



## Decomposing complete graphs into $K_r \times K_c$ 's

H.L. Fu<sup>a,1</sup>, F.K. Hwang<sup>a,2</sup>, M. Jimbo<sup>b</sup>, Y. Mutoh<sup>b,\*</sup>, C.L. Shiue<sup>c</sup>

<sup>a</sup>Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan, ROC

<sup>b</sup>Department of Mathematics, Keio University, 3-14-1, Hiyoshi, Kohoku-Ku, Yokohama 223-8522, Japan

<sup>c</sup>Institute of Mathematics, Academic Sinica, Nankang, Taipei, Taiwan, ROC

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### Abstract

We consider the problem of decomposing a complete graph into the Cartesian product of two complete graphs  $K_r$  and  $K_c$ . This problem originated from the clone library screening. We give a general method of constructing such decompositions using various sorts of combinatorial designs. In particular, for  $r = 3$  and  $c = 3$ , we show that such a decomposition exists for all  $n$  satisfying some simple necessary conditions.

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### 1. Introduction

Define  $G(r, c) = K_r \times K_c$  as the *grid-block* with  $r$  rows and  $c$  columns where each grid point is a vertex. Two vertices are collinear if they are on the same grid line (row or column). The problem is to partition the complete graph  $K_n$  into  $G(r, c)$ 's, or each pair of vertices being collinear exactly once. The special case  $r = c = \sqrt{n}$  is called a *lattice square* (see Yates, 1940). Construction of lattice squares for  $\sqrt{n}$  an odd prime power was given in Raghavarao (1971). Hwang (1995) cited an application to clone library screening where  $(n + 1)/2$  copies of  $n$  clones are stored in  $(n + 1)/2 \times \sqrt{n} \times \sqrt{n}$  grid-blocks with every clone appearing exactly once in each grid-block.

\* Corresponding author. Tel.: +81-455661439; fax: +81-455661657.

E-mail address: [yukiyasu@jim.math.keio.ac.jp](mailto:yukiyasu@jim.math.keio.ac.jp) (Y. Mutoh).

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Clones are either positive or negative. The goal of a library screening is to identify all positive clones. Economy in both time and money requires that the clones be assayed in groups (each group is called a *pool*) and the assays are conducted in parallel. A pool array gives a negative outcome if and only if all clones contained in it are negative.

It is convenient to use the rows and the columns as pools. A positive clone will yield a positive outcome for both its row and its column. Therefore, the clone situated at the intersection of a positive row and a positive column is a prime suspect for a positive clone. But, it does not have to be one if the grid-block contains more than one positive clone since  $d$  positive clones may generate  $d$  positive rows and  $d$  positive columns with a total of  $d^2$  intersections. Hence, one needs other grid-blocks on the same set of clones to remove the ambiguities. Clearly, the differentiability would be bigger if the grid-blocks are not similar. The requirement of “unique collinearity” results the need of dissimilarity (see Barillot et al., 1991 and Berger et al., 2000).

Lattice squares have also been extended to *lattice rectangles* (see Harshbarger, 1947) by allowing  $r \neq c$ . However, the requirement of  $rc = n$  is essential for both lattice squares and lattice rectangles. In most practical uses, the grid-block has size limitation and  $n$  is large. Then we have to consider storing copies of clones on  $G(r, c)$ 's,  $r < \sqrt{n}$  or  $c < \sqrt{n}$ , while preserving the unique collinearity condition. This is the subject matter of the current paper. Thus the novelty of our design is to free  $r$  and  $c$  from the requirement  $rc = n$ .

Let  $D_{r \times c}(\gamma)$  denote a decomposition of a graph  $\gamma$  into  $G(r, c)$ 's. We first give necessary conditions for the existence of  $D_{r \times c}(K_n)$ .

**Lemma 1.** *Necessary conditions for the existence of a  $D_{r \times c}(K_n)$  are*

- (i)  $(r + c - 2)|(n - 1)$ , and
- (ii)  $rc(r + c - 2)|n(n - 1)$ .

**Proof.** Each vertex of  $K_n$  has  $n - 1$  neighbors while each vertex of a  $G(r, c)$  has  $r + c - 2$  neighbors in that grid-block, which implies (i).

