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A solution to the forest leave problem for partial 6-cycle systems

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Abstract

In this paper, we find necessary and sufficient conditions for the existence of a 6-cycle system of $K_n - E(F)$ for *every* forest F of K_n.

Keywords: Cycle systems; Forest; Leave; 6-cycle

1. Introduction

An H-*decomposition* of the graph G is a partition of $E(G)$ such that each element of the partition induces a subgraph isomorphic to H. In the case where H is an m-cycle, such a decomposition is referred to as an *m-cycle system* of G. An m-cycle system of G will be formally described as an ordered pair (V,B), where V is the vertex set of G and B is the set of m-cycles.

Results in this area date back to the nineteenth century [7], but have received a lot of attention over the past 40 years. There have been many results found on H-decompositions of G for various graphs H and G, usually with $G = K_n$. One particularly enticing but difficult problem was to solve the case where H is an m-cycle (see [8,9] for surveys of results). This all culminated in two papers. Based on a result

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by Hoffman et al. [6], a paper by Alspach and Gavlas [1] and another by Šajna [10] settled the problem of finding the values of n for which there exists an m-cycle system of K_n.

What about the many cases where K_n does not have an H-decomposition into m-cycles? Recently a paper by Alspach and Gavlas [1] and another by Šajna [10] settled the problem of finding the values of n for which there exists an m-cycle system of $K_n - I$, where I is a 1-factor. This can alternatively be viewed as a *partial* m-cycle system in which the set of edges not in any m-cycle induces a 1-factor. These edges not in any m-cycle (or the subgraph they induce) are called the *leave* L.

Continuing with the theme of finding graph decompositions of graphs which are close to complete, one way to extend these results is by finding necessary and sufficient conditions for the existence of a m-cycle system of $K_n - E(L)$, where the leave L is a spanning forest (a *forest* is a graph that contains no cycle; a connected forest is a *tree*). Clearly this would generalize the result described above where the leave is a 1-factor. This generalization is so extensive that it becomes very difficult to solve unless further restrictions are made, such as fixing the value of m. Fu and Rodger [4] have obtained such a result by finding necessary and sufficient conditions for the existence of a 4-cycle system of $K_n - E(F)$, where F is *any* forest in K_n. Here we extend these results by finding necessary and sufficient conditions for the existence of a 6-cycle system of $K_n - E(F)$ for *any* spanning forest F.

As will be shown, when the leave is a forest, it must be that n is even. Results obtaining m-cycle systems of graphs that are close to complete have also been found when m is odd, in which case each vertex has even degree in the leave (for example [3] when $m = 3$, [5] when $m = 4$ and [2] when $m = 6$).

Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. On occasions arithmetic operations will be defined on $\mathbb{Z}_n$, in which case we assume they are performed modulo n.

2. The small cases

As stated in the introduction, this section will provide necessary and sufficient conditions for the existence of a 6-cycle system of $K_n - E(F)$ for *any* spanning forest F. The approach taken in this paper requires that we know the number of components in F. This can be seen in Lemma 2.1, the result being summarized in Table 1.

Lemma 2.1. *Let n be even and let F be a spanning forest of the complete graph K_n with $c(F)$ components. The number of edges in $K_n - E(F)$ is divisible by* 6 *if and only if n and $c(F)$ are related as in Table* 1.

Proof. Let $C_1, C_2, C_3, \ldots, C_{c(F)}$ be the components of F. Then

$$|E(F)| = \sum_{i=1}^{c(F)} |E(C_i)| = \sum_{i=1}^{c(F)} (V(C_i) - 1) = \sum_{i=1}^{c(F)} V(C_i) - \sum_{i=1}^{c(F)} 1 = n - c(F).$$

Therefore, $c(F) = n - |E(F)|$.

Table 1
The number of components required in F in order that 6 divides $|E(K_n - E(F))|$ when n is even

n	$12k$	$12k+2$	$12k+4$	$12k+6$	$12k+8$	$12k+10$
$c(F) \pmod 6$	0	1	4	3	4	1

So if 6 divides $|E(K_n - E(F))|$ then $c(F) \pmod 6 \equiv (n - |E(F)|) \pmod 6 \equiv (n - (n^2-n)/2) \pmod 6$.

Also, if $c(F)$ and n are related as in Table 1 then $c(F) \equiv n - (n^2-n)/2 \pmod 6$, and so $|E(F)| \equiv |E(K_n)| \pmod 6$. So the result follows. □

Another tool that we will need is from a theorem by Sotteau [11]. Sotteau proved a generalization of the following result. It is stated here for 6-cycles only.

Lemma 2.2. *There exists a* 6*-cycle system of* $K_{a,b}$ *if and only if*:

(1) *a and b are even*,
(2) 6 *divides a or b*, *and*
(3) $a, b \geqslant 4$.

The following lemma is a special case of the results in [1], but is easily proved so is included for completeness:

Lemma 2.3. *Let* $n \in \{2,6,8\}$, *and let F be a* 1*-factor in* K_n. *There exists a* 6*-cycle system of* $K_n - E(F)$.

Proof. In each case, we define the 6-cycle system on the vertex set $\mathbb{Z}_n$.

(1) If $n=2$, let $B=\phi$ be the required 6-cycle system with leave $F=\{\{0,1\}\}$.
(2) If $n=6$, let $B=\{(0,2,1,4,3,5), (0,4,2,5,1,3)\}$ be the required 6-cycle system with leave $F=\{\{0,1\}, \{2,3\}, \{4,5\}\}$.
(3) If $n=8$, let $B=\{(0,1,2,3,5,7), (0,2,5,6,3,4), (0,3,1,7,4,6), (1,5,4,2,7,6)\}$ be the required 6-cycle system with leave $F=\{\{0,5\}, \{7,3\}, \{6,2\}, \{1,4\}\}$. □

Lemma 2.4. *If F is a spanning forest of* K_{10} *in which each vertex has odd degree and for which* 6 *divides* $|E(K_{10}-E(F))|$, *then there exists a* 6*-cycle system of* $K_{10}-E(F)$.

Proof. The maximum number of components possible in a subgraph of K_{10} in which each vertex has odd degree is 5 (i.e. a 1-factor). So by Lemma 2.1 and Table 1, since we are given that 6 divides $|E(K_{10}-E(F))|$ it follows that $c(F)=1$; so F is a tree.

There are 7 possibilities for the leave F. For $1 \leqslant i \leqslant 7$, a 6-cycle system $(\mathbb{Z}_{10}, B_i)$ of $K_{10}-E(F_i)$ is given below, where F_i is the tree induced by the edges occurring in no 6-cycle in B_i.

