into $((0,0),(0,1),(c_1,d_1),(3,1),(3,0),(c_0,d_0))$, $((0,0),(0,2),(0,1),(1,1),(1,2),(1,0))$, $((1,0),(1,1),(a_1,b_1),(2,1),(2,0),(a_0,b_0))$, and $((2,0),(2,2),(2,1),(3,1),(3,2),(3,0))$. □

Lemma 2.5. *$K_2 \times K_3$ can be packed with a 6-cycle which has leave a 1-factor F.*

Lemma 2.6. *$K_2 \times K_4$ can be packed with two 6-cycles which has leave a 4-cycle C_4.*

Lemma 2.7. *$K_2 \times C_4$ can be packed with a 6-cycle which has leave an F_2, where F_2 is an odd graph with $v/2 + 2$ edges.*

In the following discussion, for convenience, let L_m and L_n be the minimum leaves which are obtained in packing K_m and K_n with 6-cycles. If L_m or L_n is F_4, we assume $F_4 = F \cup C_4$, where F is a 1-factor of K_m or K_n. (We omit many different cases with similar proof.) Since $K_m \times K_n$ and $K_n \times K_m$ are isomorphic, without loss of the generality, we also assume $m \leq n$. Now, we are ready for the main result.

Theorem 2.8. *The minimum leaves of the maximum packings of $K_m \times K_n$ with hexagons are the follows:(where m, n are modulo 12.)*

m / n	0	1	2	3	4	5	6	7	8	9	10	11
0	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F
1	F	ϕ	F	C_3	F_4	C_4	F	C_3	F	ϕ	F_4	E_7
2	ϕ	F	C_4	F	C_4	F_2	ϕ	F	C_4	F	C_4	F_2
3	F	C_3	F	ϕ	F	C_3	F	ϕ	F	C_3	F	ϕ
4	ϕ	F_4	C_4	F	ϕ	F	ϕ	F_4	C_4	F	ϕ	F
5	F	C_4	F_2	C_3	F	C_4	F	E_7	F_2	ϕ	F	E_7
6	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F
7	F	C_3	F	ϕ	F_4	E_7	F	ϕ	F	C_3	F_4	C_4
8	ϕ	F	C_4	F	C_4	F_2	ϕ	F	C_4	F	C_4	F_2
9	F	ϕ	F	C_3	F	ϕ	F	C_3	F	ϕ	F	C_3
10	ϕ	F_4	C_4	F	ϕ	F	ϕ	F_4	C_4	F	ϕ	F
11	F	E_7	F_2	ϕ	F	E_7	F	C_4	F_2	C_3	F	C_4

F is a 1-factor. C_i is a cycle of length i. F_i is an odd graph with $v/2 + i$ edges and E_7 is a simple even graph with 7 edges.

Proof. (1) $m \equiv 0$ (*mod* 12) and $n \equiv 1$ or 9 (*mod* 12); $m \equiv 1$ (*mod* 12) and $n \equiv 2, 6, 8$ (*mod* 12); $n \equiv 9$ (*mod* 12) and $m \equiv 0, 2, 6, 8$ (*mod* 12). By Theorem 1.2, $K_m \times K_n - F$ can be decomposed into 6-cycles.

(2) $m \equiv 0$ or 2 (*mod* 12) and $n \equiv 3, 7$ (*mod* 12); $m \equiv 3$ (*mod* 12) and $n \equiv 6, 8$ (*mod* 12); $m \equiv 6$ (*mod* 12) and $n \equiv 7$ (*mod* 12); $m \equiv 7$ (*mod* 12) and $n \equiv 8$ (*mod* 12). By Theorem 1.2 and Lemma 2.5, $K_m \times K_n - F$ can be decomposed into 6-cycles.

(3) $m \equiv 0$ or 6 (*mod* 12) and $n \equiv 5$ (*mod* 12). By Lemma 2.2, $K_m \times K_n - F$

can be decomposed into 6-cycles.

(4) $m \equiv 1$ $(mod\ 12)$ and $n \equiv 3, 7$ $(mod\ 12)$. By Theorem 1.3. and Lemma 2.1, $K_m \times K_n - C_3$ can be decomposed into 6-cycles.

(5) $m \equiv 1$ $(mod\ 12)$ and $n \equiv 4, 10$ $(mod\ 12)$. By Theorem 1.3, Lemma 2.2. $K_m \times K_n - F_4$ can be decomposed into 6-cycles.