Also, there are  $n$  chooses 2 (symbol) pairs of neighbors in  $K_n$  while each  $G(r, c)$  generates  $rc(r + c - 2)/2$  pairs of neighbors. The condition (ii) follows immediately.  $\square$

In this paper we give a general approach to construct a  $D_{r \times c}(K_n)$ . This general approach includes employing two-dimensional difference families to obtain cyclic constructions, and employing all sorts of combinatorial designs, like group divisible designs, Steiner 2-designs, pairwise balanced designs and resolvable designs, to expand a construction. In particular, these tools are adequate to construct a  $D_{2 \times 3}(K_n)$ , a  $D_{2 \times 4}(K_n)$  and a  $D_{3 \times 3}(K_n)$ , etc. The existence of a  $D_{2 \times 3}(K_n)$  was shown by Carter (1989) by decomposing  $K_n$  into cubic graphs. Meanwhile, Mutoh et al. (2002) showed the existence of a  $D_{2 \times 4}(K_n)$  by utilizing some direct and recursive constructions including the above. Here, we show the existence of a  $D_{3 \times 3}(K_n)$  for all  $n$  satisfying the necessary conditions of Lemma 1.

## 2. Two-dimensional difference families

Let the vertices of  $K_n$  be labeled by the numbers  $0, 1, \dots, n - 1$ . Then the  $n$  labels generate  $n - 1$  distinct differences. On the other hand, each row in a  $G(r, c)$  generates  $c(c - 1)$  differences and each column  $r(r - 1)$  differences, so a  $G(r, c)$  generates a total of  $rc(r + c - 2)$  differences.

A cyclic development of a  $G(r, c)$  yields  $n$   $G(r, c)$ 's  $G_0, G_1, \dots, G_{n-1}$ , where the  $G_i$  is obtained from  $G$  by adding  $i \pmod n$  to every vertex label.

**Lemma 2.** Suppose  $rc(r + c - 2)$  divides  $n - 1$  (implying  $n$  is odd). To decompose  $K_n$  into  $G(r, c)$ 's, it suffices to construct  $(n - 1)/rc(r + c - 2)$  base  $G(r, c)$ 's such that the differences are all distinct.

**Proof.** By cyclically developing these  $(n - 1)/rc(r + c - 2)$   $G(r, c)$ 's, we obtain  $n \cdot (n - 1)/rc(r + c - 2)$   $G(r, c)$ 's which constitute a decomposition of  $K_n$ . The reason is that two labels  $i$  and  $j$  will appear in a  $G(r, c)$  developed from the base  $G(r, c)$  containing the differences  $i - j$  and  $j - i$ .  $\square$

Such a difference will be represented by a member of its residue class nearest to zero.

**Example 1.**  $n = 19, r = 2$  and  $c = 3$ . Then  $(n - 1)/rc(r + c - 2) = 1$ . The base  $G(2, 3)$

1	-	7	-	11
9	-	6	-	4

generates differences  $\pm 6, \pm 9, \pm 4$  in the first row, differences  $\pm 3, \pm 5, \pm 2$  in the second row, and differences  $\pm 8, \pm 1, \pm 7$  in the three columns.

If  $rc(r + c - 2)$  does not divide  $n - 1$ , then at least one base  $G(r, c)$  cannot be developed to complete a full cycle.

**Example 2.**  $n = 10, r = 2$  and  $c = 3$ . Then  $(n - 1)/rc(r + c - 2) = 1/2$ .

We use the base  $G(2, 3)$  (first one) but only for half a cycle.

1	-	2	-	4	2	-	3	-	5	3	-	4	-	6
7	-	6	-	9	8	-	7	-	0	9	-	8	-	1
4	-	5	-	7	5	-	6	-	8					
0	-	9	-	2	1	-	0	-	3					

Note that the first row and the second row generate the same differences  $\pm 1, \pm 3, \pm 2$ . So the two rows in all five  $G(2, 3)$ 's together constitute a full development of the set  $\{1, 2, 4\}$ . Similarly, the first column and the second column generate the same differences  $\pm 4$ . Finally, the third column generates the differences  $\pm 5$ . Since  $5 \equiv -5 \pmod{10}$ , a half cycle yields all pairs of difference 5.

