The Maximum Genus of a Graph with Given Diameter and Connectivity

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Abstract

In this paper, we first review some of the known results about the maximum genus of a graph with given diameter or (and) connectivity. Then we prove that a 3-connected diameter 4 multigraph has Betti deficiency at most 2. Furthermore, we show this upper bound is sharp.

1 Preliminaries and Known results

The *maximum genus* of a connected graph G, $\gamma_M(G)$, is the largest genus of an orientable surface on which G has a 2-cell embedding, and the *Betti deficiency* of G, $\xi(G)$, is equal to $\beta(G)-2\gamma_M(G)$, where $\beta(G)=|E(G)|-|V(G)|+1$ is the *Betti number* (*or the cycle rank*) of G. For convenience, we shall use "deficiency" to repalce "Betti deficiency". Clearly, $\xi(G)$ and $\beta(G)$ have the same parity and the maximum genus of a graph can be determined by its deficiency. In case that $\xi(G)\leq 1$, the graph G is said to be *upper embeddable*, i.e. $\gamma_M(G)=\lfloor\frac{\beta(G)}{2}\rfloor$. For the basic definitions in graph theory readers may refer to the book "Graphs and Digraphs" [1].

In the study of maximum genus, one of the most remarkable facts is that this topological invariant can be characterized in a purely combinatorial manner. In what follows, we introduce two such works. The first one is due to Khomenko, Ostroverkhy and Kuzmenko [8]. This result was later independently proved by Xuong [14], and an essential part of it by Jungerman [6].

Theorem 1.1.[6, 8, 14] *Let G be a connected graph. Then $\xi(G)=\min\{\xi(G,T)\mid T$ is a spanning tree of $G\}$ where $\xi(G,T)$ is the number of odd size components in $G-E(T)$.*

The second one is due to Nebeský [10]. Let G be a connected graph and $A\subseteq E(G)$. Let $\nu(G,A)=c(G-A)+b(G-A)-1-|A|$ where $c(G-A)$

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denotes the number of components in $G - A$ and $b(G - A)$ denotes the number of components in $G - A$ with odd Betti number. Then we have

Theorem 1.2.[10] *Let G be a connected graph. Then*

$$\xi(G) = \max\{\nu(G, A) \mid A \subseteq E(G)\}.$$

It is easy to see that the above two theorems produce a minimax pair of finding $\xi(G)$. For convenience, we shall refer to the methods used in Theorem 1.1 and Theorem 1.2 as Xuong's and Nebeský's method respectively. We note here that each method has its advantage in finding the maximum genus of a graph. First, by Xuong's method, the following result is easy to see.

Theorem 1.3. *Every 4-edge-connected graph is upper embeddable.*

Proof. This is a direct result of Theorem 1.1 and the well-known fact that every 4-edge-connected graph contains two edge-disjoint spanning trees [9]. ■

Therefore, in what follows, we shall only consider those graphs which are k-connected where $k \leq 3$. Quite recently, Kanchi and Chen [7] obtained a good result to describe the relationship between the maximum genus and connectivity.

Theorem 1.4. [7] *Let G be a 2-connected graph. Then $\gamma_M(G) \geq \lceil \beta(G)/3 \rceil$.*

Subsequently, Chen et al. [2] improved the above result to 2-edge-connected graphs and they also constructed infinitely many 3-connected cubic graphs G such that $\gamma_M(G) = \lceil \beta(G)/3 \rceil$.

On the diameter direction, Škoviera [11] utilized Xuong's method to study the maximum genus of diameter 2 graphs and he proved

Theorem 1.5.[11] *Every diameter 2 multigraph is upper embeddable.*

In the sequel, Škoviera also proves that a 2-connected diameter 2 pseudograph has deficiency at most 4 [12]. Then, by using Nebeský's method, Fu and Tsai are able to characterize the 2-connected diameter 2 graphs with deficiency 4. This theorem concerns the extremal graphs of 2-connected diameter 2 graphs. Note here that a 2-connected graph G of diameter 2 is called *extremal* if and only if $|E(G)| = 2|V(G)| - 5$.

