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Note

C_4-saturated bipartite graphs

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Abstract

Let H be a graph. A graph G is said to be H-free if it contains no subgraph isomorphic to H. A graph G is said to be an H-saturated subgraph of a graph K if G is an H-free subgraph of K with the property that for any edge $e \in E(K) \setminus E(G)$, $G \cup \{e\}$ is not H-free. We present some general results on $K_{s,t}$-saturated subgraphs of the complete bipartite graph $K_{m,n}$ and study the problem of finding, for all possible values of q, a C_4-saturated subgraph of $K_{m,n}$ having precisely q edges.

1. Introduction

In 1951, Zarankiewicz posed the problem of determining the largest subgraph of the complete bipartite graph $K_{m,n}$ which does not contain a subgraph isomorphic to $K_{s,t}$ [11]. The problem is now commonly known as *The Zarankiewicz Problem* and the size (throughout the paper, the size of a graph refers to the number of edges it contains) of the largest such subgraph is denoted by $z(m,n;s,t)$. To avoid trivial cases, it is assumed that $2 \leqslant s \leqslant m$ and $2 \leqslant t \leqslant n$ and for brevity, $z(n,n;t,t)$ is denoted by $z(n;t)$. The literature contains several results on the Zarankiewicz problem, for example see [2,4,6,7,8,9].

While the Zarankiewicz problem is concerned only with bipartite graphs of maximum size that do not contain a given complete bipartite subgraph, there exist many smaller bipartite graphs having the property that the addition of any further edge creates the forbidden complete bipartite subgraph. Such graphs are the main subject of this paper and we make the following definitions.

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An *H-subgraph* of a graph is a subgraph that is isomorphic to a given graph H. A graph is *H-free* if it contains no H-subgraphs. An H-free subgraph G of a graph K is *H-saturated* if it has the additional property that the for any edge $e \in E(K) \setminus E(G)$, $G \cup \{e\}$ is not H-free.

A lot of work has been done on H-saturated subgraphs of the complete graph (see [1,3,5,10]). However, other than results on the Zarankiewicz problem, H-saturated subgraphs of non-complete graphs have not been studied. In this paper, we consider $K_{r,s}$-saturated subgraphs of $K_{m,n}$.

In Section 2, we describe a relationship between certain $K_{s,t}$-free subgraphs of $K_{m,n}$ and partial t-(v,K,λ) designs. This relationship is used to prove several results on C_4-saturated ($K_{2,2}$-saturated) subgraphs of $K_{m,n}$. We note that if G is a C_4-saturated subgraph of $K_{m,n}$ with bipartition $A \cup B$, then G has the nice property that its girth is at least 6 and any pair a,b of non-adjacent vertices with $a \in A$ and $b \in B$ are joined by a path of length 3. In Section 3 we focus on the problem of determining the set $S(m,n)$ of integers q for which there exists a C_4-saturated subgraph of $K_{m,n}$ having size q. Note that $z(m,n;2,2) = \operatorname{Max} S(m,n)$.

The following bounds that have been obtained for $z(n;2)$ and show that $\max S(n,n)$ is about $n^{3/2}$.

Theorem 1.1 (Bollobas [4]). *For $n = p^2 + p + 1$ where p is a prime power*, $z(n;2) = (p+1)n$.

Theorem 1.2 (Bollobas [4]). *If n is sufficiently large then*

$$n^{3/2} - n^{4/3} < z(n;2) \leqslant \tfrac{1}{2}(n + n\sqrt{4n-3}).$$

In particular, $\lim_{n\to\infty} z(n;2)n^{-3/2} = 1$.

2. General results

We now describe the connection between certain $K_{s,t}$-free subgraphs of $K_{m,n}$ and partial $t-(v,K,\lambda)$ designs. A (partial) $t-(v,K,\lambda)$ design is a pair (V,B) where V is a v-set and B is a collection of subsets, with sizes belonging to K, of V with the property that every t-set of V is in (at most) exactly λ subsets of B.