**Example 3.**  $n = 37$ .

The base  $G(3, 3)$  is

0	–	3	–	12
6	–	22	–	23
10	–	17	–	25

The construction of such base  $G(r, c)$ 's is elucidated in the next section.

**3. General constructions**

Suppose that  $st$  vertices are partitioned into  $s$  subsets of  $t$  vertices each. Let  $K_s(t)$  be the complete multipartite graph such that  $(i, j)$  is an edge if  $i$  and  $j$  are not in the same subset. It is useful to consider the decomposition of  $K_s(t)$ . We will omit the subscript  $r \times c$  in  $D_{r \times c}(G)$  in this section. Similar to Lemma 1, we have

**Lemma 3.** *Necessary conditions for a  $D(K_s(t))$  to exist are*

- (i)  $(r + c - 2)|(s - 1)t$ , and
- (ii)  $rc(r + c - 2)|(s - 1)st^2$ .

The next result shows a connection between decomposing  $K_n$  and decomposing  $K_s(t)$ .

**Theorem 4.** *A  $D(K_{st+1})$  exists if a  $D(K_{t+1})$  and a  $D(K_s(t))$  exist.*

**Proof.** Partition  $\{1, \dots, st\}$  into  $s$  subsets  $S_1, \dots, S_s$  each having  $t$  vertices. Let  $D_i(K_{t+1})$  denote the design when the  $t + 1$  vertices are from  $S_i \cup \{0\}$ . Then  $(\bigcup_{i=1}^s D_i(K_{t+1}) \cup D(K_s(t))) = D(K_{st+1})$ .  $\square$

By naming the elements in  $S_i : i, s + i, \dots, (t - 1)s + i$ , for  $i = 1, \dots, s$ , the differences which are multiples of  $s$  are taken care of in  $D_i(K_{t+1})$ s,  $i = 1, \dots, s$ . There are still  $(s - 1)t$  out of the  $st$  differences left in  $K_{st+1}$ . Therefore, we have the following lemma.

**Lemma 5.** *Suppose  $rc(r + c - 2)|(s - 1)t$ . To obtain a  $D(K_s(t))$ , it suffices to construct  $(s - 1)t/rc(r + c - 2)$   $G(r, c)$ 's with distinct differences which are not multiples of  $s$ .*

**Example 4.**  $s = 3, t = 9, r = 2$  and  $c = 3$ . Then  $(s - 1)t/rc(r + c - 2) = 1$ .

$$S_1 = \{1, 4, 7, 10, 13, 16, 19, 22, 25\},$$

$$S_2 = \{2, 5, 8, 11, 14, 17, 20, 23, 26\},$$

$$S_3 = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}.$$

The base  $G(2, 3)$  is

$$\begin{array}{cccc} 1 & - & 15 & - & 23 \\ | & & | & & | \\ 2 & - & 13 & - & 6 \end{array}$$

Again, when  $rc(r + c - 2)$  does not divide  $(s - 1)t$ , some cycles are not fully developed.

**Example 5.**  $s = 4, t = 9, r = 2$  and  $c = 3$ . Then  $(s - 1)t/rc(r + c - 2) = 3/2$ .

$$S_1 = \{1, 5, 9, 13, 17, 21, 25, 29, 33\}, \quad S_2 = \{2, 6, 10, 14, 18, 22, 26, 30, 34\},$$

$$S_3 = \{3, 7, 11, 15, 19, 23, 27, 31, 35\}, \quad S_4 = \{4, 8, 12, 16, 20, 24, 28, 32, 36\}.$$

The two base  $G(2, 3)$ 's, one goes half cycle, are

$$\begin{array}{cccccc} 1 & - & 3 & - & 18 & 5 & - & 18 & - & 8 \\ | & & | & & | & | & & | & & | \\ 23 & - & 30 & - & 29 & 36 & - & 23 & - & 26 \end{array}$$

Note that in the second  $G(2, 3)$ , the first row and the second row generate the same differences  $\pm 13, \pm 3, \pm 10$ . The first column and the second column generate the same differences  $\pm 5$ .