$B_1 = \{(1,4,5,8,2,9),\ (2,5,6,9,3,7),\ (3,6,4,7,1,8),\ (1,2,4,8,7,5),\ (2,3,5,9,8,6),\ (3,1,6,7,9,4)\}$, so $E(F_1) = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\{0,7\},\{0,8\},\{0,9\}\}$.

$B_2 = \{(1,4,5,8,2,9),\ (2,5,6,9,3,7),\ (3,6,4,0,1,8),\ (1,2,4,8,7,5),\ (2,3,5,9,8,6),\ (3,1,6,7,9,4)\}$, so $E(F_2) = \{\{0,2\},\{0,3\},\{0,5\},\{0,6\},\{0,7\},\{0,8\},\{0,9\},\{7,4\},\{7,1\}\}$.

$B_3 = \{(1,4,5,8,2,9),\ (2,5,6,9,3,7),\ (3,6,4,0,1,8),\ (1,2,4,8,7,5),\ (2,3,5,9,8,6),\ (3,0,6,7,9,4)\}$, so $E(F_3) = \{\{0,2\},\{0,5\},\{0,7\},\{0,8\},\{0,9\},\{7,4\},\{7,1\},\{1,3\},\{1,6\}\}$.

$B_4 = \{(1,4,5,8,2,9),\ (2,5,0,9,3,7),\ (3,6,4,0,1,8),\ (1,2,4,8,7,5),\ (2,3,5,9,8,6),\ (3,0,6,7,9,4)\}$, so $E(F_4) = \{\{0,2\},\{0,7\},\{0,8\},\{7,4\},\{7,1\},\{1,3\},\{1,6\},\{6,5\},\{6,9\}\}$.

$B_5 = \{(1,4,5,8,2,9),\ (2,5,0,9,3,7),\ (3,6,4,0,1,8),\ (1,2,4,8,7,5),\ (2,3,5,9,8,6),\ (3,1,6,7,9,4)\}$, so $E(F_5) = \{\{0,2\},\{0,3\},\{0,6\},\{0,7\},\{0,8\},\{7,4\},\{7,1\},\{6,5\},\{6,9\}\}$.

$B_6 = \{(1,4,5,8,2,9),\ (2,5,6,9,3,7),\ (3,6,4,0,1,8),\ (1,2,4,8,0,5),\ (2,3,5,9,8,6),\ (3,1,6,7,9,4)\}$, so $E(F_6) = \{\{0,2\},\{0,3\},\{0,6\},\{0,7\},\{0,9\},\{7,4\},\{7,1\},\{7,5\},\{7,8\}\}$.

$B_7 = \{(1,4,5,8,2,9),\ (2,5,6,9,3,7),\ (3,6,4,0,1,8),\ (1,2,0,8,7,5),\ (2,3,5,9,8,6),\ (3,0,6,7,9,4)\}$, so $E(F_7) = \{\{0,5\},\{0,7\},\{0,9\},\{7,4\},\{7,1\},\{1,3\},\{1,6\},\{4,2\},\{4,8\}\}$. □

3. Some building blocks

In this section, we provide some 6-cycle systems of small graphs which will be used to build 6-cycle systems of $K_n - E(F)$ in Section 4. For the next six lemmas, let $G_{\gamma,\beta} = G_{\gamma,\beta}(V)$ be a graph labeled with the vertices in the sequence V (the order in which the vertices are listed is important) and let B be a 6-cycle system of $G_{\gamma,\beta}(V)$ (γ and β are indices that represent the βth graph defined in the γth lemma of this section (i.e. in Lemma 3.γ)). Let G^c denote the complement of a graph G. Also let $G \vee H$ denote the *join* of two vertex disjoint graphs G and H (so $E(G\vee H)=E(G)\cup E(H)\cup \{\{u,v\}\colon u\in V(G), v\in V(H)\}$).

Lemma 3.1. *Let* $G_{1,1}(0,1,2,\ldots,11)$ *be the graph* $K_8^c \vee K_4 + \{\{0,2\},\ \{1,2\}\} - \{\{0,11\},\ \{1,10\},\ \{2,9\},\ \{2,8\}\}$ *with* $V(K_8^c)=\mathbb{Z}_8$ *and* $V(K_4)=\mathbb{Z}_{12}\setminus\mathbb{Z}_8$. *There exists a* 6-*cycle system of* $G_{1,1}$.

Proof. There exists a 6-cycle system $(\mathbb{Z}_{12},B)$ of $G_{1,1}$ defined by $B=\{(0,2,1,8,9,10),\ (2,10,5,8,4,11),\ (0,8,6,10,11,9),\ (1,9,4,10,8,11),\ (3,10,7,9,6,11),\ (3,8,7,11,5,9)\}$. □

Lemma 3.2. *Let* $G_{2,1}(0,1,2,\ldots,7)$ *be the graph* $K_4^c \vee K_4 - \{\{0,i\}\mid 4\leqslant i\leqslant 7\}$, *and let* $G_{2,2}(0,1,2,\ldots,7)$ *be the graph* $K_4^c \vee K_4 - \{\{i,7-2i\},\ \{i,6-2i\}\mid i\in\mathbb{Z}_2\}$ *with* $V(K_4^c)=\mathbb{Z}_4$ *and* $V(K_4)=\mathbb{Z}_8\setminus\mathbb{Z}_4$ *in each case. There exists a* 6-*cycle system* B *of* $G_{2,\beta}$ *for* $1\leqslant\beta\leqslant 2$.

Proof. For $1 \leqslant \beta \leqslant 2$, there exists a 6-cycle system $(\mathbb{Z}_8, B_\beta)$ of $G_{2,\beta}$ defined by $B_1 = \{(1,7,2,4,6,5), (1,6,3,7,5,4), (2,6,7,4,3,5)\}$ and $B_2=\{(0,4,6,7,3,5), (1,7,4,5,2,6), (2,4,3,6,5,7)\}$. □

Lemma 3.3. *Let* $G_{3,1}(0,1,2,\ldots,15)$ *be the graph* $K_{10}^c \vee K_6 + \{\{0,3\},\{1,4\},\{2,4\}\} - \{\{i,15-i\},\{4,10\} \mid i \in \mathbb{Z}_5\}$, *and let* $G_{3,2}(0,1,2,\ldots,15)$ *be the graph* $K_{10}^c \vee K_6 + \{\{0,3\},\{1,3\},\{2,3\}\} - \{\{i,15-i\},\{3,11\},\{3,10\} \mid i \in \mathbb{Z}_4\}$ *with* $V(K_{10}^c) = \mathbb{Z}_{10}$ *and* $V(K_6)=\mathbb{Z}_{16} \setminus \mathbb{Z}_{10}$ *in each case. There exists a* 6*-cycle system B of* $G_{3,\beta}$ *for* $1 \leqslant \beta \leqslant 2$.

