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Face 2-colorable Quadrilateral Embeddings of Complete Bipartite Graphs*

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Abstract

An embedding is said to be face 2-colorable if the faces of the embedding can be colored with two colors such that no two monochromatic faces share an edge. In this paper, it is proved that a face 2-colorable quadrilateral embedding of the complete bipartite graph $K_{m,n}$ exists if and only if m and n are even. Moreover, we obtain a different proof of $\gamma(K_{m,n}) = \lceil\frac{(m-2)(n-2)}{4}\rceil$ which does not use rotational scheme and the methods known.

A triangular (quadrilateral) embedding of a graph is an embedding such that all the faces are 3-cycles (4-cycles). Without a doubt, the search for triangular embeddings of complete graphs in closed surfaces gives the birth of modern topological graph theory, see [1]. In [2], the authors studied a more structural triangular embedding of the complete graph, called face 2-colorable embedding. A triangular embedding is said to be face 2-colorable if the triangular faces of the embedding can be colored with two colors such that no two monochromatic triangles share an edge. Hence, each edge in the complete graph is in exactly one triangle in each of the two colors. This

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implies that the embedding induces two disjoint Steiner triple systems. For the details about Steiner Triple Systems, see [3].

Their study motivates us to study the embedding of complete bipartite graphs, especially the face 2-colorable embeddings. Since the complete bipartite graphs contain no triangles and thus to determine the genus of such graphs requires quadrilateral embeddings. It is now well-known that the genus of $K_{m,n}$ is $\lceil \frac{(m-2)(n-2)}{4} \rceil$, see [4] [8] [9]. By the general Euler formula $|V(G)| - |E(G)| + |R(G)| = 2 - 2g$, the embedding with minimum genus is equivalent to the embedding with maximum number of regions. Therefore, the embedding of $K_{m,n}$ with minimum genus is obtained by using as many 4-cycles as possible.

A 4-cycle system of a graph G is a decomposition of G into 4-cycles. It is known that a 4-cycle system of K_n exists if and only if $n \equiv 1 \pmod 8$ [5] and a 4-cycle system of $K_{m,n}$ exists if and only if m and n are both even [6]. Also, it is clear that a 4-cycle system of $2K_{m,n}$ exists provided m, $n > 1$ and mn is even. Therefore, if m and n are even, then it is possible to have a 4-cycle system of $2K_{m,n}$ which is a disjoint union of two 4-cycle systems of $K_{m,n}$. Thus, as an analog of face 2-colorable triangular embeddings of complete graphs, the existence of a face 2-colorable quadrilateral embedding of $K_{m,n}$ produces a pair of disjoint 4-cycle system of $K_{m,n}$.

A pseudosurface results when finitely many identifications, of finitely many points each, are made on a given surface. (The result of each such an identification is called a singular point; a pseudosurface fails to be a 2-manifold at exactly its singular points. Note that surfaces are trivially pseudosurfaces.) The orientability criterion carries over to pseudosurfaces, since vertex identification does not affect region boundaries. [7]

For much more detail about this construction, which has become a standard tool in topological graph theory, we refer to [1]. Now, we are ready for the main results.

Theorem 1. *A face 2-colorable quadrilateral embedding of $K_{m,n}$ exists if and only if both m and n are even.*

Proof. (Necessity) If a face 2-colorable embedding exists, then each vertex must be of even degree. Therefore, the necessity follows.

(Sufficiency) Let m and n be positive integers. The proof will be by induction on n. First, a face 2-colorable quadrilateral embedding of $K_{m,2}$ can be obtained by Figure 1. Thus it is true for $n = 2$.

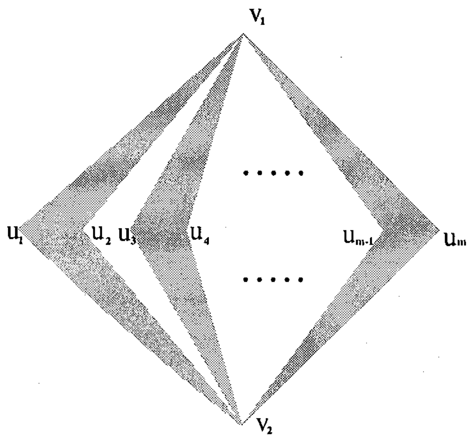

Figure 1: The face 2-colorable quadrilateral embedding of $K_{m,2}$.

