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INTERSECTION PROBLEM OF STEINER SYSTEMS $S(3, 4, 2v)$

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1. Introduction

A *Steiner quadruple system of order* v (SQS(v)) is a pair (Q, q), where Q is a v-set and q is a collection of 4-element subsets of Q, called *blocks*, such that every 3-element subset of Q is contained in exactly one block of q. Hanani [12] proved that an SQS(v) exists if and only if $v \equiv 2$ or 4 (mod 6). It is easy to see that $|q| = \frac{1}{24}v(v-1)(v-2)$, which we will denote by q_v in what follows.

Denote by $J[v]$ the set of all positive integers k such that there exists a pair of SQS(v) which have exactly k blocks in common, and set $I[v] = \{0, 1, 2, \ldots, q_v - 14\} \cup \{q_v - 12, q_v - 8, q_v\}$.

In [6], Gionfriddo and Lindner conjectured that $J[v] = I[v]$ for every $v \equiv 2$ or 4 (mod 6) and $v \geqslant 8$. Since the conjecture by Giofriddo and Lindner, a considerable amount of work has been done in an attempt to prove that $J[v] = I[v]$ [3–11]. But compared to the whole problem, we still have a lot of work to do.

In this paper, we proved that $J[2v] = I[2v]$ for certain admissible order v of Steiner quadruple systems, and $I[2v] \setminus \{q_{2v} - h: h = 17, 18, 19\} \subseteq J[2v]$ for every $v \equiv 2$ or 4 (mod 6). Moreover, with the assumption of $J[16] = I[16]$, $J[20] = I[20]$, and $J[28] = I[28]$, we are able to prove that $J[2v] = I[2v]$, where $v \equiv 2$ or 4 (mod 6) and $v \geqslant 4$.

2. The main theorems

A *partial quadruple system* (PQS) is a pair (P, p), where P is a finite set and p is a collection of 4-subsets of P (called blocks) such that every 3-subset of P is contained in at most one block of p. Two partial quadruple systems (P, p_1) and (P, p_2) are said to be *mutually* balanced if any given triple of distinct elements of P is contained in a block in p_1 if and only if it is contained in a block of p_2. Two mutually balanced PQSs are disjoint if they have no block in common. (DMB PQSs). It is easy to see if (P, p_1) and (P, p_2) are two DMB PQSs, then

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$|p_1| = |p_2|$. Also, if (P, p_1) is a PQS of an SQS(Q, q), then $(Q, (q \backslash p_1) \cup p_2)$ is also an SQS; moreover, (Q, q) and $(Q, (q \backslash p_1) \cup p_2)$ have $|q| - |p_1|$ blocks in common.

Construction A ([15]). Denote by α a permutation on $Q = \{1, 2, \ldots, v\}$, where $v \equiv 2$ or 4 (mod 6). Let (Q, q) be an SQS(v). Set $S = Q \times \{1, 2\}$, define a collection of blocks B_α on S as follows:

(1) For each block $\{x, y, z, w\} \in q$, the following collection of blocks $(B_\alpha\{x, y, z, w\})$ is contained in B_α:

$$\{(x,1),(y,1),(z,1),(w_\alpha,2)\}, \quad \{(x,2),(y,2),(z,2),(w_{\alpha^{-1}},1)\},$$
$$\{(x,1),(y,1),(z_\alpha,2),(w,1)\}, \quad \{(x,2),(y,2),(z_{\alpha^{-1}},1),(w,2)\},$$
$$\{(x,1),(y_\alpha,2),(z,1),(w,1)\}, \quad \{(x,2),(y_{\alpha^{-1}},1),(z,2),(w,2)\},$$
$$\{(x_\alpha,2),(y,1),(z,1),(w,1)\}, \quad \{(x_{\alpha^{-1}},1),(y,2),(z,2),(w,2)\}.$$

(2) For each 2-subset of Q, $\{(x,1), (y,1), (x_\alpha, 2), (y_\alpha, 2)\} \in B_\alpha$.