We can also construct a  $D(K_s(t))$  from one with smaller parameters. Let  $V$  be a set of  $v$  elements,  $\mathcal{B}$  be a collection of  $k$ -subsets of  $V$  and  $\mathcal{G}$  be a partition of  $V$  into  $k$  classes, each of size  $n$ . A triple  $(V, \mathcal{G}, \mathcal{B})$  is called a group divisible design with classes of size  $n$  and block size  $k$ , denoted by  $GD(k, n, v)$ , if each pair of elements from  $V$  is either contained in exactly one group or is contained in exactly one block, but not both. Especially, in the case of  $n = 1$ , a pair  $(V, \mathcal{B})$  instead of  $(V, \mathcal{G}, \mathcal{B})$  is called a Steiner 2-design with block size  $k$  and  $v$  elements, denoted by  $S(2, k, v)$ . Then the following theorem is obtained.

**Theorem 6.** *A  $D(K_m(nt))$  exists if a  $GD(s, n, v)$  and a  $D(K_s(t))$  exist, where  $m = v/n$ . Especially, a  $D(K_v(t))$  exists if an  $S(2, s, v)$  and a  $D(K_s(t))$  exist.*

**Proof.** Make  $t$  copies of each element of a  $GD(s, n, v)$ . Apply a  $D(K_s(t))$  on the set which is  $t$  copies of all elements in each block of the  $GD(s, n, v)$ .  $\square$

**Example 6.**  $s = 3, t = 9, v = 7, r = 2$  and  $c = 3$ . Then  $n = tv + 1 = 64$ . Partition  $\{1, \dots, 63\}$  into seven sets  $S_1, \dots, S_7$ , each with nine elements. The seven blocks of an  $S(2, 3, 7)$  are  $B_1 = \{S_1, S_2, S_4\}, B_2 = \{S_2, S_3, S_5\}, B_3 = \{S_3, S_4, S_6\}, B_4 = \{S_4, S_5, S_7\}, B_5 = \{S_5, S_6, S_1\}, B_6 = \{S_6, S_7, S_2\}, B_7 = \{S_7, S_1, S_3\}$ , which is known as Fano plane, that is, the projective plane of order 2.

**Corollary 7.** *A  $D(K_{vt+1})$  exists if an  $S(2, s, v)$ , a  $D(K_{t+1})$  and a  $D(K_s(t))$  exist.*

**Proof.** By utilizing to an  $S(2, s, v)$  and a  $D(K_s(t))$ , there is a  $D(K_v(t))$ . To this design we add a point  $\{0\}$ . Apply a  $D(K_{t+1})$  on  $G \cup \{0\}$ , where  $G$  is a group of a  $D(K_v(t))$ .  $\square$

Let  $S(2, \{k_1, \dots, k_z\}, v)$  denote a generalization of a Steiner 2-design where the block size can vary. We can generalize Theorem 6 to

**Theorem 8.** *A  $D(K_v(t))$  exists if an  $S(2, \{s_1, \dots, s_z\}, v)$  and a  $D(K_{s_i}(t))$   $i = 1, \dots, z$ , exist.*

**Example 7.**  $v = 12, s = 4, t = 9, r = 2$  and  $c = 3$ . We can construct an  $S(2, \{3, 4\}, 12)$  from an  $S(2, 4, 13)$  by dropping the element 13. Since a  $D(K_3(9))$  and a  $D(K_4(9))$  both exist, a  $D(K_{12}(9))$  exists.

The following result gives a different construction.

**Theorem 9.** *A  $D(K_{(v-1)t+1})$  exists if a  $D(K_{(s-1)t+1})$ , an  $S(2, s, v)$  and a  $D(K_s(t))$  exist.*

**Proof.** Delete the element  $v$  from an  $S(2, s, v)$ . Then some blocks become of size  $s - 1$ . Since  $v$  appears with any other element once, these blocks form a partition of  $\{1, \dots, v - 1\}$ . Let  $\mathcal{B}_1$  denote the set of size- $s$  blocks and  $\mathcal{B}_2$  the set of size- $(s - 1)$  blocks. Make  $t$  copies of each element and apply a  $D(K_{(s-1)t+1})$  on each  $B'_i \cup \{0\}$  for  $B'_i \in \mathcal{B}_2$ , and apply a  $D(K_s(t))$  on each  $B_i \in \mathcal{B}_1$ .  $\square$