Theorem 1.6.[4] *Let G be a 2-connected diameter 2 graph. Then $\xi(G) = 4$ if and only if G is an extremal 2-connected graph of diameter 2 at each vertex of which an odd number of loops are added.*

The deficiency of a connected diameter 2 pseudograph with a cut vertex may be larger than 4, but still it can be determined [5,12]. Since it is all set for diameter 2 graphs, diameter 3 graphs come next. The main result in this direction is the following theorem.

Theorem 1.7.[3] *Every diameter 3 multigraph has Betti deficiency at most 2.*

Subsequently, Fu and Tsai also characterize diameter 3 simple graphs. But for diameter 3 pseudographs, the work turns out to be much harder. A 2-connected diameter 3 pseudograph may have very large Betti deficiency as suggested by Figure 1. Thus, 3-connected graphs are what we are interested in.

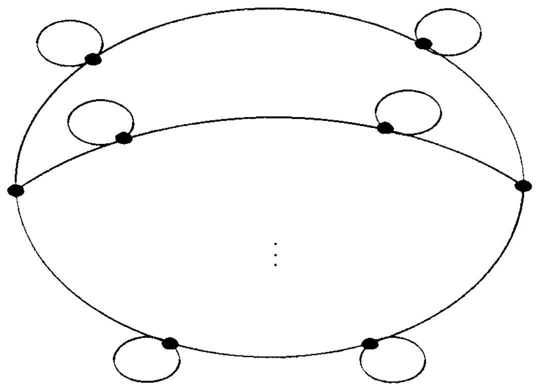

Figure 1.

With some effort, Tsai proves the following result in his thesis.

Theorem 1.8. [13] *Let G be a 3-connected diameter 3 pseudograph. Then* $\xi(G) \leq 44$.

Thus $\xi(G)$ is bounded for such graphs, nevertheless does not seem to be sharp.

If we move forward to diameter 4 graph, then we see that a 2-connected diameter 4 simple graph and a 3-connected diameter 4 pseudograph can have arbitrarily large deficiency, see [13] for details. Moreover, a 3-connected diameter 6 simple graph can be arbitrarily large, see also [13]. Thus, the graphs G with bounded deficiency are limited to the cases when G is 3-connected diameter 4 multigraph and 3-connected diameter 5 multigraph. To this end, Tsai obtained the following two results.

Theorem 1.9.[13] *Let G be a 3-connected diameter 4 multigraph. Then* $\xi(G) \leq 4$.

Theorem 1.10. [13] *Let G be a 3-connected diameter 5 simple graph. Then* $\xi(G) \leq 18$.

The purpose of this paper is to improve the result stated in Theorem 1.9.

Theorem 1.11. *Let G be a 3-connected simple graph of diameter 4. Then $\xi(G) \leq 2$, and the bound is attained by an infinite family of multigraphs.*

2 The main result

In [3], Nebeský's method has been applied and the "minimality" property of the edge subset A in this method plays an important role. For conevenience, we call a graph with $\xi(G) \geq 2$ a *deficient graph.* Next any set $A \subseteq E(G)$ such that $\nu(G, A) = \xi(G)$ will be called a *Nebeský set*; furthermore if A is minimal, then it will be called a *minimal Nebeský set.*

Lemma 2.1.[3] *Let G be a deficient graph and let A be a minimal Nebeský set of G. Then*

(a) $b(G-A) = c(G-A)$, *moreover, if* G *is a multigraph then every component of* $G - A$ *is non-trivial and if* G *is a simple graph then every component of* $G - A$ *contains at least three vertices;*
(b) *the end vertices of every edge in* A *belong to distinct components of* $G-A$*; and*
(c) *any two components of* $G - A$ *are joined by at most one edge of* A.