In what follows, we will use $B\{x, y, z, w\}$ to denote the collection of eight blocks obtained from $\{x, y, z, w\} \in q$ in (1), where α is an identity mapping. Also we let the following collection of blocks be $B'\{x, y, z, w\}$:

$$\{(x,1),(y,1),(z,1),(w,1)\}, \quad \{(x,2),(y,2),(z,2),(w,2)\},$$
$$\{(x,1),(y,1),(z,2),(w,2)\}, \quad \{(x,2),(y,2),(z,1),(w,1)\},$$
$$\{(x,1),(y,2),(z,1),(w,2)\}, \quad \{(x,2),(y,1),(z,2),(w,1)\},$$
$$\{(x,1),(y,2),(z,2),(w,1)\}, \quad \{(x,2),(y,1),(z,1),(w,2)\}.$$

It is a routine matter to check $B\{x, y, z, w\}$ and $B'\{x, y, z, w\}$ are DMB PQSs'.

Lemma 2.1. *Let u and v be admissible order of* SQS. *If there exists an* SQS(v) *which contains a subsystem of order u, then $J[2u] \subseteq J[2v]$.*

Proof. Let (Q, q) be an SQS(v) with a subsystem of order u, and α be the permutation $(1, 2, \ldots, u)(u+1, u+2, \ldots, v)$. Since $u \geq 4$ and $v \geq 2u$, it is a result of Construction A [15] that (S, B_α) and (S, B_e) have no blocks in common. (e is the identity mapping.) Also, (S, B_α) and (S, B_e) have subsystems of order $2u$ respectively, which can be replaced by any SQS$(2u)$. this concludes the proof. □

Lemma 2.2. *If there exists an* SQS(v) *which contains a subsystem of order u, then $t_v - t_u + k_1 + k_2 \in J[2v]$ for every $k_1 \in J[2u]$ and $k_2 \in (q_v - q_u) \cdot \{0, 8\}$, where $t_u = \frac{1}{2}u(u-1)$, and $(q_v - q_u) \cdot \{0, 8\}$ is the sum of $(q_v - q_u)$ copies of $\{0, 8\}$.* ($A + B = \{a + b \mid a \in A$ and $b \in B\}$).

Proof. Let (Q, q) be an SQS(v) with a subsystem of order u. By taking α as the identity mapping, we can construct an SQS($2v$)(S, B_e) which contains a subsystem of order $2u$, (P, p). Since for each $\{x, y, z, w\}$ in $q \backslash p$, $B\{x, y, z, w\}$ can be replaced by $B'\{x, y, z, w\}$, and (P, p) can be replaced by any SQS($2u$), we conclude the proof. □

We note here that since an SQS(4) can be embedded in an SQS(v), hence we have $\{t_v - 6\} + \{0, 2, 6, 14\} + (q_v - 1) \cdot \{0.8\} \subseteq J[2v]$ for every admissible order $v \geq 8$ of SQS.

Lemma 2.3. *If there exists an* SQS(v) *which contains a subsystem of order* $u \geq 8$, *and* $t_v - t_u \leq q_{2u} - 14$, *then* $J[2u] = I[2u]$ *implies that* $J[2v] = I[2v]$.

Proof. For every $k \in I[2v]$, if $k \geq t_v - t_u$, since $u \geq 8$, and $J[2u] = I[2u]$, by Lemma 2.2, $k \in J[2v]$. If $k < t_v - t_u$, then $k < q_{2u} - 14$, hence $k \in I[2u] = J[2u] \subseteq J[2v]$ (Lemma 2.1). Since it has been shown $J[2v] \subseteq I[2v]$, [6] we have the proof. □

For example: $u = 20$, $v = 58$. Since there exists an SQS(58) which contains a subsystem of order 20, [12] and $J[40] = I[40]$ [4] we conclude that $J[116] = I[116]$.

As we can see, Lemma 2.3 will provide a bunch of orders v such that $J[2v] = I[2v]$ if the following three conditions are satisfied:

(i) an SQS(u) can be embedded in an SQS(v),

(ii) $t_v - t_u \leq q_{2u} - 14$, and

(iii) $J[2u] = I[2u]$.

We will discuss these conditions separately.

From [12, 13] we have the following lemma.

Lemma 2.4. *An* SQS(u) *can be embedded in an* SQS(v) *if* (i) $v = 2u$, (ii) $v = 3u - 2$, (iii) $v = 3u - 2w$ and there exists an SQS(u) *which contains a subsystem or order* w (including $w = 2$), (iv) $v = 4u - 6$, *and* (v) $v = 12u - 10$.