Now, assume that a face 2-colorable quadrilateral embedding of $K_{m,k}$ exists for each even integer k. For convenience, we also let $K_{m,k} = (A, B)$ where $A = \{u_1, u_2, \cdots, u_m\}$, $B = \{v_1, v_2, \cdots, v_k\}$ and η be a face 2-colorable embedding. For a fixed $v \in B$, let $F_1, F_2, \cdots, F_{\frac{m}{2}}$ be $\frac{m}{2}$ monochromatic 4-cycles which are incident to v.

Because the orientable surface is a 2-manifold, every point in it can be regarded as an interior point of the open disk. Without loss of generality, consider an open neighborhood of v and the $\frac{m}{2}$ black faces which are incident with v as depicted in Figure 2, where $v_{i_j} \in A \backslash \{v\}$ and overlap is possible (for example, if the vertex v is the vertex v_1 of the embedding in the Figure 1, then all vertices v_{i_j} are the vertex v_2).

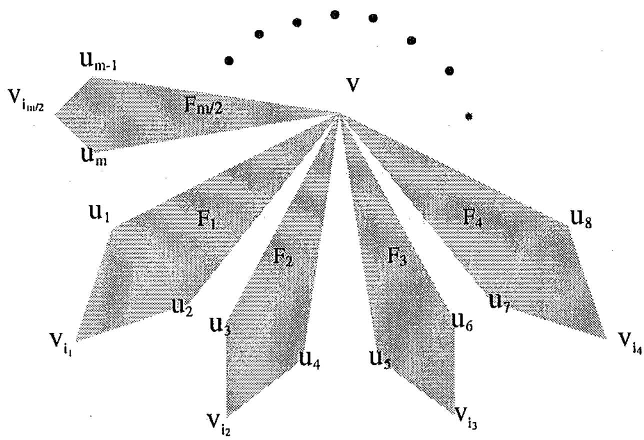

Figure 2:

For these $\frac{m}{2}$ black faces $F_1,\ F_2,\ \cdots F_{\frac{m}{2}}$, consider the auxiliary face 2-colored embedding ψ in a pseudosurface, as depicted in Figure 3.

Now we can observe that the regions $F'_1,\ F'_2,\ \cdots F'_{\frac{m}{2}}$ in ψ have the same corners with $F_1,\ F_2,\ \cdots F_{\frac{m}{2}}$ but with reverse orientation. So, by the "surgery" technique, remove from the embedding η the open quadrilateral faces $F_1,\ F_2,\ \cdots F_{\frac{m}{2}}$ and from the embedding ψ the regions $F'_1,\ F'_2,\ \cdots F'_{\frac{m}{2}}$. Identify the closed curve $C(F_i)$ and $C(F'_i)$ created after removing the faces F_i and F'_i for each $i \in \{1, \cdots, \frac{m}{2}\}$. The above procedure indeed attaches the pseudosurface in Figure 3 to the orientable surface in Figure 2. Thus we obtain a face 2-colorable quadrilateral embedding of $K_{m,k+2}$ on an orientable surface. This completes the proof.

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Note that by attaching a pseudosurface to an orientable surface, we have a new surface which has more handles than the original one. For example,