With the support of Lemma 2.1, we are able to construct a new graph based on the choice of A. Let G be a deficient graph and let A be a minimal Nebeský set of G. G_A is called a ***testable graph*** of G (with respect to A) if $V(G_A)$ is the set of components of $G - A$ and two vertices in G_A are adjacent if and only if they are joined in G by an edge of A. For convenience, if G_A is a testable graph, then A is called a ***testable edge set*** of G. In order to avoid ambiguity, we shall refer the vertices of G_A to as the "nodes" of G_A, and $u_A, v_A, \cdots$ are typical notation for the nodes. Accordingly, the following two lemmas are easy to prove. For completeness, the proofs are included here.

Lemma 2.2.[13] *If* G *is a deficient graph and* A *is a minimal Nebeský set of* G, *then*

$$\xi(G) = 2p(G_A) - q(G_A) - 1, \text{ where}$$

$p(G_A)$ *and* $q(G_A)$ *are the number of nodes and edges of* G_A *respectively.*

Proof. By the definition of G_A, $p(G_A) = c(G-A)$ and $q(G_A) = |A|$. Applying Theorem 1.2 and Lemma 2.1,

$$\begin{aligned} \xi(G) &= c(G-A) + b(G-A) - 1 - |A| \\ &= 2c(G-A) - 1 - |A| \\ &= 2p(G_A) - q(G_A) - 1. \end{aligned}$$

■

Lemma 2.3.[13] *If* G *is a deficient graph and* A *is a minimal Nebeský set of* G, *then the minimum degree of* G_A *is not greater than 3, that is,* $\delta(G_A) \leq 3$.

Proof. Suppose the contrary. Then $\deg_{G_A}(v_A) \geq 4$ for each node v_A in G_A. Thus $q(G_A) \geq 2p(G_A)$. By Lemma 2.2,

$$\xi(G) = 2p(G_A) - q(G_A) - 1 \leq 2p(G_A) - 2p(G_A) - 1 = -1.$$

This is a contradiction. So we have the proof. ■

In what follows, the degree of a node v_A in G_A is denoted by $\deg(v_A)$, and the degree of a vertex x in G is denoted by $\deg(x)$ if no confusion occurs.

Theorem 2.4. *Let* G *be a 3-connected diameter 4 multigraph. Then* $\xi(G) \leq 2$.

Proof. Let $\mathcal{G} = \{H \mid H \text{ is a 3-connected diameter 4 multigraph with } \xi(H) > 2\}$. We shall claim that $\mathcal{G}$ is an empty set. Suppose the contrary. Let $G \in \mathcal{G}$ be with minimum order. Clearly, G is a deficient graph. Now let A be a minimal Nebeský's set. By Lemma 2.1 (a), each component of $G - A$ has odd Betti number. Thus, each component of $G - A$ is either a triangle or two vertices which are joined by multiple edges. Otherwise, there exists a graph $G' \in \mathcal{G}$ with $|V(G')| < |V(G)|$. Now let T_{x_A} denote the component of $G - A$ which corresponds to x_A in G_A for each node $x_A \in V(G_A)$. Clearly, if T_{x_A} is a triangle, then each vertex of T_{x_A} is incident with at least one edge of A and thus $\deg(x_A) \geq 3$. On the other hand, if T_{x_A} is a component with two vertices, then each vertex of T_{x_A} is incident with at least two edges of A and thus $\deg(x_A) \geq 4$.

Now since G is a deficient graph, Lemma 2.2 gives $\xi(G) = 2p(G_A) - 1 - q(G_A)$. Furthermore Lemma 2.3 shows that there exist some nodes in $V(G_A)$ whose degrees are equal to 3. Accordingly, let z_A be one of the nodes of degree 3. And, let $D_0 = \{z_A\}$, $D_1 = N(z_A)$ and $D_2 = V(G_A) \setminus N[z_A]$. We call $x \in V(G)$ a *distance k vertex* if $\min\{d(x,z) \mid z \in V(T_{z_A})\} = k$ and we denote $E(D_i, D_j) = \{x_A y_A \in E(G_A) \mid x_A \in D_i \text{ and } y_A \in D_j\}$, where $0 \leq i, j \leq 2$. (Note here that the order of x_A and y_A is important throughout of the proof.) In order to figure out $q(G_A)$, we also need the following definitions:

$\mathcal{A}_1 = \{x_A y_A \in E(D_2, D_1) \mid$ there exists a distance 1 vertex of T_{y_A} which is adjacent to a distance 2 vertex of T_{x_A} or a distance 2 vertex of T_{y_A} which is adjacent to a distance 3 vertex of T_{x_A} and a distance 1 vertex of T_{w_A} for some $w_A \in D_1 \setminus \{y_A\}\}$.

$\mathcal{A}_2 = \{x_A y_A \in E(D_2, D_2) \mid x_A$ is not incident with any edge of $\mathcal{A}_1$ and y_A is incident with one edge of $\mathcal{A}_1$ and T_{y_A} contains a vertex which is both adjacent to a vertex of T_{x_A} and a vertex of T_{u_A} for some $u_A \in D_1\} \cup \{x_A y_A \in E(D_2, D_2) \mid x_A$ is not incident with any edge of $\mathcal{A}_1$ and y_A is incident with at least two edges of $\mathcal{A}_1\}$.

$\mathcal{A}_3 = \{x_A y_A \in E(D_1, D_1) \mid$ there exists a distance 2 vertex of T_{x_A} which is adjacent to a distance 1 vertex of $T_{y_A}\}$;

Now, according to the above edge subsets of $E(G_A)$, we define a directed graph $\overrightarrow{G_A}$ based on G_A:

(i) $V(\overrightarrow{G_A}) = V(G_A)$,

(ii) if $x_A y_A \in E' = (\bigcup^{3} \mathcal{A}_i) \cup E(D_1, D_0)$, then join two arcs from y_A to x_A, and

(iii) if $x_A y_A \in E(G_A) \setminus E'$, then we let (x_A, y_A) and (y_A, x_A) be arcs of $\overrightarrow{G_A}$.

By the definition of $\overrightarrow{G_A}$, it is easy to see that each edge of G_A is corresponding to two arcs of $\overrightarrow{G_A}$ and

$$\sum_{x_A \in V(G_A)} \deg(x_A) = \sum_{x_A \in V(\overrightarrow{G_A})} \deg^-(x_A),$$

where $\deg^-(x_A)$ denotes the indegree of x_A in $\overrightarrow{G_A}$. Therefore, the indegree sum of $\overrightarrow{G_A}$ gives $2q(G_A)$.

The reason why we use indegree sum of $\overrightarrow{G_A}$ instead of the degree sum of G_A can be explained as follows. In order that we have a graph with very small deficiency, $q(G_A)$ must be very close to $2p(G_A)$, i.e., the average degree of the nodes in G_A is about 4. Hence, if there are only a small number of nodes with degree 3 in G_A then we are done. But, this may not be true, since we may have a node with large degree node and a bunch of degree 3 nodes. Thus, by defining $\overrightarrow{G_A}$ properly, we expect to obtain "more" nodes with indegree larger than 3.

Now, we count the indegree sum of $\overrightarrow{G_A}$. Let x_A be an arbitrary node in $V(\overrightarrow{G_A})$. Then, clearly $\deg^-(x_A) = 0$ if $x_A = z_A$. Hence we have two cases $x_A \in D_1$ and $x_A \in D_2$ to consider. First, let $x_A \in D_2$. We claim that if x_A is incident with an edge of $\mathcal{A}_1 \cup \mathcal{A}_2$, then $\deg^-(x_A) \geq 4$.