Given any partial $t-(v,K,\lambda)$ design (V,B), the *variety-block graph* of the design is the bipartite graph $G_{V,B}$ with vertex partition V,B defined by joining $v \in V$ to $b \in B$ if and only if $v \in b$. Clearly, if $G_{V,B}$ contains a $K_{t,\lambda+1}$-subgraph, with the part of size t in V and the part of size $\lambda+1$ in B, then there will be a t-set which occurs in $\lambda+1$ blocks. Conversely, given a $K_{t,\lambda+1}$-free bipartite graph $G_{V,B}$, with vertex partition V,B, we can define a partial $t-(v,K,\lambda)$ design (V,B) by letting $v \in b$ if and only if $v \in V$ is joined to $b \in B$ in $G_{V,B}$.

Here, we are mostly interested in C_4-free subgraphs of $K_{m,n}$ and hence in partial $2-(v,K,1)$ designs. In order to study C_4-saturated subgraphs, we now introduce the notion of a *non-extendable* partial $2-(v,K,1)$ design.

A partial $2-(v,K,1)$ design (V,B') is said to be an *extension* of a partial $2-(v,K,1)$ design (V,B) if $B\neq B'$ and for each $b\in B$ there exists a $b'\in B'$ with $b\subseteq b'$. A partial $2-(v,K,1)$ design is said to be *non-extendable* if it has no extension. We have the following lemma.

Lemma 2.1. *The variety-block graph $G_{V,B}$ of a partial $2-(v,K,1)$ design (V,B) is a C_4-saturated subgraph of $K_{|V|,|B|}$ if and only if (V,B) is non-extendable.*

Proof. If (V,B) is extendable, then let (V,B') be an extension and let $x\in b'\backslash b$ where $b\subseteq b'$, $b\in B$ and $b'\in B'$. Then $G_{V,B}\cup\{x,b\}$ is C_4-free, since (V,B') is a partial $2-(v,K,1)$ design; so $G_{V,B}$ is not C_4-saturated.

If $G_{V,B}$ is not C_4-saturated, then let $\{x,b\}$ be an edge such that $G_{V,B}\cup\{x,b\}$ is C_4-free. Then (V,B'), where B' is the set of blocks obtained from B by replacing b with $b'=b\cup\{x\}$, is an extension of $G_{V,B}$. $\square$

We first consider C_4-saturated subgraphs of smallest possible size. The following two lemmas show that for all $m,n\geqslant 2$, $m+n-1$ is the smallest element of $S(m,n)$.

Lemma 2.2. *A C_4-saturated subgraph of $K_{m,n}$ contains at least $m+n-1$ edges.*

Proof. If G is a spanning subgraph of $K_{m,n}$ with fewer than $m+n-1$ edges then G is disconnected. Hence, there is an edge of $K_{m,n}$ which is not in G and which joins two components of G. Clearly, the addition of such an edge to G does not create a C_4 and so G is not C_4-saturated. $\square$

Lemma 2.3. *For all $m,n\geqslant 2$, there exists a C_4-saturated subgraph of $K_{m,n}$ with $m+n-1$ edges.*

Proof. Let V be a set of size m with $m\geqslant 2$ and let $x\in V$. Also, let B be the collection of n subsets of V given by $B=\{V,\{x\},\{x\},\dots,\{x\}\}$. It is easy to check that (V,B) is a non-extendable $2-(m,\{1,m\},1)$ design and so by Lemma 2.1, its variety-block graph is a C_4-saturated subgraph of $K_{m,n}$ with $m+n-1$ edges. $\square$

We now consider C_4-saturated subgraphs of $K_{m,n}$ with the maximum possible number of edges. We begin by proving the following lemma. It shows that in the case $K=\{k,k+1\}$, a partial $2-(m,K,1)$ design is non-extendable whenever the number of pairs which do not occur in blocks is less than k.