Proof. For $1 \leqslant \beta \leqslant 2$, there exists a 6-cycle system $(\mathbb{Z}_{16}, B_\beta)$ of $G_{3,\beta}$ defined by

$B_1 = \{(2,4,1,12,11,10), (10,3,11,15,14,0), (3,0,11,7,12,13), (4,13,1,15,10,12), (5,10,9,14,2,11), (5,15,6,11,13,14), (6,13,7,15,8,12), (3,15,13,10,7,14), (4,15,12,9,11,14), (5,13,9,15,2,12), (6,14,8,11,1,10), (0,13,8,10,14,12)\}$ and

$B_2 = \{(2,3,1,12,11,10), (10,4,11,15,14,0), (3,0,11,7,12,13), (4,13,1,15,10,12), (5,10,9,14,2,11), (5,15,6,11,13,14), (6,13,7,15,8,12), (3,15,13,10,7,14), (4,15,12,9,11,14), (5,13,9,15,2,12), (6,14,8,11,1,10), (0,13,8,10,14,12)\}$. □

Lemma 3.4. *Let* $G_{4,1}(0,1,2,\ldots,15)$ *be the graph* $K_8^c \vee K_8 - \{\{0,i\} \mid 8 \leqslant i \leqslant 15\}$,

let $G_{4,2}(0,1,2,\ldots,15) = K_8^c \vee K_8 - \{1,8\},\{1,9\},\{0,i\} \mid 10 \leqslant i \leqslant 15\}$,

let $G_{4,3}(0,1,2,\ldots,15) = K_8^c \vee K_8 - \{i,15-4i\},\{i,14-4i\},\{i,13-4i\},\{i,12-4i\} \mid i \in \mathbb{Z}_2\}$,

let $G_{4,4}(0,1,2,\ldots,15) = K_8^c \vee K_8 - \{0,i\},\ \{1,j\},\ \{2,k\} \mid 12 \leqslant i \leqslant 15, 10 \leqslant j \leqslant 11, 8 \leqslant k \leqslant 9\}$, *and*

let $G_{4,5}(0,1,2,\ldots,15)=K_8^c \vee K_8 - \{\{i,15-2i\},\{i,14-2i\} \mid i \in \mathbb{Z}_4\}$, *with* $V(K_8^c)=\mathbb{Z}_8$ *and* $V(K_8) = \mathbb{Z}_{16} \setminus \mathbb{Z}_8$ *for each case. There exists a* 6*-cycle system* (Z_{16}, B_β) *of* $G_{4,\beta}$ *for* $1 \leqslant \beta \leqslant 5$.

Proof. Let $(\mathbb{Z}_{16}/\mathbb{Z}_7, B)$ be a 6-cycle system of K_9 (this is easy to do; or see [1], or see [8] for a survey). For $1 \leqslant \beta \leqslant 5$, there exists a 6-cycle system $(\mathbb{Z}_{16}, B_\beta)$ of $G_{4,\beta}$ defined by

$B_1 = B \cup \{(9,1,8,6,15,4), (11,1,10,5,14,4), (9,2,8,5,11,6), (13,3,12,6,10,4), (13,6,14,3,10,2), (13,5,15,2,12,1), (11,3,15,1,14,2), (8,4,12,5,9,3)\}$,

$B_2 = B \cup \{(9,0,8,6,15,4), (11,1,10,5,14,4), (9,2,8,5,11,6), (13,3,12,6,10,4), (13,6,14,3,10,2), (13,5,15,2,12,1), (11,3,15,1,14,2), (8,4,12,5,9,3)\}$,

$B_3 = B \cup \{(9,0,8,6,15,4), (11,0,10,5,14,4), (9,2,8,5,11,6), (13,3,12,6,10,4), (13,6,14,3,10,2), (13,5,15,2,12,1), (11,3,15,1,14,2), (8,4,12,5,9,3)\}$,

$B_4 = B \cup \{(9,0,8,6,15,4), (11,0,10,5,14,4), (9,1,8,5,11,6), (13,3,12,6,10,4), (13,6,14,3,10,2), (13,5,15,2,12,1), (11,3,15,1,14,2), (8,4,12,5,9,3)\}$, and

$B_5 = B \cup \{(9,0,8,6,15,4), (13,0,10,5,14,4), (9,1,8,5,13,6), (11,0,12,6,10,4), (11,6,14,3,10,1), (11,5,15,2,12,3), (13,3,15,1,14,2), (8,4,12,5,9,2)\}$. □

Lemma 3.5. *Let* $G_{5,1}(0,1,2,\ldots,15)$ *be the graph* $K_8^c \vee K_8$—$\{\{0,15\},\ \{0,14\},\ \{12,14\},\ \{13,14\},\ \{1,11\},\ \{1,10\},\ \{2,9\},\ \{2,8\}\}$ *and let* $G_{5,2}(0,1,2,\ldots,15)$ *be the graph.* $K_8^c \vee K_8$—$\{\{0,15\},\ \{0,14\},\ \{12,14\},\ \{13,14\},\ \{1,11-i\} \mid i \in \mathbb{Z}_4\}$ *with* $V(K_8^c) = \mathbb{Z}_8$ *and* $V(K_8) = \mathbb{Z}_{16} \setminus \mathbb{Z}_8$. *There exists a* 6*-cycle system B of G for* $1 \leqslant \beta \leqslant 2$.

Proof. For $1 \leqslant \beta \leqslant 2$, there exists a 6-cycle system $((\mathbb{Z}_{16}, B_\beta)$ of $G_{5,\beta}$ defined by

$B_1 = \{(8,1,9,0,11,5), (1,13,6,14,5,15), (3,10,2,15,6,11), (3,14,7,13,8,9), (4,8,0,12,7,11), (4,9,5,12,15,10), (4,15,13,11,8,12), (4,14,9,11,10,13), (13,12,11,14,15,9), (1,12,10,6,8,14), (3,12,6,9,7,8), (3,13,5,10,7,15), (2,14,10,8,15,11), (2,12,9,10,0,13)\}$ and

$B_2 = \{(8,2,9,0,11,5), (1,13,6,14,5,15), (3,10,2,15,6,11), (3,14,7,13,8,9), (4,8,0,12,7,11), (4,9,5,12,15,10), (4,15,13,11,8,12), (4,14,9,11,10,13), (13,12,11,14,15,9), (1,12,10,6,8,14), (3,12,6,9,7,8), (3,13,5,10,7,15), (2,14,10,8,15,11), (2,12,9,10,0,13)\}$. □