By a direct computation, we have Table 1 to show that under certains kinds of embeddings, how large u is in order to get $t_v - t_u \leq q_{2u} - 14$ and $t_v - t_u \leq q_{2u} - 20$.

Since in [4], it has been shown that $J[4s] = I[4s]$ for every admissible order

Table 1

Embeddings	$t_v - t_u \leq q_{2u} - 14$	$t_v - t_u \leq q_{2u} - 20$
$v = 2u$	$u \geq 8$	$u \geq 8$
$v = 3u - 2$	$u \geq 14$	$u \geq 14$
$v = 3u - 2w$	$u \geq 14$	$u \geq 14$
$v = 4u - 6$	$u \geq 22$	$u \geq 22$
$v = 12u - 10$	$u \geq 218$	$u \geq 218$

$s \geqslant 10$ of Steiner quadruple systems, we can, of course, put $u = 2s$. Thus, we obtain an infinite class of orders v such that $J[2v] = I[2v]$. For example $v = 6s - 2$, $s \geqslant 10$, and s is an admissible order of Steiner quadruple systems.

Theorem 2.5. *If*

(i) $v = 6u - 2$, $u \geqslant 10$; *or*

(ii) $v = 8u - 6$, $u \geqslant 14$; *or*

(iii) $v = 6u - 2w$, $u \geqslant 10$, *and there exists as* SQS$(2u)$ *which contains a subsystem of order* w *(including* $(w = 2)$; *or*

(iv) $v = 24u - 10$, $u \geqslant 110$;

then $J[2v] = I[2v]$.

Proof. This is a direct result from Table 1, Lemma 2.3, and $J[4u] = I[4u]$ for every $u \geqslant 10$ [4]. □.

For the situation $t_v - t_u > q_{2u} - 14$, it is possible that we can use the following lemma to show that $J[2v] = I[2v]$.

Lemma 2.6. *If there exists an* SQS(v), (Q, q), *which contains a subsystem* (P, p) *of order* $u \geqslant 8$, *and there exists at least* $\lceil \frac{1}{6}(t_v - t_u - q_{2u} + 14) \rceil + 1$ *blocks in* $q \backslash p$ *such that no two blocks have more than one element in common, and each block contains at most one element of* p, *then* $J[2u] = I[2u]$ *implies that* $J[2v] = I[2v]$. ($\lceil a \rceil$ *is the greatest integer not greater than* a.)

Proof. The proof is similar to that of Lemma 2.2. For each block $\{x, y, z, w\} \in q \backslash p$, it is not difficult to see $B\{x, y, z, w\}$ (as in Lemma 2.2) and the collection of blocks

$$
\begin{array}{ll}
\{(x,1),(y,1),(x,2),(y,2)\}, & \\
\{(x,1),(z,1),(x,2),(z,2)\}, & \\
\{(x,1),(w,1),(x,2),(w,2)\}, & \{(y,1),(z,1),(y,2),(z,2)\}, \\
\{(y,1),(w,1),(y,2),(w,2)\}, & \\
\{(z,1),(w,1),(z,2),(w,2)\}, &
\end{array}
$$

form an SQS(8) based on the set $\{(x, 1), (y, 1), (z, 1), (w, 1), (x, 2), (y, 2), (z, 2), (w, 2)\}$. Since $J[8] = \{0, 2, 6, 14\}$ we have

$$
\begin{aligned}
&J[2u] + t_v - t_u - 6(\lceil \tfrac{1}{6}(t_v - t_u - q_{2u} + 14) \rceil + 1) \\
&\quad + (q_v - q_u - \lceil \tfrac{1}{6}(t_v - t_u - q_{2u} + 14) \rceil - 1) \cdot \{0, 8\} \\
&\quad + (\lceil \tfrac{1}{6}(t_v - t_u - q_{2u} + 14) \rceil + 1) \cdot \{0, 2, 6, 14\} \subseteq J[2v].
\end{aligned}
$$

Also $t_v - t_u - 6(\lceil \frac{1}{6}(t_v - t_u - q_{2u} + 14) \rceil + 1)$ is less than $q_{2u} - 14$, hence we have $I[2v] \setminus \{0, 1, \ldots, q_{2u} - 14\} \subseteq J[2v]$. By Lemma 2.1, we conclude the proof. □

With Lemma 2.6, we can have $J[44] = I[44]$ and $J[52] = I[52]$ if $J[16] = I[16]$ and $J[20] = I[20]$ respectively.