Lemma 2.4. *Suppose (V,B) is a partial $2-(v,K,1)$ design with $B=\{b_1,b_2,\dots,b_n\}$. If $m=v$, $K=\{k,k+1\}$ for some positive integer k, and $\binom{v}{2}-\sum_{i=1}^{n}\binom{|b_i|}{2}<k$ then the variety-block graph of (V,B) is a C_4-saturated subgraph of $K_{m,n}$ having maximum size.*

Proof. By Lemma 2.1, it suffices to show that there is no partial $2-(v,K',1)$ design (V,A) with $A=\{a_1,a_2,\dots,a_n\}$ and $\sum_{i=1}^{n}|a_i|>\sum_{i=1}^{n}|b_i|$ (if no such design exists then clearly (V,B) is non-extendable). Suppose otherwise and for $i=1,2,\ldots,n$ let $|a_i|=y_i$, $|b_i|=x_i$, $d_i=y_i-x_i$ and $d_i^*=\binom{y_i}{2}-\binom{x_i}{2}$. Then $\sum_{i=1}^n d_i=\sum_{i=1}^n (y_i-x_i)=\sum_{i=1}^n y_i-\sum_{i=1}^n x_i>0$ and

$$d_i^* = \binom{y_i}{2} - \binom{x_i}{2} = \frac{y_i(y_i-1)}{2} - \frac{x_i(x_i-1)}{2} = \frac{y_i^2-x_i^2+x_i-y_i}{2}$$

$$= \frac{(y_i-x_i)(x_i+y_i-1)}{2} = (y_i-x_i)x_i + \frac{(y_i-x_i)(y_i-x_i-1)}{2}$$

$$\geqslant d_i k \quad \left(\text{since } x_i \geqslant k \text{ and } \frac{(y_i-x_i)(y_i-x_i-1)}{2} \geqslant 0\right).$$

Now,

$$\binom{v}{2} - \sum_{i=1}^{n}\binom{|B_i|}{2} = \binom{v}{2} - \sum_{i=1}^{n}\left(\binom{|y_i|}{2} - d_i^*\right)$$

$$= \binom{v}{2} - \sum_{i=1}^{n}\binom{|y_i|}{2} + \sum_{i=1}^{n} d_i^*$$

$$\geqslant \binom{v}{2} - \sum_{i=1}^{n}\binom{|y_i|}{2} + \sum_{i=1}^{n} d_i k$$

$$\geqslant \sum_{i=1}^{n} d_i k \quad \left(\text{since } \binom{v}{2} \geqslant \sum_{i=1}^{n}\binom{|y_i|}{2}\right)$$

$$\geqslant k \quad \left(\text{since } \sum_{i=1}^{n} d_i > 0\right).$$

This is a contradiction and we have the proof. □

The following corollary is immediate.

Corollary 2.1. *Let $G_{V,B}$ be the variety-block graph of a $2-(m,\{k,k+1\},1)$ design (v,B) with $|B|=n$. Then $G_{V,B}$ is a C_4-saturated subgraph of $K_{m,n}$ with maximum size.*

We note that Theorem 1.1 follows immediately from Corollary 2.1 since there exists a $2-(p^2+p+1,\{p+1\},1)$ design whenever p is a prime power. Also, it follows from Corollary 2.1 that the maximum size of a C_4-saturated subgraph of $K_{n,n}$ is no greater than that achieved if there exists a symmetric $2-(n,k,\lambda)$ design (a symmetric design has an equal number of points and blocks). Hence, it follows that $z(n;2)\leqslant nk$ where $k=\frac{1}{2}(1+\sqrt{4n-3})$ is size of the blocks in a symmetric $2-(v,k,\lambda)$ design, and so we have the upper bound in Theorem 1.2.

3. Possible sizes of C_4-saturated subgraphs of $K_{m,n}$

Let $S(m,n)$ denote the set of integers q for which there exists a C_4-saturated subgraph of $K_{m,n}$ having size q. In this section we illustrate, by considering the example $K_{9,12}$, how the ideas developed in the preceding section may be used to determine $S(m,n)$.

