

THE EXISTENCE OF 2×4 GRID-BLOCK DESIGNS AND THEIR APPLICATIONS*

YUKIYASU MUTOH[†], TOSHIO MORIHARA[†], MASAKAZU JIMBO[†], AND HUNG-LIN FU[‡]

Abstract. Fu, Hwang, Jimbo, Mutoh, and Shiue [*J. Statist. Plann. Inference*, to appear] introduced the concept of a grid-block design, which is defined as follows: For a v -set V , let \mathcal{A} be a collection of $r \times c$ arrays with elements in V . A pair (V, \mathcal{A}) is called an $r \times c$ grid-block design if every two distinct points i and j in V occur exactly once in the same row or in the same column. This design has originated from the use of DNA library screening. They gave some general constructions and proved the existence of 3×3 grid-block designs. Meanwhile, the existence of 2×3 grid-block designs was shown by Carter [*Designs on Cubic Multigraphs*, Ph.D. thesis, McMaster University, Hamilton, ON, Canada, 1989] by decomposing K_v into cubic graphs. In this paper, we show the existence of 2×4 grid-block designs.

Key words. graph decomposition, graph design, grid-block

AMS subject classifications. 05B05, 05C70

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1. Introduction. A graph G is a pair of sets (V, E) , where V is a finite set, and E is a set of unordered pairs of elements of V . The elements of V are called *vertices* of G and the elements of E are called *edges* of G . If x and y are vertices of a graph G , we say that x is *adjacent* to y if there is an edge between x and y . K_v is the graph with v vertices such that every vertex is adjacent to every other vertex. For a v -set V , let \mathcal{A} be a collection of $r \times c$ arrays with elements in V . Each array in \mathcal{A} is called a *grid-block*. For a graph $G = (V, E)$, a pair (V, \mathcal{A}) is called an $r \times c$ grid-block design with respect to G denoted by $D_{r \times c}(G)$ if every two distinct points i and j in V such that $\{i, j\} \in E$ occur exactly once in the same row or in the same column. We used the terminology “grid-block design” to avoid the confusion with the “grid design” defined by Lamken and Wilson [9]. Here we show an example of a $D_{3 \times 3}(K_9)$.

EXAMPLE 1. *The following two grid-blocks form a $D_{3 \times 3}(K_9)$.*

1	2	3	1	6	8
4	5	6	9	2	4
7	8	9	5	7	3

A grid-block design was introduced by Fu et al. [7]. It is easy to show the following necessary conditions for the existence of a $D_{r \times c}(K_v)$.

LEMMA 1.1. *Necessary conditions for the existence of a $D_{r \times c}(K_v)$ are*

- (i) $(r + c - 2)|(v - 1)$ and
- (ii) $rc(r + c - 2)|v(v - 1)$.

Combinatorial designs were used as an efficient way of group testing in fields such as medical science and pharmaceutical science (see Du and Hwang [6]). Recently, a

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[†]Department of Mathematics, Keio University, Yokohama, Kanagawa, Japan 223-8522 (yukiyasu@jim.math.keio.ac.jp, toshio@jim.math.keio.ac.jp, jimbo@math.keio.ac.jp).

[‡]Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan, R.O.C. (hlfu@math.nctu.edu.tw).

combinatorial design has come to be applied to DNA library screening to discover the required DNA sequences by testing every row and every column in a microtiter plate at the same time.

In DNA library screening, a popular group testing method is a two-stage test. In this method, every row and every column in a microtiter plate is tested at the same time in the first stage, and each individual segment with positive response is tested in the second stage. See Figure 1.1 for demonstration. To reduce the number of tests and to improve the efficiency of experiments, several methods of screening have been studied by many authors.

Berger, Mandell, and Subrahmanya [1] evaluated the efficiency for the two-stage test from the point of view of information theory, while Fu et al. [7] introduced a combinatorial method based on a grid-block design.