Lemma 3.6. *Let $G_6(0,1,2,\ldots,15)$ be the graph $K_8^c \vee K_8$—$\{\{0,15\}, \{0,14\}, \{12,14\}, \{13,14\}, \{1,11\}, \{1,10\}, \{8,10\}, \{9,10\}\}$ with $V(K_8^c) = \mathbb{Z}_8$ and $V(K_8) = \mathbb{Z}_{16} \setminus \mathbb{Z}_8$. There exists a 6-cycle system B of G_6.*

Proof. There exists a 6-cycle system $(\mathbb{Z}_{16}, B)$ of G_6 defined by $B = \{(2,14,11,10,6,12), (2,15,4,10,7,13), (3,15,9,6,8,12), (3,14,1,8,9,13), (0,11,5,9,7,12), (4,9,11,8,5,13), (5,10,12,13,11,15), (1,9,14,15,8,13), (2,10,0,9,3,11), (2,9,12,5,14,8), (3,8,7,15,13,10), (0,8,4,11,6,13), (4,14,6,15,1,12), (7,11,12,15,10,14)\}$. □

4. Main result

Finally, we are ready to prove the main result of this paper. Let $G[W]$ denote the subgraph of G induced by W.

Theorem 4.1. *Let F be a forest in the complete graph K_n with $|E(F)| \geqslant 1$. There exists a 6-cycle system of $G = K_n - E(F)$ if and only if*

(1) *all vertices in F have odd degree,*
(2) *$|E(K_n - E(F))|$ is divisible by 6 and*
(3) *n is even.*

Remark. Note that condition (1) requires F to be a spanning forest of K_n.

Proof. Suppose that there exists a 6-cycle system (V, B) of $G = K_n - E(F)$. Then for each $v \in V$, the 6-cycles in B partition the edges incident with v into pairs, so $d_G(v)$ is even. Since $|E(F)| \geqslant 1$ and F is a forest, F contains at least one vertex, say w, with $d_F(w) = 1$, so $d_G(w) = n - 2$. Therefore, n is even. Also, for each $v \in V$, $d_F(v) = (n-1) - d_G(v)$ so $d_F(v)$ is odd. Then clearly F spans K_n. Since the 6-cycles in B partition the edges of G, it follows that 6 must divide $|E(K_n - F)|$.

To prove the sufficiency, first note that the necessary conditions prevent the possibility that $n = 4$.

Next observe that if $n \in \{2,6,8\}$ then conditions (1)–(3) require that F be a 1-factor. Lemma 2.3 provides such a 6-cycle system of $K_n - E(F)$ for each $n \in \{2,6,8\}$. If $n = 10$, then conditions (1)–(3) require that F be a spanning tree, thus giving seven possibilities for F, and each is considered in Lemma 2.4. Therefore we can now assume that $n \geqslant 12$.

The remaining cases are proved by induction. So now suppose that for each positive integer α with $2 \leqslant \alpha < n$ and for any forest F' in K_α that together satisfy:

(1′) all vertices in F' have odd degree (so F' is spanning),
(2′) $|E(K_\alpha - E(F'))|$ is divisible by 6 and
(3′) α is even,

there exists a 6-cycle system of $K_\alpha - E(F')$. We can assume that $V(K_n) = \mathbb{Z}_n$. We obtain a 6-cycle system $(\mathbb{Z}_n, B)$ of $K_n - E(F)$ by considering several cases in turn: F has at least three components which are isomorphic to K_2; $n=6t$; $n=6t+2$; $n=12k+4$; and $n = 12k + 10$ (3 subcases). We regularly make use of Table 1, since it is easier to find the number of components $c(F')$ in F' than it is to check that condition (2′) is satisfied.

Case 1: Suppose F has three components isomorphic to K_2.

Let the vertex sets of these three components be $\{n-i, n-i-1\}$, where $i \in \{1,3,5\}$. Let $F' = F[\mathbb{Z}_{n-6}]$ and let $\alpha = n - 6$.

We must check to see that F' and $\alpha = n-6$ satisfy conditions (1′)–(3′). Since F' is formed by removing the three components of F isomorphic to K_2, $d_{F'}(v) = d_F(v)$ for each $v \in \mathbb{Z}_{n-6}$. So all vertices in F' have odd degree, and thus (1′) is satisfied. Also $c(F') = c(F) - 3$ and $|V(F')| = n-6$. Since we assumed that 6 divides $|(K_n - E(F))|$, by Table 1, we have that 6 divides $|E(K_{n-6} - E(F'))|$, so (2′) is satisfied. Clearly, $n - 6$ is even since n is even, so (3′) is satisfied. We apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-6}, B_1)$ of $K_{n-6} - E(F')$.

Since $n \geqslant 12, n - 6 \geqslant 6$ and $|\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}| = 6$, by Lemma 2.2 there exists a 6-cycle system $(\mathbb{Z}_n, B_2)$ of $K_{n-6,6}$ with bipartition $\{\mathbb{Z}_{n-6}, \mathbb{Z}_n \setminus \mathbb{Z}_{n-6}\}$ of the vertex set.

By Lemma 2.3, there exists a 6-cycle system B_3 of $K_6 - \{\{n-i, n-i-1\} | i \in \{1,3,5\}\}$ defined on the vertex set $\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}$. Then $(\mathbb{Z}_n,\ B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

In view of Case 1, we can assume that in the remaining cases, F contains at most 2 components that are isomorphic to K_2. The remaining cases depend on the congruence of $n \pmod{12}$.

Case 2: Suppose $n = 6t$, so $n \geqslant 12$.

By Table 1, we know that $c(F) \equiv 0$ or $3 \pmod 6$ so in particular $c(F) \geqslant 3$. Let C_0, C_1, and C_2 be three components in F. In view of Case 1, we can assume that one of the components, say C_2, is not K_2.

For $0 \leqslant i \leqslant 2$, let the second vertex in a maximum length path $P_i \in C_i$ be named i; then vertex i is adjacent to a vertex of degree 1 in F, namely the first vertex in P_i, which we call $n - i - 1$. Since we know that C_2 is not K_2, and since we know that each vertex in F has odd degree, vertex $i = 2$ is incident with at least two additional edges in F. At least one of these edges is incident to a leaf (a *leaf* is a vertex of degree one) or else the path P_2 would not be maximal. Name this leaf $n - 4$ (see Fig. 1).