If x_A is incident with an edge of $\mathcal{A}_2$, but not incident with any edge of $\mathcal{A}_1$, then x_A is incident with either at least two edge of $\mathcal{A}_2$ or one edge of $\mathcal{A}_2$ and at least two edges of $E(G_A) \setminus E'$. Thus, by definition of $\overrightarrow{G_A}$, $\deg^-(x_A) \geq 4$. Otherwise, x_A is incident with one edge of $\mathcal{A}_1$. If x_A is also incident with $\mathcal{A}_2$, then by the definition of $\mathcal{A}_2$, x_A is incident with at least two edges of $\mathcal{A}_1$ or x_A is also incident with at least two edges of two edges of $E(G_A) \setminus E'$. Again, by the definition of $\overrightarrow{G_A}$, $\deg^-(x_A) \geq 4$, and we have the claim.

Now, let $M = \{x_A \in D_2 \mid \deg(x_A) = 3$ and x_A is not incident with any edge of $\mathcal{A}_1 \cup \mathcal{A}_2\}$. By the definition of $\overrightarrow{G_A}$, $\deg^-(x_A) = \deg(x_A)$. Hence we have

$$\sum_{x_A \in D_2} \deg^-(x_A) \geq 4|D_2| - |M|. \tag{1}$$

Next, we claim that if $x_A \in M$, then $|N(x_A) \cap D_1| \geq 2$. Suppose the contrary. Then x_A has at most one neighbor in D_1. First, if $|N(x_A) \cap D_1| = 0$, then in order that $d(x,z) \leq 4$ for each $x \in V(T_{x_A})$ and each $z \in V(T_{z_A})$, the node x_A has to be incident with some edge in $\mathcal{A}_2$. This is a contradiction. Hence $|N(x_A) \cap D_1| = 1$. Let w_A be the common node and x_1w_1 be an edge of $E(G)$ which corresponds to the edge x_Aw_A of $E(G_A)$. Since x_A is not incident with any edge of $\mathcal{A}_1$, x_1 and w_1 must be distance 3 and distance 2 vertices respectively, and w_1 is not adjacent to any distance 1 vertex except those of T_{w_A}. However, since $\deg(x_A) = \deg(z_A) = 3$, T_{x_A} and T_{z_A} are triangles. Let $V(T_{x_A}) = \{x_1, x_2, x_3\}$ and $V(T_{z_A}) = \{z_1, z_2, z_3\}$. Then, at most one of the paths from x_1 to a vertex of T_{z_A}, x_1- (vertices in) $T_{w_A} - z_1$, is of length 3 and the other two paths $x_1 - T_{w_A} - z_2$, $x_1 - T_{w_A} - z_3$ are of length 4. Now we consider $d = \max\{d(x_2, z) | z \in V(T_{z_A})\}$. In order that $d \leq 4$, x_A has to be adjacent to a node in D_2, say y_A, and x_2 is adjacent to a vertex of T_{y_A}, say y_1. Moreover, $N(y_A) \cap N(z_A) \neq \emptyset$ and y_A is incident with at most one edge of $\mathcal{A}_1$. Let v_A be the common node with $v_Ay_A \in \mathcal{A}_1$. By the fact that x_A is not incident with the edges of $\mathcal{A}_2$, one of the two paths, $x_2 - T_{y_A} - T_{v_A} - z_2$ and $x_2 - T_{y_A} - T_{v_A} - z_3$, are of length 5. Furthermore, since $\deg(x_A) = 3$, $\deg(x_2) = 3$. Thus x_2 may not be dajacent to any vertex of $V(G_A) \setminus \{y_1, x_1, x_3\}$. Accordingly, $d(x_2, z_i) \geq 5$ for $i = 2$ or 3. This is a contradiction, again. Thus we have the claim:

$$|N(x_A) \cap N(z_A)| \geq 2 \quad \text{for each } x_A \in M. \tag{2}$$

Finally, let $x_A \in D_1$. x_A must be incident with an edge of $E(D_1, D_0)$. Furthermore, if x_A is incident with an edge of $\mathcal{A}_1$ such that T_{x_A} contains a distance 2 vertex which is adjacent to a distance 3 vertex and a distance 1 vertex of T_{w_A} for some $w_A \in D_1 \setminus \{x_A\}$, then $x_Aw_A \in \mathcal{A}_3$. Accordingly, x_A is also incident with at least one edge of $\mathcal{A}_3$ or at least two edges of $E(G_A) \setminus E'$. By the definition of $\overrightarrow{G_A}$, $\deg(x_A) \geq 4$. Hence