**Example 8.**  $v = 7, s = 3, t = 9, r = 2$  and  $c = 3$ . Then  $(v - 1)t + 1 = 55$ .

Let  $B_1, \dots, B_7$  be the seven blocks of an  $S(2, 3, 7)$  as given in Example 6. After element 7 is deleted,  $\mathcal{B}_1 = \{(S_1, S_2, S_4), (S_2, S_3, S_5), (S_3, S_4, S_6), (S_5, S_6, S_1)\}$  and  $\mathcal{B}_2 = \{(S_4, S_5), (S_6, S_2), (S_1, S_3)\}$ . Use  $D(K_3(9))$  on each block of  $B$ , and use a  $D(K_{19})$  on  $S_4 \cup S_5 \cup \{0\}, S_6 \cup S_2 \cup \{0\}$  and  $S_1 \cup S_3 \cup \{0\}$ .

While Theorem 9 deals with deleting an element from an  $S(2, s, v)$ , it is also possible to add an element. A Steiner 2-design  $(V, \mathcal{B})$  is said to be resolvable if  $\mathcal{B}$  is partitioned into subcollections  $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_u$ , called a parallel class or a resolution class such that each points in  $V$  is contained in  $\mathcal{R}_i$  exactly once for any  $i$ .

**Theorem 10.** *A  $D(K_{(v+1)t+1})$  exists if a resolvable  $S(2, s, v)$  system, a  $D(K_{t+1})$ , a  $D(K_s(t))$  and a  $D(K_{s+1}(t))$  exist.*

**Proof.** Add a new element  $v + 1$  to every block in a parallel class of the resolvable  $S(2, s, v)$ . Then some blocks become of size  $s + 1$ , which are handled by a  $D(K_{s+1}(t))$ .  $\square$

**Example 9.**

- (1 2 3) (4 5 6) (7 8 9)
- (1 4 7) (2 5 8) (3 6 9)
- (1 5 9) (2 6 7) (3 4 8)
- (1 6 8) (2 4 9) (3 5 7)

is a resolvable  $S(2, 3, 9)$  (see Hall, 1986). Add element 10 to the three blocks of the first parallel class to obtain blocks (1 2 3 10), (4 5 6 10), (7 8 9 10) of size 4. Apply a  $D(K_4(9))$  on the size-4 blocks and a  $D(K_3(9))$  on the size-3 blocks to obtain a  $D(K_{91})$ .

**Corollary 11.** *A  $D(K_{(v+i)t+1})$  exists if a resolvable  $S(2, s, v)$  system with at least  $i$  parallel classes, a  $D(K_{t+1})$ , a  $D(K_s(t))$ , a  $D(K_{s+1}(t))$  and a  $D(K_{it+1})$  exist.*

**Proof.** Let  $(V, \mathcal{B})$  be a resolvable  $S(2, s, v)$  with  $u \geq i$  parallel classes  $\{\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_u\}$ . Add  $i$  new elements to  $V$  and let  $\bar{V} = V \cup \{v + 1, v + 2, \dots, v + i\}$ . To each block  $B$  in  $\mathcal{R}_j$  add an element  $v + j$  and let  $\bar{B} = B \cup \{v + j\}$  for  $j = 1, 2, \dots, i$ . Make  $t$  copies of each element of  $\bar{V}$  and let  $S_j$  be the set of  $t$  copies of each element  $j$  for  $j = 1, 2, \dots, v + i$ . Moreover, add an element  $\{0\}$ . Apply a  $D(K_{s+1}(t))$  on each  $\bigcup_{j \in \bar{B}} S_j$  for all  $B \in \mathcal{R}_j$  and  $j = 1, 2, \dots, i$ , and a  $D(K_s(t))$  on the elements  $\bigcup_{j \in B} S_j$  for blocks  $B$  in  $\mathcal{B} \setminus \bigcup_{j=1}^i \mathcal{R}_j$ . In addition, apply a  $D(K_{t+1})$  on  $S_j \cup \{0\}$  for  $j = 1, 2, \dots, v$  and a  $D(K_{it+1})$  on  $(\bigcup_{j=1}^i S_{v+j}) \cup \{0\}$ .  $\square$

In case of  $r = c$ , we may use ‘affine geometry’ to obtain the decomposition.