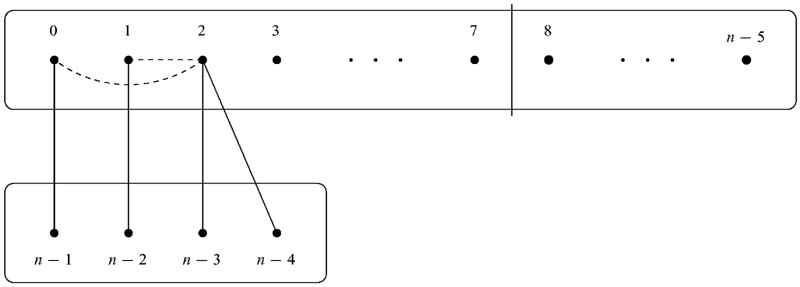


Fig. 1. $n = 6t$.

Now that we have selected 4 special vertices, we proceed as follows. Let F' be formed from $F[\mathbb{Z}_{n-4}]$ and adding edges $\{0,2\}$ and $\{1,2\}$, and let $\alpha = n-4$. We check to see that conditions ($1'$–$3'$) are satisfied.

Since $d_{F'}(i) = d_F(i) - 1 + 1$ for $0 \leqslant i \leqslant 1$ and $d_{F'}(2) = d_F(2) - 2 + 2$, clearly $d_{F'}(i) = d_F(i)$ for $1 \leqslant i \leqslant n-4$, so all vertices in F' have odd degree, so ($1'$) is satisfied. Furthermore, the edges $\{0,2\}$ and $\{1,2\}$ connect the three components C_0, C_1, and C_2 in F to create a single component in F', so we know that $c(F') = c(F) - 2$. Since $|V(F')| = n-4$ and we are assuming 6 divides $|E(K_n - E(F)|$, by Table 1, 6 divides $|E(K_{n-4} - E(F')|$. It follows that ($2'$) is satisfied. Clearly since n is even, $n-4$ is even, so ($3'$) is satisfied. So, we can apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-4}, B_1)$ of $K_{n-4} - E(F')$.

For convenience, if $n = 12$ then $B_2 = \emptyset$. If $n \geqslant 18$ then $n - 12 \geqslant 6 > 3$. Clearly $n - 12 = 6(t-2)$ which is divisible by 6 (in particular we are interested in $t > 2$). So by Lemma 2.2, there exists a 6-cycle system $(\mathbb{Z}_n \setminus \mathbb{Z}_8, B_2)$ of $K_{n-12,4}$ with bipartition $\{\mathbb{Z}_{n-4} \setminus \mathbb{Z}_8, \mathbb{Z}_n \setminus \mathbb{Z}_{n-4}\}$ of the vertex set.

By Lemma 3.1, there exists a 6-cycle system B_3 of $G_1 = G_1(0,1,2,3,4,5,6,7,n-4,n-3,n-2,n-1)$. Recall from the definition of G in Lemma 3.1 that $\{\{0,2\},\{1,2\}\} \subseteq E(G)$. These two edges are in F', so they do not occur in F nor in any 6-cycle in B_1, so they are placed in a 6-cycle in B_3. Also, $\{\{0,n-1\},\{1,n-2\},\{2,n-3\},\{2,n-4\}\} \cap E(B_3) = \emptyset$, which means those edges are not in a 6-cycle in B_3. This is good because these edges occur in F. Therefore $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

Case 3: Suppose $n = 6t + 2$, so $n \geqslant 14$.

This proof requires carefully selecting four vertices (named $n-4, n-3, n-2$, and $n-1$) in F, which we do for several cases. By Table 1, we know that $c(F) \equiv 1$ or $4 \pmod 6$.

Suppose $c(F) = 1$. If F is a star centered at vertex say 0, then it has at least 13 leaves, so choose any 4 and call them $n-1, n-2, n-3$, and $n-4$. If F is not a star, let P be a maximum length path in F. Since $n \geqslant 14$, and since F is not a star, P has length at least 3. Label the second vertex in P with 0 and the second to last vertex in P with 1. Since P is maximal and each vertex in F has odd degree, we know that vertex 0 is adjacent to at least two vertices of degree 1, call them $n-1$ and $n-2$, and similarly vertex 1 is adjacent to at least two vertices of degree 1, call them $n-3$ and $n-4$.

Finally, suppose $c(F) > 1$. Then F has at least 4 components. Since at most 2 components are isomorphic to K_2, there are at least two components not isomorphic to K_2, call them C_0 and C_1. For $0 \leqslant i \leqslant 1$, let vertex i be the second vertex in a maximum length path $P_i \in C_i$. Since P_i is a maximum length path, vertex i is adjacent to at least 2 vertices of degree 1 call them $n-2i-1$ and $n-2i-2$.

Now that we have selected four special vertices, we proceed as follows. Let $F' = F[\mathbb{Z}_{n-4}]$. Therefore, F' spans $\mathbb{Z}_{n-4}$. Since either

(i) $d_{F'}(i) = d_F(i) - 4$ for $i = 0$ and $d_{F'}(i) = d_F(i)$ for $1 \leqslant i \leqslant n-5$ or
(ii) $d_{F'}(i) = d_F(i) - 2$ for $0 \leqslant i \leqslant 1$ and $d_{F'}(i) = d_F(i)$ for $2 \leqslant i \leqslant n-5$,

all vertices in F' have odd degree and thus (1′) is satisfied. Since F' is formed from F by deleting vertices of degree 1, $c(F') = c(F)$. Therefore, since $|V(F')| = n-4$ and we are assuming 6 divides $|E(K_n - E(F))|$, by Table 1, 6 divides $|E(K_{n-4} - E(F'))|$ so (2′) is satisfied. Clearly, since n is even, so is $n-4$, so (3′) is satisfied. We can apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-4}, B_1)$ of the vertex set $K_{n-4} - E(F')$.

Since $n \geqslant 14$, $n-8 \geqslant 6$, clearly $n-8 = 6(t-1)$ which is divisible by 6 (in particular, we are interested when $t > 1$). By Lemma 2.2, there exists a 6-cycle system $(\mathbb{Z}_n \backslash \mathbb{Z}_4, B_2)$ of $K_{n-8,4}$ with bipartition $\{\mathbb{Z}_{n-4} \setminus \mathbb{Z}_4, \mathbb{Z}_n \setminus \mathbb{Z}_{n-4}\}$ of the vertex set.

By Lemma 3.2, there exists a 6-cycle system B_3 of $G_{2,i} = G_{2,i}(0,1,2,3,n-4,n-3,n-2,n-1)$, where $i = 1$ when F is a star and $i = 2$ otherwise.

It follows from the definition of $G_{2,1}$ that $\{\{0,n-1\},\{0,n-2\},\{0,n-3\},\{0,n-4\}\} \cap E(B_3) = \emptyset$ which is good because these edges occur in F when F is a star. Similarly, the definition of $G_{2,2}$ ensures that $\{\{0,n-1\}, \{0,n-2\}, \{1,n-3\}, \{1,n-4\}\} \cap E(B_3) = \emptyset$ which is good because these edges occur in F when F is not a star. Therefore $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

Case 4: Suppose $n = 12k + 4$, so $n \geqslant 16$.