Lemma 2.7. *If* $J[16] = I[16]$, *then* $J[44] = I[44]$.

Proof. Since there exists an SQS(22) which contains a subsystem of order 8 [12], and there are $\lceil \frac{1}{6}(t_{22} - t_8 - q_{16} + 14) \rceil + 1 = 13$ blocks which are not in the subsystem such that no two blocks have more than one element in common, hence by Lemma 2.6, we have $J[44] = I[44]$. (We omit the tedious work on finding those 13 blocks.) □

Lemma 2.8. *If* $J[20] = I[20]$, *then* $J[52] = I[52]$.

Proof. Since there exists an SQS(26) which contains a subsystem of order 10 [12], and it is easy to find $\lceil \frac{1}{6}(t_{26} - t_{10} - q_{20} + 14 \rceil + 1 = 2$ blocks which are not in the subsystem such that these two blocks don't have more than one element in common, hence by Lemma 2.5, we have the proof. □

As we can see, with $J[2u] = I[2u]$ for small orders u, we can have $J[2v] = I[2v]$ for larger orders. In what follows we are going to use recursive method to prove that $J[2v] = I[2v]$ for every $v \equiv 2$ or $4 \pmod 6$, $v \geqslant 4$, provided $J[16] = I[16]$, $J[20] = I[20]$, and $J[28] = I[28]$. (As far as the author can tell, these three orders have not been totally solved yet.)

Theorem 2.9. *If* $J[16] = I[16]$, $J[20] = I[20]$, *and* $J[28] = I[28]$, *then* $J[2v] = I[2v]$ *for every* $v \equiv 2$ *or* $4 \pmod 6$, $v \geqslant 4$.

Proof. Since $J[2v] = I[2v]$ for each $v = 8$, 10, 14, 16, 20 [4], 22 (Lemma 2.7), 26 (Lemma 2.8), 28 [4], 32 [4], $34 = 3 \cdot 14 - 8$, $38 = 3 \cdot 14 - 4$, 40 [4], 44 [4], $46 = 3 \cdot 16 - 2$, 50 [5], 52 [4], 56 [4], $58 = 3 \cdot 20 - 2$, $62 = 3 \cdot 22 - 4$, 64 [4], 68 [4], $70 = 3 \cdot 26 - 8$, $74 = 3 \cdot 26$, 76 [4], 80 [4], $82 = 3 \cdot 28 - 2$, $86 = 3 \cdot 38 - 2$ 14, 88 [4], 92 [4], $94 = 3 \cdot 32 - 2$, $98 = 3 \cdot 34 - 4$, 100 [5]. It is not difficult to see, we can use the recursive method now, if v is obtained from the embeddings (i), (ii), (iii), and (iv) in Lemma 2.4 with $u \geqslant 8$ or 14 or 22 as the case may be. All we have to show is the case $v = 12u - 10$. If $u = 6n + 4$, then $v = 12(6n + 4) - 10 = 72n + 38 = 3(24n + 14) - 4$, this in (iii). If $u = 6n + 2$, we use Table 2 to show that we can change v from embedding (v) to one of (i), (ii), (iii), and (iv). (We note here that if $v \equiv 2$ or $4 \pmod 6$, then v is in at least one of the five forms of Lemma 2.4.). □