**Theorem 12.** *For an even integer  $n$  and an odd prime power  $q$ , a  $D_{q \times q}(K_{q^n})$  exists.*

**Proof.** Let  $\alpha$  be a primitive element of  $GF(q^n)$ . Then each point of  $GF(q^n)$  is represented by  $\alpha^i$ . For convenience, let  $\alpha^\infty = 0$ . Here, for a prime power  $q$ , let  $AG_i(n, q)$  be the set of  $i$ -dimensional subspaces and their cosets of  $GF(q)$ . We define a base  $G(q, q)$   $G_0$  as follows:

$\alpha^\infty$	$\alpha^0$	$\alpha^{2u}$	...	$\alpha^{(2q-4)u}$
$\alpha^u$	$\alpha^0 + \alpha^u$	$\alpha^{2u} + \alpha^u$	...	$\alpha^{(2q-4)u} + \alpha^u$
$\alpha^{3u}$	$\alpha^0 + \alpha^{3u}$	$\alpha^{2u} + \alpha^{3u}$	...	$\alpha^{(2q-4)u} + \alpha^{3u}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\alpha^{(2q-3)u}$	$\alpha^0 + \alpha^{(2q-3)u}$	$\alpha^{2u} + \alpha^{(2q-3)u}$	...	$\alpha^{(2q-4)u} + \alpha^{(2q-3)u}$

where

$$u = \frac{q^n - 1}{2(q - 1)}.$$

Then  $G_0$  is a 2-flat (called plane) in  $AG_2(n, q)$  and rows and columns are 1-flats (called lines) in  $AG_1(n, q)$ . Thus,  $G_0$  generates a  $D_{q \times q}(K_{q^n})$  together with its cyclic shifts  $\alpha^i G_0$  for  $i = 0, 1, \dots, u - 1$  and with parallel 2-flats of them.

In fact, let  $\alpha^i$  and  $\alpha^j$  be two points in  $AG(n, q)$ . To count the number of rows and columns of  $G(q, q)$ s containing  $\alpha^i$  and  $\alpha^j$  simultaneously, we have only to count the number of rows/columns such that  $0 (= \alpha^\infty)$  and  $\alpha^i - \alpha^j$  occur together. We can represent  $\alpha^i - \alpha^j = \alpha^l$  for some integer  $l$ . There is one line passing through the origin  $0$  and  $\alpha^l$ , which proves the theorem.  $\square$

A Steiner 2-design  $(V, \mathcal{B})$  is said to be *cyclic* if there exists an automorphism  $\sigma$  of order  $v = |V|$  which acts cyclically on  $V$ . For a cyclic Steiner 2-design  $(V, \mathcal{B})$ , the collection  $\mathcal{B}$  of blocks is partitioned into orbits by  $\sigma$ . We choose block arbitrary from each orbit and call it a base block. Next, by combining base blocks of a cyclic Steiner 2-design, we obtain the following theorem:

**Theorem 13.** *Let  $p$  be an odd prime and  $v \equiv p \pmod{2p(p - 1)}$ . If there exists a cyclic  $S(2, p, v)$ , then there exists a  $D_{p \times p}(K_{pv})$ .*

**Proof.** Let  $t = (v - p)/2p(p - 1)$  and  $u = v/p$ . Firstly, we consider a cyclic  $S(2, p, v)$ . This design has  $2t$  base blocks with cycle length  $v$  and a single base block with cycle length  $u$ . It is known that we can construct an  $S(2, p, pv)$  from an  $S(2, p, v)$  for a prime  $p$  (see, Jimbo and Kuriki, 1983; Colbourn and Colbourn, 1984; Grannell and Griggs, 1986). Let  $a_x = (0, a_{x1}, \dots, a_{x(p-1)})$  be base blocks with cycle length  $v$  in  $S(2, p, v)$  for  $x = 0, 1, \dots, 2t - 1$ . According to the construction in Colbourn and Colbourn (1984), Grannell and Griggs (1986) and Jimbo and Kuriki (1983), we can obtain some base blocks of an  $S(2, p, pv)$  as follows for any  $j = 0, 1, \dots, p - 1$  and  $x = 0, 1, \dots, 2t - 1$ .

$$(0, a_{x1} + jv, a_{x2} + 2jv, \dots, a_{x(p-1)} + (p - 1)jv) \pmod{pv}. \tag{1}$$