By Table 1, we know that $c(F) \equiv 4 \pmod 6$, so in particular $c(F) \geqslant 4$. Let C_0, C_1, C_2, and C_3 be components in F. We also know that one of the components, say C_3, is not a K_2. For $0 \leqslant i \leqslant 3$, let P_i be a maximum path in C_i, and let $n-i-1$ be the first vertex in P_i and let i be the second vertex in P_i. If C_3 is a star then let $n-5$ and $n-6$ be two additional vertices of degree one adjacent to vertex 3. If C_3 is not a star then let vertex 4 be the second to last vertex on P_3. Since P_3 is maximal, vertex 4 is adjacent to at least 2 vertices of degree 1, call them $n-5$ and $n-6$ (see Fig. 2).

Now that we have selected 6 special vertices, namely $n-6$, $n-5$, $n-4$, $n-3$, $n-2$, $n-1$, we proceed as follows. Let F' be formed from $F[\mathbb{Z}_{n-6}]$ by adding edges

(i) $\{0,3\}, \{1,4\}$ and $\{2,4\}$ if C_3 is not a star and
(ii) $\{0,3\}, \{1,3\}$ and $\{2,3\}$ if C_3 is a star.

Clearly F' spans K_{n-6}. Since either

(i) $d_{F'}(i) = d_F(i) + 1 - 1$ for $0 \leqslant i \leqslant 3$ and $d_{F'}(i) = d_F(i) + 2 - 2$ for $i = 4$ or
(ii) $d_{F'}(i) = d_F(i) + 1 - 1$ for $0 \leqslant i \leqslant 2$ and $d_{F'}(i) = d_F(i) + 3 - 3$ for $i = 3$,

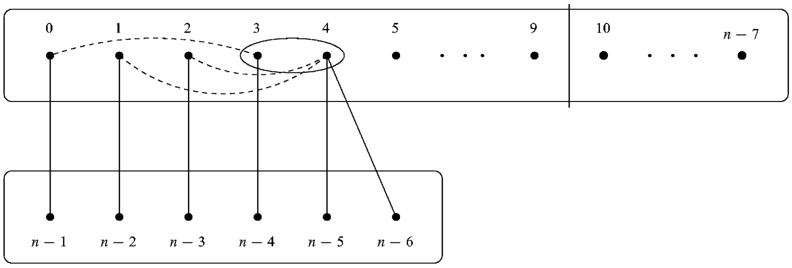


Fig. 2. Naming the vertices when $n = 12k + 4$ (if C_3 is a star then vertex 3 and vertex 4 are the same vertex).

all of the vertices in F' have odd degree so $(1')$ is satisfied. Furthermore, by connecting the four components C_0, C_1, C_2, and C_3 in F to create a single component in F', we know that $c(F') = c(F) - 3$. Therefore, since $c(F) \equiv 4 \pmod 6$, we have $c(F') \equiv 1 \pmod 6$. Since $|V(F')| = n - 6$ and we are assuming 6 divides $|E(K_n - E(F))|$ by Table 1, 6 divides $|E(K_{n-6} - E(F'))|$ so $(2')$ is satisfied. Clearly, since n is even, so is $n - 6$, so $(3')$ is satisfied. Therefore, we can apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-6}, B_1)$ of $K_{n-6} - E(F')$.

For convenience, if $n = 16$ then $B_2 = \emptyset$. If $n \geqslant 28$, then clearly $n - 16 = 6(2k - 2)$ which is divisible by 6. So, by Lemma 2.2, there exists a 6-cycle system $(\mathbb{Z}_n \setminus \mathbb{Z}_{10}, B_2)$ of $K_{n-16,6}$ with bipartition $\{\mathbb{Z}_{n-6} \setminus \mathbb{Z}_{10},\ \mathbb{Z}_n \setminus \mathbb{Z}_{n-6}\}$ of the vertex set.

Let $\beta = 1$ if C_3 is not a star and $\beta = 2$ if C_3 is a star. By Lemma 3.3, there exists a 6-cycle system B_3 of $G_{3,\beta} = G_{3,\beta}(0,1,2,3,4,5,6,7,8,9,n-6,n-5,n-4,n-3,n-2,n-1)$. Notice that the edges of $G_{3,\beta}$ joining vertices in $\mathbb{Z}_{n-6}$ are precisely the edges in $E(F') \setminus E(F)$, and do not occur in a 6-cycle in B_1 but do occur in a 6-cycle in B_3. Also, the edges joining a vertex in $\mathbb{Z}_{n-6}$ to a vertex in $\mathbb{Z}_n \setminus \mathbb{Z}_{n-6}$ that occur in no 6-cycle in B_3 are precisely the edges in $E(F) \setminus E(F')$. Therefore, $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

Case 5: Suppose $n = 12k + 10$, so $n \geqslant 22$.

By Table 1, we know that $c(F) \equiv 1 \pmod 6$. This proof requires the term leaf pair to be defined. A *leaf pair* is a set Y of two vertices each of degree 1 in F that have a common neighbor, $N(Y)$. We call $N(Y)$ the *center* of Y. Additionally, 2, 3, or 4 leaf pairs that have a common center are called a *double*, *triple*, or *quadruple* leaf pair respectively. This proof requires carefully selecting 8 vertices (named: $n-8, n-7, n-6, n-5, n-4, n-3, n-2$, and $n-1$) in F which we do for 3 subcases.

Subcase 1: Suppose F has at least four disjoint leaf pairs, call them $\{n-(2i+1), n-(2i+2)\}$ for $0 \leqslant i \leqslant 3$.