Lemma 3.1. *There exists a C_4-saturated subgraph of $K_{9,12}$ of size q if and only if $q \in \{20, 21, \ldots, 36\}$.*

Proof. Clearly, the variety-block graph of the $2-(9,3,1)$ design yields the largest possible C_4-saturated subgraph of $K_{9,12}$. This graph has 36 edges. Also, by Lemmas 2.2 and 2.3 the smallest possible C_4-saturated subgraph of $K_{9,12}$ has 20 edges. Hence, we know that $S(9,12) \subseteq \{20, 21, \ldots, 36\}$ and it remains to construct a C_4-saturated subgraph of $K_{9,12}$ of size q for $q = 21, 22, \ldots, 35$. By Lemma 2.1 it is sufficient to construct a non-extendable $2-(9,K,1)$ design (V,B) with $|B| = 12$ and $\sum_{b \in B} |b| = q$ for $q = 21, 22, \ldots, 35$. Such designs are given below,

21: $\{1,2,\ldots,8\}, \{1,9\}, \{2,9\}, \{1\}, \ldots, \{1\}$,

22: $\{1,2,\ldots,8\}, \{1,9\}, \{2,9\}, \{3,9\}, \{1\}, \ldots, \{1\}$,

23: $\{1,2,\ldots,8\}, \{1,9\}, \{2,9\}, \{3,9\}, \{4,9\}, \{1\}, \ldots, \{1\}$,

24: $\{1,2,\ldots,8\}, \{1,9\}, \{2,9\}, \ldots, \{5,9\}, \{1\}, \ldots, \{1\}$,

25: $\{1,2,\ldots,8\}, \{1,9\}, \{2,9\}, \ldots, \{6,9\}, \{1\}, \ldots, \{1\}$,

26: $\{1,2,\ldots,8\}, \{1,9\}, \{2,9\}, \ldots, \{7,9\}, \{1\}, \ldots, \{1\}$,

27: $\{1,2,\ldots,8\}, \{1,9\}, \{2,9\}, \ldots, \{8,9\}, \{1\}, \{1\}, \{1\}$,

28: $\{1,2,3,\ldots,7\}, \{1,8,9\}, \{2,9\}, \{3,9\}, \ldots, \{7,9\}, \{2,8\}, \{3,8\}, \{1\}, \{1\}$,

29: $\{1,2,3,\ldots,7\}, \{1,8,9\}, \{2,9\}, \{3,9\}, \ldots, \{7,9\}, \{2,8\}, \{3,8\}, \{4,8\}, \{1\}$,

30: $\{1,2,3,\ldots,7\}, \{1,8,9\}, \{2,9\}, \{3,9\}, \ldots, \{7,9\}, \{2,8\}, \{3,8\}, \{4,8\}, \{5,8\}$,

31: $\{1,2,3,\ldots,6\}, \{1,7,8\}, \{1,9\}, \{9,7,2\}, \{9,8,3\}, \{7,3\}, \{7,4\}, \{7,5\}$,
$\{7,6\}, \{8,2\}, \{8,4\}, \{8,5\}$,

32: $\{1,2,3,4,5\}, \{1,6,7\}, \{1,8,9\}, \{6,8,2\}, \{6,9,3\}, \{7,8,4\}, \{7,9,5\}$,
$\{6,4\}, \{6,5\}, \{7,2\}, \{7,3\}, \{1\}$,

33: $\{1,2,3,4\}, \{5,6,7\}, \{8,9\}, \{8,5,1\}, \{8,6,2\}, \{8,7,3\}, \{9,5,2\}$,
$\{9,6,3\}, \{9,7,1\}, \{4,5\}, \{4,6\}, \{4,7\}$,

34: $\{1,2,3,4\}, \{1,5,6\}, \{7,8,9\}, \{5,7,1\}, \{5,8,2\}, \{5,9,3\}, \{6,7,2\}$,
$\{6,8,3\}, \{6,9,1\}, \{4,7\}, \{4,8\}, \{4,9\}$,

35: $\{1,2,3,4\}, \{1,5,6,7\}, \{1,8,9\}, \{8,5,2\}, \{8,6,3\}, \{8,7,4\}, \{9,5,3\}, \{9,6,4\}$,
$\{9,7,2\}, \{5,4\}, \{6,2\}, \{7,3\}$. □

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