Making  $t$  pairs of base blocks  $(a_x, a_y)$  in the  $S(2, p, v)$  by utilizing two base blocks  $a_x$  and  $a_y$ , we obtain the following base  $G(p, p)$ 's:

$0$	$a_{x1}$	$\dots$	$a_{x(p-1)}$	(mod $pv$ )	(2)
$a_{y1}$	$\dots$				
$\vdots$	$a_{xi} + a_{yj} + (i \cdot j)v$	$\vdots$			
$a_{y(p-1)}$	$\dots$				



The rows and columns of the  $G(p, p)$  contain each base block of (1) exactly once for  $j = 0, 1, \dots, p - 1$ . By checking the base  $G(p, p)$ 's, we find all differences except for  $\pm u, \pm 2u, \dots, \pm(p^2 - 1)u/2 \pmod{pv}$ .

By Theorem 12, it is known that a  $D_{p \times p}(K_{p^2})$  exists. Multiplying each element of a  $D_{p \times p}(K_{p^2})$  by  $u$ , and by making its cyclic shift of length  $u$ , we obtain a  $D_{p \times p}(K_{pv})$  together with the base  $G(p, p)$ s (2).  $\square$

#### 4. The existence of a $D_{3 \times 3}(K_n)$

In this section, we give the existence theorem of decomposition of complete graphs into  $G(3, 3)$ 's by utilizing direct constructions. The necessary condition for the existence of a  $D_{3 \times 3}(K_n)$  is  $n \equiv 1, 9 \pmod{36}$ . To prove sufficiency, we need the following proposition (see, Peltesohn, 1938).

**Proposition 14.** *If  $v \equiv 1, 3 \pmod{6}$  and  $v \neq 9$ , then there exists a cyclic  $S(2, 3, v)$ .*

By virtue of Theorem 13 and Proposition 14, we obtain the following:

**Lemma 15.** *If  $n \equiv 9 \pmod{36}$ , then a  $D_{3 \times 3}(K_n)$  exists.*

Note that the existence of a  $D_{3 \times 3}(K_9)$  is shown by Theorem 12. And by utilizing a  $D_{3 \times 3}(K_9)$  and a Steiner 2-design  $S(2, 9, n)$ , we can obtain a  $D_{3 \times 3}(K_n)$ . That is, for  $n \equiv 1, 9 \pmod{72}$  if an  $S(2, 9, n)$  exists, then a  $D_{3 \times 3}(K_n)$  exists. Unfortunately the existence problem for an  $S(2, 9, n)$  is not completely solved yet. Thus, in this paper, we construct a  $D_{3 \times 3}(K_n)$  for  $v \equiv 1 \pmod{36}$  directly.

**Lemma 16.** *If  $n \equiv 1 \pmod{36}$ , then a  $D_{3 \times 3}(K_n)$  exists.*

**Proof.** Firstly, in case of  $n = 72t + 1$ , Peltesohn (1938) showed that there exists a cyclic  $S(2, 3, n)$  (see also Beth et al. (1986, pp. 318–319)). According to his result,

$$(0, 1 + 2x, 33t + 1 + x), \quad x = 0, 1, \dots, 3t - 1, \tag{3}$$

$$(0, 2 + 2x, 24t + 2 + x), \quad x = 0, 1, \dots, 3t - 2, \tag{4}$$

$$(0, 9t + 1 + 2x, 27t + 1 + x), \quad x = 0, 1, \dots, 3t - 1, \tag{5}$$

$$(0, 9t + 2 + 2x, 18t + 2 + x), \quad x = 0, 1, \dots, 3t - 1, \tag{6}$$

$$(0, 6t, 24t + 1), \tag{7}$$

are base blocks of a cyclic  $S(2, 3, n)$ .