Table 2

n	v	Embeddings
1	86	$3\cdot 38-2\cdot 14\,(38=3\cdot 14-4)$
2	158	$3\cdot 70-2\cdot 26\,(70=3\cdot 26-8)$
3	230	$3\cdot 86-2\cdot 14\,(\mathrm{n}=1)$
4	302	$3\cdot 110-2\cdot 14\,(110=3\cdot 38-4, 38=3\cdot 14-4)$
5	374	$3\cdot 182-2\cdot 86\,(182=3\cdot 86-2\cdot 38, n=1)$
6	446	$3\cdot 158-2\cdot 14\,(158=12\cdot 14-10)$
7	518	$3\cdot 182-2\cdot 14\,(n=5)$
8	590	$3\cdot 254-2\cdot 86\,(254=3\cdot 86-4)$
9	662	$3\cdot 230-2\cdot 14\,(n=3)$
10	734	$3\cdot 254-2\cdot 14\,(n=8, n=1)$
11	806	$3\cdot 350-2\cdot 122\,(350=3\cdot 122-2\cdot 8, 122=4\cdot 32-6, 32=4\cdot 8)$
12	878	$3\cdot 302-2\cdot 14\,(n=4)$
13	950	$3\cdot 326-2\cdot 14\,(326=3\cdot 110-4, n=4)$
14	1022	$3\cdot 446-2\cdot 158\,(n=6)$
15	1094	$3\cdot 374-2\cdot 14\,(n=5)$
16	1166	$3\cdot 398-2\cdot 14\,(398=12\cdot 34-10, 34=3\cdot 14-8)$
17	1238	$3\cdot 422-2\cdot 14\,(422=3\cdot 142-4, 142=3\cdot 50-8, 50=4\cdot 14-6)$
18	1310	$3\cdot 446-2\cdot 14\,(n=6)$
19	1382	$3\cdot 470-2\cdot 14\,(470=3\cdot 158-4, n=6)$
20	1454	$3\cdot 494-2\cdot 14\,(494=3\cdot 166-4, 166=3\cdot 56-2, 56=4\cdot 14)$
21	1526	$3\cdot 518-2\cdot 14\,(n=7)$
22	1598	$3\cdot 542-2\cdot 14\,(542=3\cdot 182-4, n=7)$
23	1670	$3\cdot 566-2\cdot 14\,(566=3\cdot 190-4, 190=7\cdot 28-6)$ [13]
24	1742	$3\cdot 590-2\cdot 14\,(n=8)$
25	1814	$3\cdot 614-2\cdot 14\,(614=3\cdot 230-2\cdot 38, n=3)$
26	1886	$3\cdot 638-2\cdot 14\,(638=3\cdot 274-2\cdot 92, 274=3\cdot 92-2, 92=3\cdot 40-2\cdot 14, 40=3\cdot 14-2)$
27	1958	$3\cdot 662-2\cdot 14\,(n=9)$
28	2030	$3\cdot 686-2\cdot 14\,(686=3\cdot 230-4, n=3)$
29	2102	$3\cdot 710-2\cdot 14\,(710=3\cdot 238-4, 238=3\cdot 80-2, 80=2\cdot 40, 40=3\cdot 14-2)$
30	2174	$3\cdot 734-2\cdot 14\,(n=10)$
31	2246	$3\cdot 758-2\cdot 14\,(758=3\cdot 254-4, 254=3\cdot 86-4, n=1)$
32	2318	$3\cdot 782-2\cdot 14\,(782=3\cdot 266-2\cdot 8, 266=3\cdot 94-2\cdot 8, 94=3\cdot 32-2, 32=4\cdot 8, 94=3\cdot 34-8, 34=3\cdot 14-8)$
33	2390	$3\cdot 806-2\cdot 14\,(806=3\cdot 278-2\cdot 14, 278=3\cdot 94-8, 94=3\cdot 34-8, 34=3\cdot 14-8)$
34	2462	$3\cdot 830-2\cdot 14\,(830=3\cdot 278-4, 278=3\cdot 94-4, 94=3\cdot 34-8, 34=3\cdot 14-8)$

Without assuming $J[16]=I[16]$, $J[20]=I[20]$, and $J[28]=I[28]$, we still have the following corollary.

Corollary 2.10. $J[2v]\supseteq I[2v]\backslash\{q_{2v}-h\colon h=17, 18, 19\}$, $v\geqslant 4$, $v\equiv 2$ *or* 4 (mod 6).

Proof. By a similar argument as above and the results from [6–11] we have $J[2v]\supseteq I[2v]\backslash\{q_{2v}-h\colon h=17, 18, 19\}$ except $v=22$ and 26 (Table 1). With one more block in Lemma 2.7 and Lemma 2.8, we should be able to prove this corollary. (We omit again the tedious work.) □

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