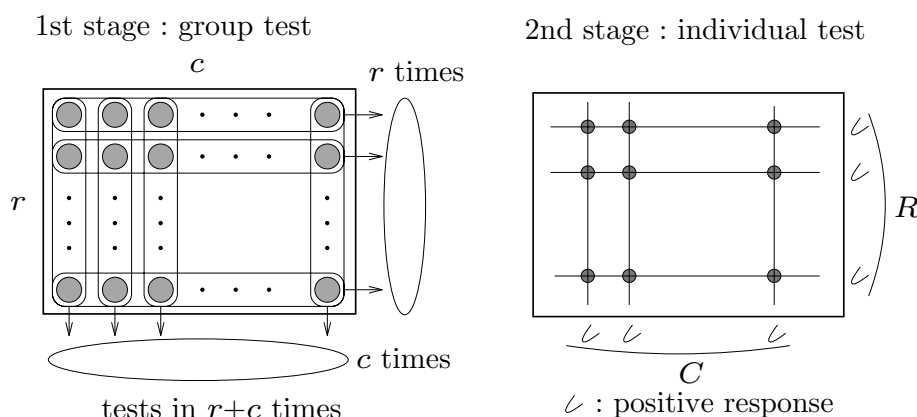


FIG. 1.1. The demonstration of DNA library screening.

In this paper, we start with a recursive construction for a grid-block design. Then, by utilizing this recursive construction together with those given by Fu et al. [7], we will prove the existence of 2×4 grid-block designs which satisfy the necessary condition $v \equiv 1 \pmod{32}$.

2. General constructions. In this section, we prepare a proposition and lemmas to use in the next section. First, we define a block design. For sets of positive integers K and M , let V be a set of v points, let \mathcal{G} be a partition of V such that each G has m points for $m \in M$, and let \mathcal{B} be a collection of k -subsets (*blocks*) of V for $k \in K$. A triple $(V, \mathcal{G}, \mathcal{B})$ is called a *group divisible design*, denoted by $GD[K, \lambda, M; v]$, if every two distinct points contained in different groups occur in exactly λ blocks and if every two distinct points contained in the same group do not occur together in any blocks. Especially, a $GD[\{k\}, \lambda, \{m\}; v]$ is written by $GD[k, \lambda, m; v]$ for simplicity of notation.

Suppose that the set of st vertices are partitioned into s subsets of size t each. Let $K_s(t)$ be the complete multipartite graph such that (i, j) is an edge if i and j are not in the same subset. A grid-block design $D_{r \times c}(K_s(t))$ is called a *group divisible grid-block design*. It is easy to see that the following lemma holds.

LEMMA 2.1. *Necessary conditions for a $D_{r \times c}(K_s(t))$ to exist are*

- (i) $(r + c - 2)|(s - 1)t$ and
- (ii) $rc(r + c - 2)|(s - 1)st^2$.

Fu et al. [7] proved the following construction.

TABLE 3.1
 Table of the existence of group divisible designs.

v	K	Group type	u	Exceptions	Ref.
$0, 1 \pmod{4}$	$\{4, 5\}$	1^u	$0, 1 \pmod{4}$	12	[2]
12	4	3^4	—	—	[4]
$12 \pmod{12}$	4	2^u	$1 \pmod{3}$	—	[4]
$3 \pmod{12}$	4	3^u	$1 \pmod{4}$	—	[4]
$6 \pmod{12}$	4	6^u	Anything	18	[4]
$7 \pmod{12}$	4	$7^1 1^u$	$0 \pmod{12}$	19	[3]
$10 \pmod{12}$	4	$7^1 1^u$	$3 \pmod{12}$	—	[3]
$11 \pmod{12}$	4	$5^1 2^u$	$0 \pmod{3}$	—	[3]

PROPOSITION 2.2 (Fu et al. [7]). *A $D_{r \times c}(K_{st+1})$ exists if a $D_{r \times c}(K_{t+1})$ and a $D_{r \times c}(K_s(t))$ exist.*

We give a recursive construction by utilizing a group divisible design, group divisible grid-block designs, and grid-block designs.