$$\sum_{x_A \in D_1} \deg^-(x_A) \geq 4|D_1|. \tag{3}$$

On the other hand, each node of D_1 is incident with one edge of $E(D_1, D_0)$ and each node of M is adjacent to at least two nodes of D_1 (By (2)), hence we have

$$\sum_{x_A \in D_1} \deg^-(x_A) \geq 2|D_1| + 2|M|. \tag{4}$$

Combining (3) and (4), we have

$$\sum_{x_A \in D_1} \deg^-(x_A) \geq 3|D_1| + |M|. \tag{5}$$

Now by (1) and (5), we have

$$\begin{aligned} 2q(G_A) &= \sum_{x_A \in V(\overrightarrow{G}_A)} \deg^-(x_A) \\ &\geq 4|D_2| - |M| + 3|D_1| + |M| \\ &= 4p(G_A) - |D_1| - 4 \qquad (p(G_A) = 1 + |D_1| + |D_2|) \\ &= 4p(G_A) - 7. \qquad (|D_1| = 3) \end{aligned}$$

By Lemma 2.2, $\xi(G) = 2p(G_A) - 1 - q(G_A) \leq 2$, a contradiction. This concludes the proof. ■

To see that the upper bound presented in Theorem 2.4 is best possible, let us consider the following infinite family of graphs, as depicts in Figure 5. It is not difficult to check that their Betti deficiency are equal to 2. On the other hand, we have the following observation which shows the existence of upper embeddable 3-connected diameter 4 graphs.

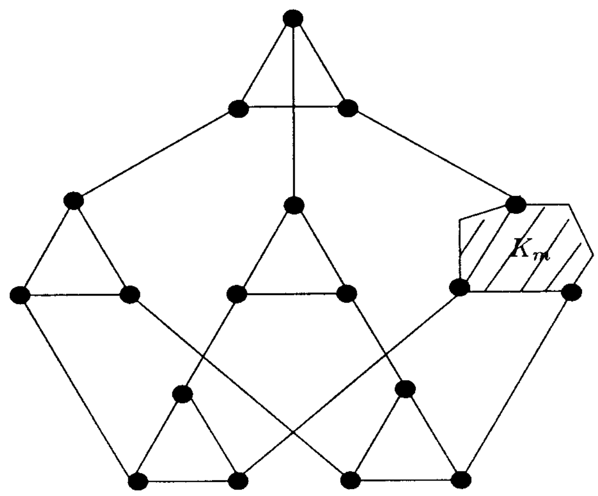

Figure 5. ($m \equiv 0$ or 3 (mod 4))

Corollary 2.5. *Let G be a 3-connected diameter 4 multigraph such that* $|V(G)| \equiv |E(G)| \pmod 2$. *Then G is upper embeddable and* $\gamma_M(G) = \frac{1}{2}(|E(G)| - |V(G)|)$.

To conclude, we present two tables to give a picture of the study of the maximum genus via graph's diameter and connectivity. For the deficiency of a 3-connected diameter 5 multigraph whether the deficiency is bounded or not remains unknown so far.

Type / Diameter	Simple Graph	Multigraph	Pseudograph
2	≤ 1	≤ 1	≤ 4
3	≤ 2	≤ 2	unbounded
≥ 4	unbounded		

Figure 6. The deficiency of a 2-connected graph

Diameter \ Type	Simple Graph	Multigraph	Pseudograph
3	≤ 1	≤ 2	≤ 44
4	≤ 2	≤ 2	unbounded
5	≤ 18	?	unbounded
≥ 6	unbounded		

Figure 7. The deficiency of a 3-connected graph

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