Table 1  
The correspondence of the base  $G(3,3)$ 's and base blocks

	Base block #			$x$			Adding constants		
Rows in $G_x$	(3)	(3)	(4)	1	0	0	0	$9t + 3 + 2x$	$27t + 2 + x$
Columns in $G_x$	(5)	(5)	(6)	1	0	0	0	$3 + 2x$	$33t + 2 + x$
Rows in $H_x$	(4)	(4)	(3)	2	1	2	2	$9t + 6 + 2x$	$18t + 4 + x$
Columns in $H_x$	(6)	(6)	(5)	2	1	2	0	$6 + 2x$	$24t + 4 + x$
Rows in $R_1$	(3)	(3)	(4)	$3t - 2$	$3t - 3$	$3t - 3$	0	$15t - 3$	$30t - 1$
Columns in $R_1$	(5)	(5)	(6)	$3t - 2$	$3t - 3$	$3t - 3$	0	$6t - 3$	$36t - 1$
Rows in $R_2$	(6)	(7)	(6)	$3t - 1$	—	$3t - 2$	$66t$	$15t - 1$	$30t$
Columns in $R_2$	(5)	(3)	(4)	$3t - 1$	$3t - 1$	$3t - 2$	0	$15t$	$39t$

By adding some constants for these base blocks and arranging them in  $G(3,3)$ 's as follows, we obtain base  $G(3,3)$ 's for a  $D_{3 \times 3}(K_{72t+1})$

$$G_x = \begin{pmatrix} 0 & 3 + 2x & 33t + 2 + x \\ 9t + 3 + 2x & 9t + 4 + 4x & 42t + 4 + 3x \\ 27t + 2 + x & 27t + 4 + 3x & 51t + 4 + 2x \end{pmatrix},$$

$$H_x = \begin{pmatrix} 0 & 6 + 2x & 24t + 4 + x \\ 9t + 6 + 2x & 9t + 10 + 4x & 33t + 9 + 3x \\ 18t + 4 + x & 18t + 9 + 3x & 51t + 7 + 2x \end{pmatrix}$$

for  $x = 0, 3, \dots, 3t - 6$ , and

$$R_1 = \begin{pmatrix} 0 & 6t - 3 & 36t - 1 \\ 15t - 3 & 21t - 8 & 51t - 5 \\ 30t - 1 & 36t - 5 & 57t - 2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 15t & 66t \\ 15t - 1 & 21t - 1 & 39t \\ 30t & 51t & 45t - 2 \end{pmatrix}.$$

In fact the rows in  $G_x$  are obtained by adding 0,  $9t + 3 + 2x$  and  $27t + 2 + x$  to (3) for  $x = 1$ , (3) for  $x = 0$  and (4) for  $x = 0$  in Table 1. And, the columns in  $G_x$  are obtained by adding 0,  $3 + 2x$  and  $33t + 2 + x$  to (3) for  $x = 1$ , (3) for  $x = 0$  and (4) for  $x = 0$ .

Similarly, for  $H_x$ ,  $R_1$  and  $R_2$ , the rows and columns are constructed by (3)–(7). Moreover, note that  $x \equiv 0, 1$  and  $2 \pmod{3}$  occurs exactly once for each of (3) to (6) in  $G_x$  and  $H_x$  of Table 1. Thus by considering  $G_x$ ,  $H_x$  for  $x = 0, 3, 6, \dots, 3t - 6$  and  $R_1$  and  $R_2$ , the base blocks in (3)–(7) occur exactly once.

Similarly, in case of  $n = 72t + 37$ , the following  $G(3,3)$ 's generate a  $D_{3 \times 3}(K_n)$  for  $x = 0, 3, \dots, 3t - 3$ :

$$G_x = \begin{pmatrix} 0 & 33t + 16 - x & 33t + 17 + x \\ 9t + 7 + 2x & 42t + 22 + x & 42t + 25 + 3x \\ 27t + 15 + x & 51t + 26 & 51t + 28 + 2x \end{pmatrix},$$

$$H_x = \begin{pmatrix} 0 & 5 + 2x & 33t + 19 + x \\ 18t + 7 - x & 18t + 11 + x & 42t + 21 \\ 27t + 16 + x & 27t + 22 + 3x & 51t + 31 + 2x \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 18t + 9 & 24t + 12 \\ 15t + 7 & 45t + 23 & 9t + 6 \\ 30t + 15 & 9t + 4 & 3t + 2 \end{pmatrix}.$$

Thus we proved the lemma.

By Lemmas 15 and 16, the following theorem is obtained.

**Theorem 17.** *The necessary condition  $n \equiv 1, 9 \pmod{36}$  for the existence of a  $D_{3 \times 3}(K_n)$  is sufficient.*

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