(6) $m \equiv 1$ $(mod\ 12)$ and $n \equiv 5$ $(mod\ 12)$. By Lemma 2.2. $K_m \times K_n - C_4$ can be decomposed into 6-cycles.

(7) $m \equiv 2$ or 8 $(mod\ 12)$ and $n \equiv 5$, $(mod\ 12)$. By Theorem 1.3, Lemma 2.2, and Lemma 2.7, $K_m \times K_n - F_2$ can be decomposed into 6-cycles.

(8) $m \equiv 2$ $(mod\ 12)$ and $n \equiv 2, 8$ $(mod\ 12)$; $m \equiv 8$ $(mod\ 12)$ and $n \equiv 8$ $(mod\ 12)$, $L_m = L_n = F$. So, we can decompose $K_m \times K_n$ which has leaves $\frac{n}{2}C_4$. Then by Lemma 2.3, $K_m \times K_n - C_4$ can be decomposed into 6-cycles.

(9) $m \equiv 2$ $(mod\ 12)$ and $n \equiv 4$ or 10 $(mod\ 12)$; $m \equiv 4$ $(mod\ 12)$ and $n \equiv 8$ $(mod\ 12)$; $m \equiv 8$ $(mod\ 12)$ and $n \equiv 10$ $(mod\ 12)$. By Lemma 2.3 and Lemma 2.6. $K_m \times K_n - C_4$ can be decomposed into 6-cycles.

(10) $m \equiv 3$ $(mod\ 12)$ and $n \equiv 4$ or 10 $(mod\ 12)$. By Theorem 1.3.and Lemma 2.2, and Lemma 2.5. $K_m \times K_n - F$ can be decomposed into 6-cycles.

(11) $m \equiv 3$ $(mod\ 12)$ and $n \equiv 5$ $(mod\ 12)$. By Theorem 1.3, Lemma 2.2, $K_m \times K_n$ can be decomposed into 6-cycles which has leave $C_3 \cup (C_4 \times C_3)$. By Lemma 2.2, $C_4 \times C_3$ can be decomposed into 6-cycles which has leave

$2C_3$. So $K_m \times K_n - 3C_3$ can be decomposed into 6-cycles. Let $C_1' = (a_0, a_1, a_5, a_2, a_3, a_4)$, and $C_2' = (c_0, c_1, c_5, c_2, c_3, c_4)$ be two 6-cycles in the 6-cycle system of $K_m \times K_n - 3C_3$. Let $L' = (a_i, b_i, c_i)$, $i \in Z_3$ and $H = (a_3, b_3, c_3) \cup (d_3, e_3, f_3)$. By Theorem 1.3 and Lemma 2.4, $C_1' \cup C_2' \cup L' \cup H - (d_3, e_3, f_3)$ can be decomposed into four 6-cycles. So, $K_m \times K_n - C_3$ can be decomposed into 6-cycles.

(12) $m \equiv 3$ or 7 $(mod\ 12)$ and $n \equiv 9$ $(mod\ 12)$. By Theorem 1.3 and Lemma 2.1, $K_m \times K_n - 3C_3$ can be decomposed into 6-cycles. Similar to (11), $K_m \times K_n - C_3$ can be decomposed into 6-cycles.

(13) $m \equiv 4$ or 10 $(mod\ 12)$ and $n \equiv 5$ $(mod\ 12)$. By Theorem 1.3, Lemma 2.2, $K_m \times K_n$ can be decomposed into 6-cycles which has leaves $F \cup 5K_4 \cup 4C_4$. Let $V(5K_4) = \{a_i, b_i, c_i, d_i | i \in Z_5\}$, and $4C_4 = (a_0, a_1, a_2, a_3) \cup (c_0, c_1, c_2, c_3) \cup (b_1, b_2, b_3, b_4) \cup (d_1, d_2, d_3, d_4)$. By Theorem 2.2, then $5K_4 \cup C_4 - F$ can be decomposed into six 6-cycles and $(b_1, b_2, d_2, d_1, d_4, b_4) \cup (a_4, c_4, b_4, b_3, d_3, d_4)$. So, $K_m \times K_n - F$ can be decomposed into 6-cycles.

(14) $m \equiv 4$ or 10 $(mod\ 12)$ and $n \equiv 7$ $(mod\ 12)$. By Theorem 1.3, Lemma 2.2, and Lemma 2.5, $K_m \times K_n - F_4$ can be decomposed into 6-cycles.