Name each of the centers of the four leaf pairs with a vertex in $\{0,1,2,3\}$ in such a way that for $0 \leqslant i < j \leqslant 3$, vertex i is the center of at least as many leaf pairs as vertex j. Naming the vertices in this manner gives rise to five possibilities for the set L consisting of these leaf pairs (see Fig. 3):

(1) L is 1 quadruple leaf pair, in which case let $\beta = 1$,
(2) L is 1 leaf pair and a triple leaf pair, in which case let $\beta = 2$,

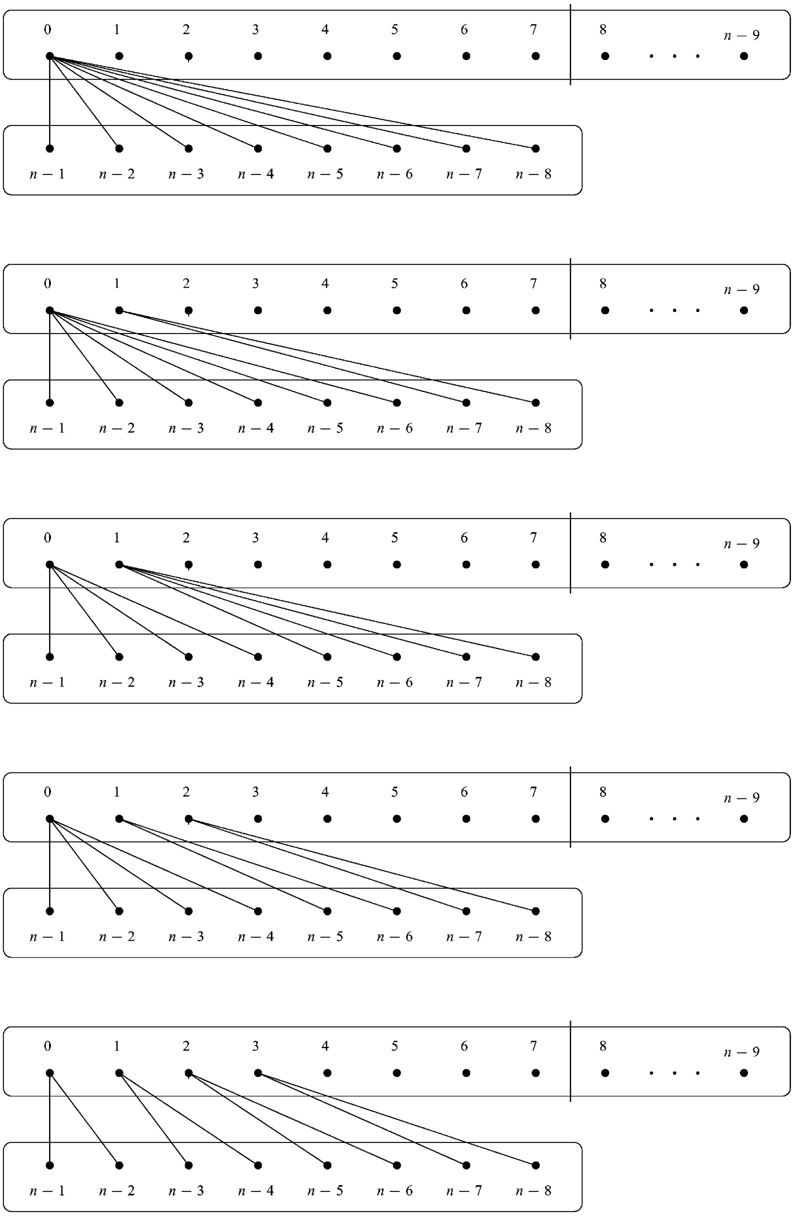


Fig. 3. Naming the vertices when $n = 12k + 10$ (subcase 1).

(3) L is 2 double leaf pairs, in which case let $\beta = 3$,
(4) L is 2 leaf pairs and a double leaf pair, in which case let $\beta = 4$ and
(5) L is 4 leaf pairs, in which case let $\beta = 5$.

Now that we have selected 8 special vertices, we proceed as follows. Let $F' = F[\mathbb{Z}_{n-8}]$. In each case, all $n-8$ vertices in F' have odd degree so $(1')$ is satisfied. Since F' is formed from F by deleting vertices of degree 1 in F, $c(F') = c(F)$. Since $|V(F')| = n-8$ and we are assuming 6 divides $|E(K_n - E(F))|$, by Table 1, we see that 6 divides $|E(K_{n-8} - E(F'))|$ so $(2')$ is satisfied. Clearly, since n is even $n-8$ is even, so $(3')$ is satisfied. So we can apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-8}, B_1)$ of $K_{n-8} - E(F')$.

Since $n \geqslant 22$, $n - 16 \geqslant 6$. Clearly $n - 16 = 6(2k - 1)$ which is divisible by 6, so by Lemma 2.2 there exists a 6-cycle system $(\mathbb{Z}_{n-8}, B_2)$ of $K_{n-16,8}$ with bipartition $\{\mathbb{Z}_{n-8} \setminus \mathbb{Z}_8, \mathbb{Z}_n \setminus \mathbb{Z}_{n-8}\}$ of $\mathbb{Z}_{n-8}$.

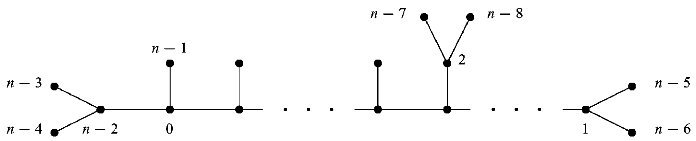
Finally, by Lemma 3.4, there exists a 6-cycle system B_3 of $G_{4,\beta}(0, 1, 2, 3, 4, 5, 6, n-8, n-7, n-6, n-5, n-4, n-3, n-2, n-1)$. Notice the edges joining vertices in $\mathbb{Z}_{n-8}$ to vertices in $\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}$ that occur in no 6-cycle in B_3 are precisely the edges in $E(F) \setminus E(F')$. Thus $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

Subcase 2: Notice that in Case 5, if $c(F) > 1$ then by Table 1 it is clear that $c(F) \geqslant 7$. Since F has at most two components that are isomorphic to K_2, if $c(F) > 1$ then there are at least 4 components each of which contains a leaf pair. So in view of subcase 1, we can now assume that F is a tree.

Suppose F has exactly three leaf pairs. Let P be a maximum length path in F. We know the second and second to last vertices are the centers of leaf pairs. Since P has only one other leaf pair, all other internal vertices in P, except possibly one, have degree 3 in F and are adjacent to a leaf. Since $n \geqslant 22$, P must have length at least 8. Therefore, by changing the direction of P if necessary, we can assure ourselves that the third leaf pair is at least as close to the end of P as it is to the beginning. Now let $n-2$ be the second vertex of P, and let the two vertices of degree 1 adjacent to it in P be $n-3$ and $n-4$. Now let 0 be the third vertex of P and call the leaf adjacent to it $n-1$. Let 1 be the second to last vertex of P, and let the two vertices of degree 1 adjacent to it in P be $n-5$ and $n-6$. Finally, call the last leaf pair $n-7$ and $n-8$ and name its center 2 if it is not vertex 1 (see Fig. 4). If vertex 1 is a single or double leaf pair let $\beta = 1$ or 2, respectively.