LEMMA 2.3. *A $D_{r \times c}(K_{vt+1})$ exists if a $GD[K, 1, M; v]$ exists and if a $D_{r \times c}(K_k(t))$ and a $D_{r \times c}(K_{mt+1})$ exist for any $k \in K$ and for any $m \in M$.*

Proof. For a v -set V , let a triple $(V, \mathcal{G}, \mathcal{B})$ be a $GD[K, 1, M; v]$, where $\mathcal{B} = \{B_1, B_2, \dots, B_b\}$ is a collection of blocks and $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ is a family of group sets. Let $T = \{0, 1, \dots, t-1\}$ and $V^* = (V \times T) \cup \{\infty\}$. For each block B_i of size $k \in K$, let $(B_i \times T, \mathcal{H}_i, \mathcal{E}_i)$ be the ingredient design $D_{r \times c}(K_k(t))$, where \mathcal{E}_i is a collection of grid-blocks and \mathcal{H}_i is a family of group sets $\{\{b_{i1}\} \times T, \{b_{i2}\} \times T, \dots, \{b_{ik}\} \times T\}$ for $b_{ij} \in B_i$. We define a collection of grid-blocks $\mathcal{A}_1 = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots \cup \mathcal{E}_b$. Also, for each group G_i of size $m \in M$, let $((G_i \times T) \cup \{\infty\}, \mathcal{F}_i)$ be the ingredient design $D_{r \times c}(K_{mt+1})$, where \mathcal{F}_i is a collection of grid-blocks. We define another collection of grid-blocks $\mathcal{A}_2 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_n$ and let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Then a pair (V^*, \mathcal{A}) is the desired $D_{r \times c}(K_{vt+1})$.

In fact, if two distinct elements x and y in V are not contained in the same group set G_j , then x and y occur together exactly once in a B_i . Hence (x, α_1) and (y, α_2) occur exactly once in the same row or in the same column of a grid-block in \mathcal{A}_1 and do not occur in \mathcal{A}_2 for any $\alpha_1, \alpha_2 \in T$. Otherwise, two elements x and y in V are contained in the same group set G_j including the case of $x = y$. In this case, (x, α_1) and (y, α_2) occur exactly once in the same row or in the same column of a grid-block in \mathcal{A}_2 and do not occur in \mathcal{A}_1 . Finally, ∞ and (x, α) for any $x \in V$ and $\alpha \in T$ occur exactly once in the same row and in the same column of a grid-block in \mathcal{A}_2 . \square

3. The existence of a 2×4 grid-block design. In this section we apply the results obtained in the previous section to prove the following theorem.

THEOREM 3.1. *The necessary condition $v \equiv 1 \pmod{32}$ for the existence of a $D_{2 \times 4}(K_v)$ is also sufficient.*

This existence theorem is shown by utilizing a recursive construction. First, we give an existence of a group divisible design.

LEMMA 3.2. *For any integer $v \geq 12$, there exists a $GD[K, 1, M; v]$, where $K = \{4, 5\}$ and $M = \{1, 2, \dots, 7\}$.*

Proof. According to Brouwer [3], Brouwer, Schrijver, and Hanani [4], and Beth, Jungnickel, and Lenz [2], we know the existence of a $GD[K, 1; M; v]$ for any $v \geq 12$ except for $v = 18$ and 19 as is listed in Table 3.1 (see also Kreher and Stinson [8] and Mullin and Gronau [10]). In Table 3.1, the notation $t_1^{u_1} t_2^{u_2}$ of a group type implies that V is divided into u_1 groups with group size t_1 and u_2 groups with group size t_2 .

TABLE 3.2
Table of the base grid-blocks of group divisible grid-block designs.

	Base grid-blocks											
$D_{2 \times 4}(K_4(32))$	0	1	6	15	0	21	58	47	0	25	74	55
	13	30	3	48	22	63	20	97	63	56	17	122
$D_{2 \times 4}(K_5(32))$	0	1	7	3	0	31	17	63	0	66	47	133
	11	27	48	39	22	73	129	30	13	149	105	51
	0	111	52	23								
	84	15	141	102								

TABLE 3.3
Table of the base grid-blocks of grid-block designs.