(15) $m \equiv 4$ or 10 $(mod\ 12)$ and $n \equiv 9$ $(mod\ 12)$. By Theorem 1.3, Lemma 2.2, and Lemma 2.5, $K_m \times K_n - F$ can be decomposed into 6-cycles.

(16) $m \equiv 5$ $(mod\ 12)$ and $n \equiv 5$ $(mod\ 12)$. By Theorem 1.3, Lemma 2.2, $K_m \times K_n$ can be decomposed into 6-cycles, which has leaves $2C_4 \cup (C_4 \times C_4)$. Similar to (13), $K_m \times K_n$ can be decomposed into 6-cycles, which has leave

a C_4.

(17) $m \equiv 5 \ (mod\ 12)$ and $n \equiv 7 \ (mod\ 12)$. By Theorem 1.3, Lemma 2.2, and Lemma 2.4, $K_m \times K_n - E_7$ can be decomposed into 6-cycles.

(18) $m \equiv 0, 1, 2, 4, 5, 6, 8, 9, 10 \ (mod\ 12)$ and $n \equiv 11 \ (mod\ 12)$. Then the proof is similar to $m \equiv 0, 1, 2, 4, 5, 6, 8, 9, 10 \ (mod\ 12)$ and $n \equiv 3 \ (mod\ 12)$, and $n \equiv 5 \ (mod\ 12)$.

(19) $m \equiv 7$ or $11 \ (mod\ 12)$ and $n \equiv 11 \ (mod\ 12)$. By Theorem 1.3, Lemma 2.1, Lemma 2.2, $K_m \times K_n$ can be decomposed into 6-cycles which has leave $C_4 \cup (C_3 \times C_3) \cup 8C_3 \cup 2(C_4 \times C_3 - C_3)$. By Lemma 2.1, Lemma 2.2 and Lemma 2.4, $K_m \times K_n - C_4$ can be decomposed into 6-cycles.

By (1) to (19), we conclude the proof. □

3 Remark on minimum coverings

Let G denote a graph, and $E(G)$ denote the collection of edges in the graph G. If E and P are collections of edges, then $E + P$ denote the union of the two collections (so if e occurs in E and occurs y times in P, then it occurs $1 + y$ times in $E + P$).

A *covering* of G with hexagons is a triple (S, C, P), where S is the vertex set of G, $P \subseteq E(G)$ called the padding, and C is a collection of hexagons that partition $E(G) + P$. The number n is called the order of the covering. So that there is no confusion: an edge $\{a, b\}$ belongs to exactly

$x + 1$ hexagons of C, where x is the number of times $\{a, b\}$ belongs to the padding P. If $|P|$ is as small as possible, then (S, C, P) is called a minimum covering of G with hexagons. So, a 6-cycle system is a minimum covering with hexagons. with padding $P = \emptyset$. For other cases, we can find a suitable P to use up the leaves (with some existent 6-cycles in packing). Thus, the minimum covering of $K_m \times K_n$ with hexagons can be obtained:

m / n	0	1	2	3	4	5	6	7	8	9	10	11
0	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F
1	F	ϕ	F_4	C_3	F_4	D	F	C_3	F_4	ϕ	F_4	C_5
2	ϕ	F_4	D	F	D	F	ϕ	F_4	D	F	D	F
3	F	C_3	F	ϕ	F	C_3	F	ϕ	F	C_3	F	ϕ
4	ϕ	F_4	D	F	ϕ	F_4	ϕ	F_4	D	F	ϕ	F_4
5	F	D	F	C_3	F_4	D	F	C_5	F	ϕ	F_4	C_5
6	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F	ϕ	F
7	F	C_3	F_4	ϕ	F_4	C_5	F	ϕ	F_4	C_3	F_4	D
8	ϕ	F_4	D	F	D	F	ϕ	F_4	D	F	D	F
9	F	ϕ	F	C_3	F	ϕ	F	C_3	F	ϕ	F	C_3
10	ϕ	F_4	D	F	ϕ	F_4	ϕ	F_4	D	F	ϕ	F_4
11	F	C_5	F	ϕ	F_4	C_5	F	D	F	C_3	F_4	D

where $D = \{\{v_1, v_2\}, \{v_1, v_2\}\}$, v_1, $v_2 \in V(D)$, F_i is an odd graph with $v/2 + i$ edges, C_i is a cycle of length i.

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