Now that we have selected at least 10 special vertices, we proceed as follows. Let $F' = F[\mathbb{Z}_{n-8}]$. Therefore, F' spans $\mathbb{Z}_{n-8}$. Since, when $\beta = 1$, $d_{F'}(i) = d_F(i) - 2$ for $i \in \mathbb{Z}_3$ and $d_{F'}(i) = d_F(i)$ for $3 \leqslant i \leqslant n-9$ or if $\beta = 2$, $d_{F'}(0) = d_F(0) - 2$, $d_{F'}(1) = d_F(1) - 4$ and $d_{F'}(i) = d_F(i)$ for $2 \leqslant i \leqslant n-9$, then all vertices in F' have odd degree so $(1')$ is satisfied. Since F' is formed from F by deleting vertex $n-2$ and its leaf pair (each of degree 1 in F), $c(F') = c(F)$. Since $|V(F')| = n-8$ and we are assuming 6 divides $|E(K_n - E(F))|$, by Table 1, 6 divides $|E(K_{n-8} - E(F'))|$, so $(2')$ is satisfied. Clearly since n is even we know $n-8$ is even, so $(3')$ is satisfied. We can apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-8}, B_1)$ of $K_{n-8} - E(F')$.

Since $n \geqslant 22$, $n - 16 \geqslant 6$. Clearly $n - 16 = 6(2k - 1)$ which is divisible by 6, so by Lemma 2.2, there exists a 6-cycle system $(\mathbb{Z}_{n-8}, B_2)$ of $K_{n-16,8}$ with bipartition $\{\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}, \mathbb{Z}_{n-8} \setminus \mathbb{Z}_8\}$ of the vertex set.



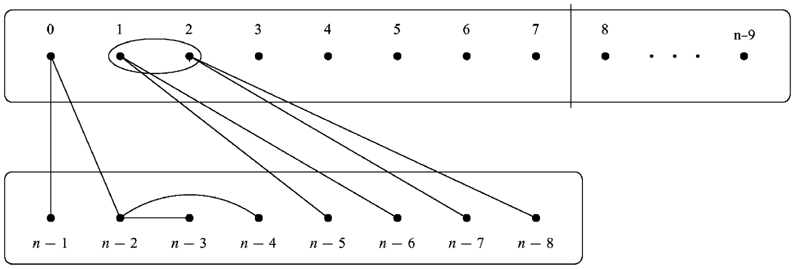


Fig. 4. Naming the vertices when $n = 12k + 10$ (subcase 2).

By Lemma 3.5, there exists a 6-cycle system B_3 of $G_{5,\beta}(0,1,2,3,4,5,6,7,n-8,n-7,n-6,n-5,n-4,n-3,n-2,n-1)$. Notice the edges joining a vertex in $\mathbb{Z}_{n-8}$ to a vertex in $\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}$ that occur in no 6-cycle in B_3 are precisely the edges in $E(F) \setminus E(F')$. Then $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

Subcase 3: Suppose F has exactly two leaf pairs. Let P be a maximum length path in F. We know the second and second to last vertices are the centers of the two leaf pairs. Now let $n-2$ be the second vertex of P and call its adjacent leaves $n-3$ and $n-4$, call the third vertex 0, and call the leaf adjacent to it $n-1$. Similarly, let $n-6$ be the second to last vertex of P and call its adjacent leaves $n-7$ and $n-8$, call the third to last vertex 1 and call the leaf adjacent to it $n-5$, these vertices are all distinct since $n \geqslant 22$ (See Fig. 5).

Now that we have selected 8 special vertices, we proceed as follows. Let $F' = F[\mathbb{Z}_{n-8}]$. Therefore, F' spans $\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}$. Since $d_{F'}(i) = d_F(i) - 2$ for $0 \leqslant i \leqslant 1$ and $d_{F'}(i) = d_F(i)$ for $2 \leqslant i \leqslant n-9$, then all vertices in F' have odd degree so $(1')$ is satisfied. Since F' is formed from deleting vertex $n-6$ and vertex $n-2$ whose leaf pairs have degree 1 and deleting $n-1$ and $n-2$ in F, $c(F') = c(F)$. Since $|V(F')| = n-8$ and we are assuming 6 divides $|E(K_n - E(F))|$, so by Table 1, 6 divides $|E(K_{n-8} - E(F'))|$ so $(2')$ is satisfied. Clearly since n is even $n-8$ is even so $(3')$ is satisfied. We can apply induction to obtain a 6-cycle system $(\mathbb{Z}_{n-8}, B_1)$ of $K_{n-8} - E(F')$.

Since $n \geqslant 22, n-16 \geqslant 6$. Clearly $n-16 = 6(2k-1)$ which is divisible by 6, so by Lemma 2.2, there exists a 6-cycle system $(\mathbb{Z}_{n-8}, B_2)$ of $K_{n-16,8}$ with bipartition $\{\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}, \mathbb{Z}_{n-8} \setminus \mathbb{Z}_8\}$ of the $K_{n-16,18}$ vertex set.

By Lemma 3.6, there exists a 6-cycle system B_3 of $G_6 = G_6(0,1,2,3,4,5,6,7,n-8,n-7,n-6,n-5,n-4,n-3,n-2,n-1)$. Notice the edges joining a vertex in

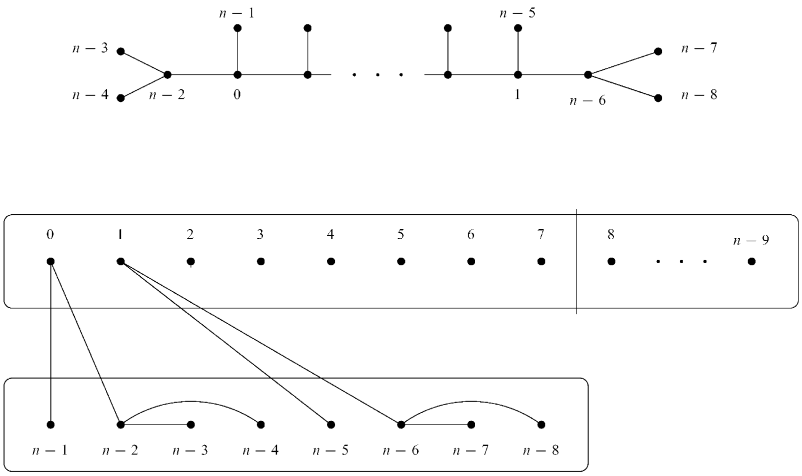


Fig. 5. Naming the vertices when $n = 12k + 10$ (subcase 3).

$\mathbb{Z}_{n-8}$ to a vertex in $\mathbb{Z}_n \setminus \mathbb{Z}_{n-8}$ that occur in no 6-cycle in B_3 are precisely the edges in $E(F) \setminus E(F')$. Then $(\mathbb{Z}_n, B_1 \cup B_2 \cup B_3)$ is a 6-cycle system of $K_n - E(F)$.

This completes the proof. □

Acknowledgements

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