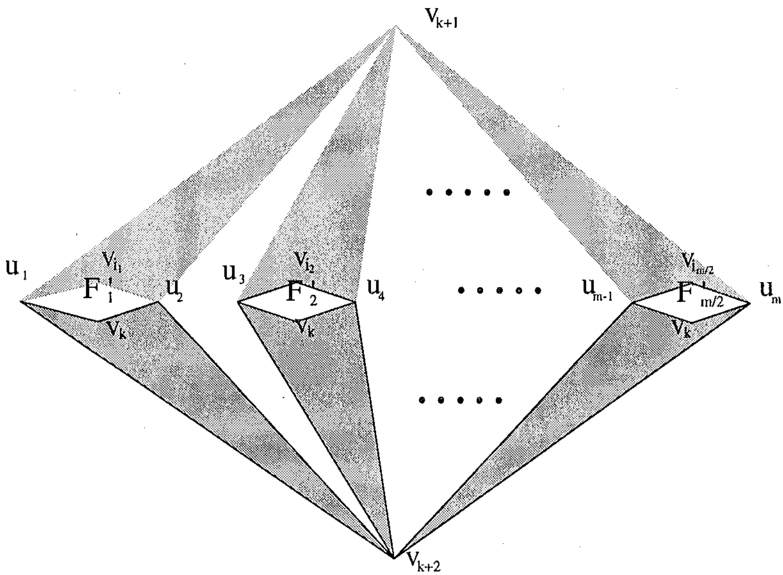

Figure 3: The auxiliary face 2-colored embedding ψ in a pseudosurface.

the one in Figure 3 creates $\frac{m}{2}-1$ new handles. Thus, by choosing a proper pseudosurface in our embedding technique, we are able to determine the genus of $K_{m,n}$ without using the well-known rotational scheme and the methods known in [8] [9]. Moreover, this method is more intuitive.

Theorem 2. $\gamma(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$.

Proof. For the case when m and n are even, Theorem 1 provides the proof ($\lceil \frac{(m-2)(k-2)}{4} \rceil + (\frac{m}{2}-1) = \lceil \frac{(m-2)(k)}{4} \rceil$ while m and n are even). Hence, we consider the other cases.

(1). $m \equiv 2 \pmod 4$ and n is odd.

First, we claim that the quadrilateral embedding of $K_{4k-2,3}$ can be constructed. This proof will be by induction on k. Obviously, there is a quadrilateral embedding of $K_{2,3}$ on the sphere and the genus of sphere is zero that satisfies the statement of the theorem. Assume that the statement is true for $k = n$, i.e., there exists a quadrilateral embedding ϕ of $K_{4n-2,3}$. By deleting a given vertex u_{4n-2} of ϕ, we have an embedding of $K_{4n-3,3}$. Furthermore, this embedding is almost quadrilateral except a 6-gon F as depicted in Figure 4.

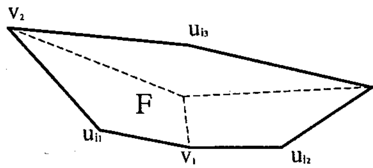

Figure 4:

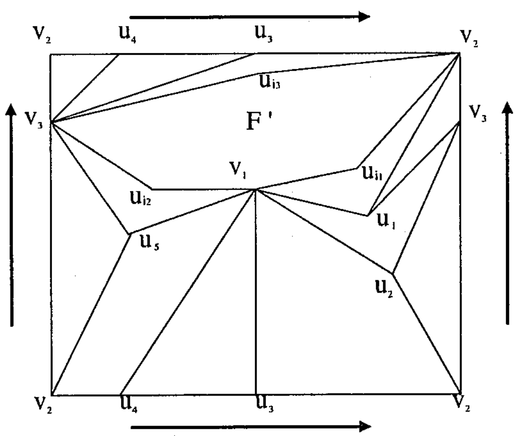

Figure 5:

Now consider the auxiliary graph which is embedded in a torus as depicted in Figure 5. Attach the chosen region F' of the embedding in the torus to the 6-gon F and identify the closed curve $C(F)$ and $C(F')$. This procedure attaches a torus to the original surface such that the new embedding in the created surface is a quadrilateral embedding of $K_{4n+2,3}$. So, we have the claim, $\gamma(K_{4k-2,3}) = k - 1$.

The next step in this case, we need to prove $\gamma(K_{4k-2,n}) =$

$\lceil \frac{(4k-4)(n-2)}{4} \rceil = (k-1)(n-2)$ as n is odd. The procedure of the construction of the quadrilateral embedding of $K_{4k-2,n}$ is similar to the proof of Theorem 1. Let $V(K_{4k-2,n}) = (A, B)$ and consider a $v \in B$, then find an auxiliary embedding in a pseudosurface. It follows from the previous proof that we create $\frac{4k-2}{2} - 1$ new handles. Because $\lceil \frac{(4k-4)(n-2)}{4} \rceil + (\frac{4k-2}{2} - 1) = \lceil \frac{(4k-4)n}{4} \rceil$, we have constructed the quadrilateral embedding of $K_{4k-2,3}$. This concludes the proof of this case.