	Base grid-blocks											
$D_{2 \times 4}(K_{33})$	0	1	3	9								
	12	5	23	28								
$D_{2 \times 4}(K_{65})$	0	1	3	7	0	10	21	45				
	5	13	22	38	47	32	60	9				
$D_{2 \times 4}(K_{97})$	0	1	3	7	0	10	23	41	0	15	37	61
	5	13	22	33	33	65	86	3	39	55	84	12
$D_{2 \times 4}(K_{193})$	0	36	65	60	0	46	180	153	0	55	108	73
	89	155	152	153	186	23	71	169	114	77	133	81
	0	14	97	165	0	105	54	44	0	76	67	148
	102	52	40	134	75	34	178	55	39	189	73	174
$D_{2 \times 4}(K_{225})$	0	104	76	167	0	189	223	92	0	221	77	194
	67	137	121	209	156	74	167	199	41	94	42	16
	0	122	177	140	0	15	220	111	0	7	10	24
	212	190	106	67	95	76	55	46	38	82	206	32
	0	87	161	99								
	79	192	102	13								

Moreover, it is known that $GD[5, 1, 4; 20]$ exists, which is obtained by deleting one parallel class of lines and five points on a line in the parallel class from $AG(2, 5)$. By deleting a single point of a $GD[5, 1, 4; 20]$, we can show the existence of a $GD[\{4, 5\}, 1, \{3, 4\}; 19]$. Similarly, by deleting two points from the same group of a $GD[5, 1, 4; 20]$, we obtain a $GD[\{4, 5\}, 1, \{2, 4\}; 18]$, which proves the case of $v = 18$ and 19. Thus, the lemma is proved. \square

Second, we give two group divisible grid-block designs which are obtained by computer.

LEMMA 3.3. *There exists a $D_{2 \times 4}(K_k(32))$ for $k = 4$ and 5.*

Proof. For $V = \mathbf{Z}_{128}$, let

$$A_0 = \begin{bmatrix} 0 & 1 & 6 & 15 \\ 13 & 30 & 3 & 48 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 21 & 58 & 47 \\ 22 & 63 & 20 & 97 \end{bmatrix}, \quad \text{and}$$

$$C_0 = \begin{bmatrix} 0 & 25 & 74 & 55 \\ 63 & 56 & 17 & 122 \end{bmatrix},$$

which are listed in Table 3.2. Here A_0 , B_0 , and C_0 are called a *base grid-block* or a *starting grid-block*. For each base grid-block, let $A_i = A_0 + i \pmod{128}$, $B_i = B_0 + i \pmod{128}$, and $C_i = C_0 + i \pmod{128}$. Now we define

$$A = \{A_0, A_1, \dots, A_{127}, B_0, B_1, \dots, B_{127}, C_0, C_1, \dots, C_{127}\};$$

TABLE 3.4
Table of the base grid-blocks of grid-block designs (continued).

	Base grid-blocks											
$D_{2 \times 4}(K_{257})$	0	51	168	216	0	22	230	37	0	58	61	234
	148	147	81	37	30	211	187	193	200	118	101	154
	0	107	73	14	0	169	42	98	0	132	246	124
	50	79	202	176	63	61	96	216	20	41	72	162
$D_{2 \times 4}(K_{289})$	0	171	210	65	0	75	178	247				
	202	190	197	206	72	255	210	185				
	0	217	34	207	0	199	54	19	0	228	8	13
	28	188	253	168	105	282	236	183	86	35	165	189
$D_{2 \times 4}(K_{321})$	0	179	122	4	0	241	47	244	0	27	256	218
	209	37	211	284	124	191	110	98	248	182	225	98
	0	185	148	163	0	133	271	227	0	25	32	213
	128	186	216	180	166	14	150	206	77	255	266	164
$D_{2 \times 4}(K_{321})$	0	235	247	257	0	3	101	281	0	35	186	37
	310	101	228	133	76	105	212	309	244	138	264	16
	0	160	1	265	0	26	317	9	0	7	157	25
	158	66	291	221	269	178	228	315	23	205	143	74
	0	146	61	16	0	315	211	33	0	279	200	255
	283	288	174	115	206	78	146	254	34	105	272	308
$D_{2 \times 4}(K_{353})$	0	240	165	294								
	313	59	255	175								
	0	286	267	129	0	133	95	248	0	81	72	26
	198	149	219	118	22	20	275	113	82	257	147	261
	0	294	142	15	0	88	76	247	0	337	109	217
	34	173	198	1	71	222	144	194	66	150	2	211
$D_{2 \times 4}(K_{353})$	0	340	7	343	0	169	254	122	0	193	8	44
	195	5	234	264	316	229	17	59	352	103	127	76
	0	52	23	154	0	186	40	83				
	45	192	134	4	236	298	201	293				

then $(\mathbf{Z}_{128}, \mathcal{A})$ is the desired $D_{2 \times 4}(K_4(32))$. In fact, by calculating the differences of two elements in the same row or in the same column of A_0 , B_0 , and C_0 , any difference except for multiples of 4 occurs exactly once.