(2). $m \equiv 0 \pmod 4$ and n is odd.

Check the maximum number of regions in the embedding of $K_{4k,n}$ on the orientable surface. Since $|V(K_{4k,n})| = 4k+n$ and $|E(K_{4k,n})| = 4kn$, we have $|F(K_{4k,n})| \leq \frac{2|E(K_{4k,n})|}{4} = 2kn$. By the general Euler formula $|V(G)| - |E(G)| + |R(G)| = 2 - 2g$, $|F(K_{4k,n})|$ must be odd. So the embedding of $K_{4k,n}$ with minimum genus must satisfy that all regions except one R_0 are 4-cycles. And the region R_0 is a 8-gon because $2|E(K_{4k,n})| - 4(|F(K_{4k,n})| - 1) = 2 \cdot 4kn - 4(2kn - 1 - 1) = 8$.

For this embedding, we start with a quadrilateral embedding of $K_{4k-2,n} = (A, B)$ obtained in (1) and consider a fixed $v \in A$. Because the orientable surface is a 2-manifold, consider an open neighborhood of v and the $\frac{n+1}{2}$ lightly shaded regions which are incident with v where $v_{i_j} \in A \backslash \{v\}$ and the v_{i_j} may overlap, see Figure 6. First, we use the auxiliary pseudosurface depicted in Figure 7 and attach the regions F_1', F_2', $\cdots F_{\frac{n-1}{2}}'$ in the auxiliary pseudosurface to the regions F_1, F_2, $\cdots F_{\frac{n-1}{2}}$ of Figure 6. This can be done by identifying the closed curve $C(F_i)$ and $C(F_i')$, $i = 1,2 \cdots, \frac{n-1}{2}$. This "surgery" technique makes sure that the new embedding in the created surface is a quadrilateral embedding of $K_{4k,n} \backslash \{ \overline{u_n v_{4k-1}}, \overline{u_n v_{4k}} \}$. Now, we use the auxiliary pseudosurface technique again. Attach the regions

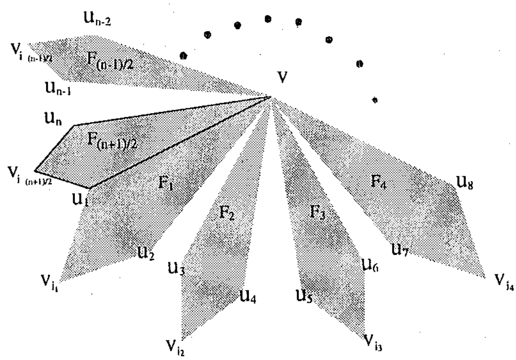

Figure 6:

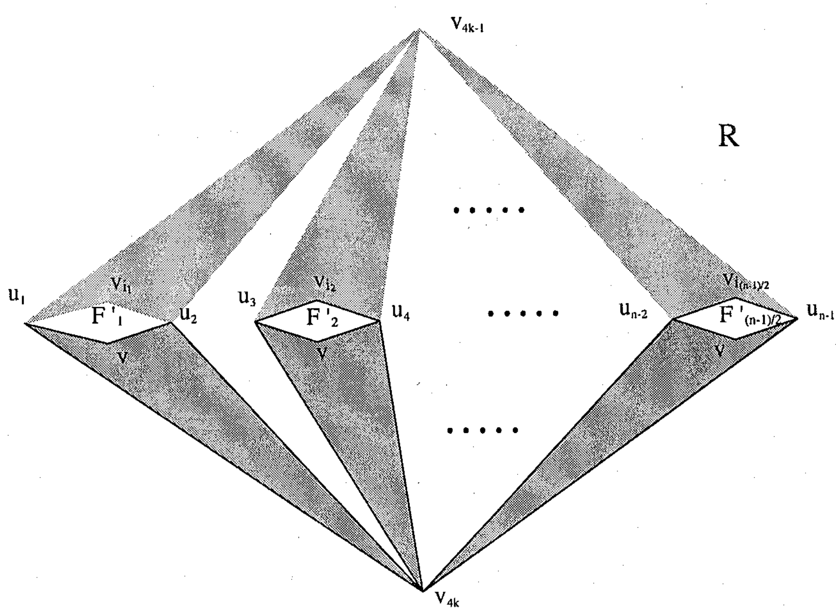

Figure 7: Attaching this first pseudosurface to the original surface.

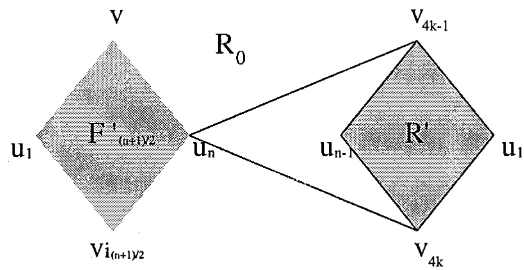

Figure 8: Attaching the 2nd pseudosurface to the above created surface.

$F'_{\frac{n-1}{2}}$ and R' in the pseudosurface depicted in Figure 8 to the regions $F_{\frac{n-1}{2}}$ and R in the above created surface. Therefore, we make sure that the new embedding in the created surface is almost a quadrilateral embedding of $K_{4k,n}$ except one region which is an 8-gon R_0. That means the number of regions in this embedding is

$$|F(K_{4k,n})| = \frac{2 \cdot 4k \cdot n - 8}{4} + 1 = 2kn - 2 + 1 = 2kn - 1.$$

By the general Euler formula, we have

$$\begin{aligned}
\gamma(K_{4k,n}) &= \frac{1}{2} \cdot (2 - (4k + n) + 4kn - (2kn - 1)) \\
&= \frac{3-n}{2} - 2k + kn = \lceil \frac{2-n}{2} - 2k + kn \rceil \\
&= \lceil \frac{(4k-2)(n-2)}{4} \rceil.
\end{aligned}$$

Hence, by induction, we have the proof of this case.

(3). $m \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$.

Let $m = 4k + 1$ and $n = 4h + 3$. We start with the quadrilateral embedding of $K_{4k+1,4h+2}$ = (A,B) obtained in (2). Let $a \in B$ and consider an open neighborhood of a as depicted in Figure 9 (i). Then replace a with two vertices a and b as depicted in Figure 9 (ii). Furthermore, create k handles in sphere which are obtained by Figure

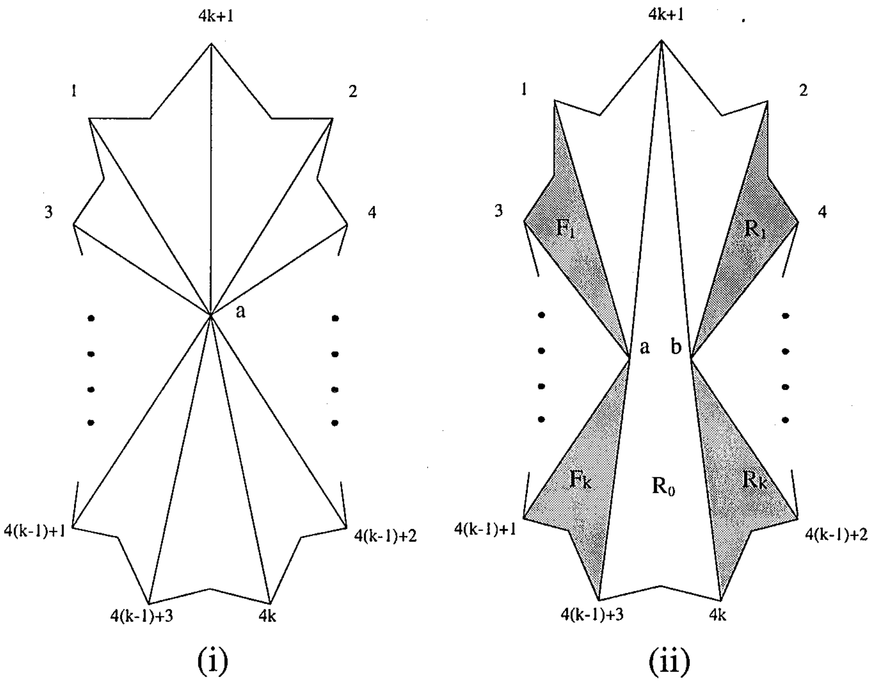

Figure 9:

10 where $i \in \{1, 2, \cdots, k\}$. Attach the shaded regions F'_i, R'_i of each sphere embeddings to the regions F_i and R_i respectively. Then, this is almost a quadrilateral embedding of $K_{4k+1,4h+3}$ except one region R_0 which is a 6-gon.

(4). $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$.

By the checking similar to (2), the embedding with minimum orientable genus must be with all regions except one 10-gon are 4-cycles. Let $m = 4k+1$ and $n = 4h+1$. By (3), we have an embedding of $K_{4k+1,4(h-1)+3}$ =(A,B) which is almost a quadrilateral embedding except one 6-gon. Similar to the embedding in (2), we have the desired embedding by using Figures 11, 12 and 13. As a matter of fact, we

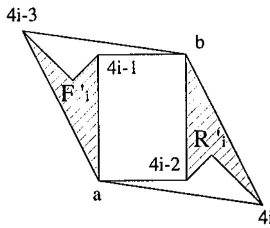

Figure 10:

have an embedding with quadrilateral faces except one 10-gon R_0.

Finally, since $m, n \equiv 3 \pmod 4$ case can be obtained in a similar way as in (4), we conclude the proof.

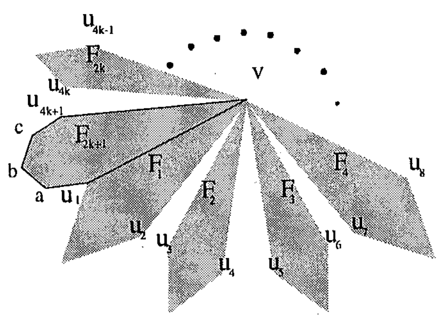

Figure 11:

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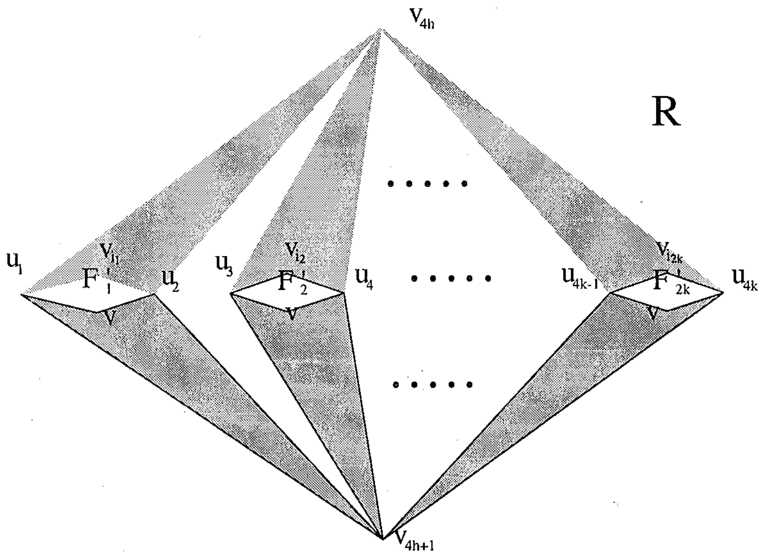

Figure 12:

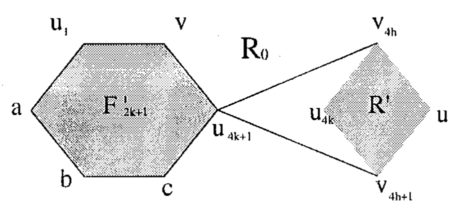

Figure 13:

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