Similarly, for $V = \mathbf{Z}_{160}$, by utilizing four base grid-blocks in Table 3.2, we obtain a $D_{2 \times 4}(K_5(32))$. In fact, by calculating the differences of two elements in the same row or in the same column of A_0 , B_0 , C_0 , and D_0 any difference except for multiples of 5 occurs exactly once. \square

Third, we give some grid-block designs which are obtained by computer.

LEMMA 3.4. *There exists a $D_{2 \times 4}(K_{32m+1})$ for any $m = 1, 2, \dots, 11$.*

Proof. By utilizing the base grid-blocks in Tables 3.3 and 3.4, we obtain the desired $D_{2 \times 4}(K_{32m+1})$'s for $m = 1, 2, 3, 6, 7, \dots, 11$. By applying Proposition 2.2 to a $D_{2 \times 4}(K_4(32))$ and a $D_{2 \times 4}(K_5(32))$ in Lemma 3.3 and a $D_{2 \times 4}(K_{33})$, $D_{2 \times 4}(K_{32m+1})$'s are obtained for $m = 4$ and 5. \square

Now we will show the main theorem.

Proof of Theorem 3.1. By Lemma 1.1, it is easy to show that the necessary condition for the existence of a $D_{2 \times 4}(K_v)$ is $v \equiv 1 \pmod{32}$. Now we write $v = 32w + 1$; then there exists a $D(K_{32w+1})$ for $w \leq 11$ by Lemma 3.4. By Lemma 3.2, a $GD[K, 1, M; w]$ exists for $w \geq 12$, where $K = \{4, 5\}$ and $M = \{1, 2, \dots, 7\}$. And a $D(K_k(32))$ exists for $k = 4$ and 5 by Lemma 3.3. Thus by Lemma 2.3 a $D(K_{32w+1})$ exists for any $w \geq 12$, which prove the main theorem. \square

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REFERENCES

- [1] T. BERGER, J. W. MANDELL, AND P. SUBRAHMANYA, *Maximally efficient two-stage screening*, *Biometrics*, 56 (2000), pp. 833–840.
- [2] T. BETH, D. JUNGNIKEL, AND H. LENZ, *Design Theory*, Cambridge University Press, Cambridge, UK, 1986.
- [3] A. E. BROUWER, *Optimal packings of K_4 's into a K_n* , *J. Combin. Theory Ser. A*, 26 (1979), pp. 278–297.
- [4] A. E. BROUWER, A. SCHRIJVER, AND H. HANANI, *Group divisible designs with block size 4*, *Discrete Math.*, 20 (1977), pp. 1–10.
- [5] J. E. CARTER, *Designs on Cubic Multigraphs*, Ph.D. thesis, McMaster University, Hamilton, ON, Canada, 1989.
- [6] D.-Z. DU AND F. K. HWANG, *Combinatorial Group Testing and Its Application*, World Scientific, River Edge, NJ, 2000.
- [7] H. L. FU, F. K. HWANG, M. JIMBO, Y. MUTOH, AND C. L. SHIUE, *Decomposing complete graphs into $K_r \times K_e$'s*, *J. Statist. Plann. Inference*, to appear.
- [8] D. L. KREHER AND D. R. STINSON, *Small group-divisible designs with block size four*, *J. Statist. Plann. Inference*, 58 (1997), pp. 111–118.
- [9] E. R. LAMKEN AND R. M. WILSON, *Decompositions of edged-colored complete graphs*, *J. Combin. Theory Ser. A*, 89 (2000), pp. 149–200.
- [10] R. C. MULLIN AND H.-D.O.F. GRONAU, *PBDs and GDDs: The basics*, in *The CRC Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz, eds., CRC Press, Boca Raton, FL, 1996, pp. 185–193.