A Note on the Block Intersection Problem of Steiner Quadruple Systems of Order u · v

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1. Introduction.

A Steiner quadruple system of order v (SQS(v)) is a pair (Q, q) where Q is a v-set and q is a collection of 4-element subsets of Q, called blocks, such that every 3-element subset of Q is contained in exactly one block of q. H. Hanani [9] proved that an SQS(v) exists if and only if $v \equiv 2$ or 4 (mod 6). It is easy to see that $|q| = v(v-1)(v-2)/24$, which is defined as q_v.

Define J[v] the set of all positive integers k such that there exists a pair of Steiner quadruple systems of order v which have exactly k blocks in common, and set $I[v] = \{0, 1, 2, \cdots, q_v-14\} \cup \{q_v-12, q_v-8, q_v\}$.

In [3], M. Gionfriddo and C.C. Lindner conjectured that J[v]=I[v] for every $v \equiv 2$ or 4 (mod 6) and $v \geqslant 8$. Since then, a considerable amount of work has been done in an attempt to prove that J[v]=I[v]. [1, 2, 4, 5, 6, 7, 8, 10]. In this paper we generalize the result in [2], prove that J[uv]=I[uv] for every u and $v \equiv 2$ or 4 (mod 6) with $\max\{u, v\} \geqslant 10$.

2. The basic constructions.

A 1-factor of the finite set V is a set of 2-element subsets of V which partition V. (A 2-element subset of V is called an edge.) A collection F of 1-factors, which partition all the edges of V, is called a 1-factorization of V. We now give the well-known doubling construction for SQSs.

Let (X, q_1) and (Y, q_2) be any two SQS(v) with $X \cap Y = \phi$. Let $F = \{F_1, F_2, \cdots, F_{v-1}\}$ and $G = \{G_1, G_2, \cdots, G_{v-1}\}$ be any two 1-factorizations of X and Y respectively, and let α be any permutation on the set $\{1, 2, \cdots, v-1\}$. Define a collection q of blocks as follows:

(1) Any block belonging to q_1 or q_2 belongs to q; and

(2) If $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $\{x_1, x_2, y_1, y_2\} \in q$ if and only if $\{x_1, x_2\} \in F_i$, $\{y_1, y_2\} \in G_j$, and $i\alpha = j$.

It is a routine matter to see that $(X \cup Y, q)$ is an SQS(2v). We will refer to the blocks obtained from (1), (2) as the blocks of first and second type respectively.

The doubling construction has been one of the principal tool in obtaining the results. In [2], another construction is used to count the intersections of two SQS(4v), which is 2v to 4v construction.

A latin cube of order v is a v-tuple $(L_1, L_2, \cdots, L_v)$ of pairwise disjoint latin squares of order v. Let $Q_i = \{(x, i): x \in S, |S| = v\}$, i = 1, 2, 3, and 4, be four disjoint v-sets, and $(Q_1 \cup Q_2, p_1)$, $(Q_3 \cup Q_4, p_2)$ be two SQS(2v). Also let F_1, F_2, F_3, and F_4 be 1-factorizations of Q_1, Q_2, Q_3, and Q_4 respectively. Define a collection of blocks q on $Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ as follows:

(1) The blocks belonging to p_1 or p_2 belong to q;

(2) The blocks of second type from the doubling construction of Q_1 and Q_3, Q_1 and Q_4, Q_2 and Q_3, and Q_2 and Q_4 belong to q; and

(3) If C is a latin cube of order v, $C = (L_1, L_2, \cdots, L_v)$, then $\{(x, 1), (y, 2), (z, 3), (w, 4)\}$ is a block in q if and only if the (x, y) entry in L_z is w. (We will refer to these blocks as the third type of blocks.)

It is not difficult to see that (Q, q) is an SQS(4v) with two disjoint subsystems of order 2v.

In order to compute the intersections of two SQS(uv), we need the following 2v to uv construction.

Let $Q_i = \{(x, i): x \in S, |S| = v\}$, $i = 1, 2, \cdots, u = 2m$, be u disjoint v-sets, $u \equiv 2$ or 4 (mod 6), and $(Q_1 \cup Q_2, p_1)$, $(Q_3 \cup Q_4, p_2)$, $\cdots$, $(Q_{u-3} \cup Q_{u-2}, p_{m-1})$, $(Q_{u-1} \cup Q_u, p_m)$ be m SQS(2v). Also let $F_1, F_2, \cdots, F_u$ be 1-factorizations of $Q_1, Q_2, \cdots, Q_u$ respectively. Define a collection of block q on $Q = \bigcup_{i=1}^{u} Q_i$ as follows:

(1) The blocks belonging to $p_1, p_2, \cdots, p_m$ belong to q;

(2) The blocks of second type from the doubling construction of Q_i and Q_j, $1 \leqslant i, j \leqslant u$, except i is odd and j = i+1; and

(3) Let (P, p) be an SQS(u), and $p = \{b_1, b_2, \cdots, b_{q_u}\}$. Also let C_t be a latin cube of order v corresponding to the block b_t, $t = 1, 2, \cdots, q_u$. Then $\{(x, a), (y, b), (z, c), (w, d)\}$ $(a < b < c < d)$ is a block in q if $b_t = \{a, b, c, d\} \in p$, and in C_t, the (x, y) entry in L_z is w.

It is a routine matter to check (Q, q) is an SQS(uv).

3. The main theorem.

Before we go any further we need the following definitions. Let $C = (L_1, L_2, \cdots, L_v)$ and $D = (M_1, M_2, \cdots, M_v)$ be two latin cubes of order v, then the *intersection* of C and D is defined to be the number

$$|C \cap D| = \sum_{i=1}^{v} |L_i \cap M_i|;$$

where $|L_i \cap M_i|$ is the number of common entries of L_i and M_i. Moreover, we define K[v] as the set of all positive integers k such that there exist two latin cubes of order v with intersection k, and $T[v] = \{0, 1, 2, \cdots, v^3-14\} \cup \{v^3-12, v^3-8, v^3\}$.

In what follows, we will use A+B to denote the set of all numbers of the form a+b, where a∈A and b∈B, and kA to denote the sum of k copies of A.

If we have two SQS(uv) obtained from the construction above, it is easy to see that the common blocks of these two SQS(uv) are the common blocks in type (1), (2), or (3). Hence we count the common blocks of these three types.

LEMMA 3.1. $J[2v] \supseteq \{0, q_{2v}-25, q_{2v}-21, q_{2v}-15, q_{2v}-14, q_{2v}\}$. [2, 3, 6, 8]

In [1], it has been shown that there exist two SQS(2v) which have exactly k blocks in common of the second type for every

$$k \in \{0, \tfrac{v}{2}, 2\cdot\tfrac{v}{2}, \cdots, (t_v-6)\tfrac{v}{2}, (t_v-4)\tfrac{v}{2}, t_v\tfrac{v}{2}\},$$

where $t_v = \frac{v(v-1)}{2}$. Hence we have the following lemma.

LEMMA 3.2. There exist two SQS(uv) which have exactly k blocks in common for each

$$k \in \frac{u\cdot(u-2)}{2}\cdot \{0, \tfrac{v}{2}, 2\cdot\tfrac{v}{2}, \cdots, (t_v-6)\tfrac{v}{2}, (t_v-4)\tfrac{v}{2}, t_v\cdot\tfrac{v}{2}\},$$

where $t_v = \frac{v(v-1)}{2}$, and u, v are admissible order of SQS.

PROOF. Since there are $\frac{u(u-1)}{2} - \frac{u}{2}$ copies of blocks of second type from doubling construction, hence we have the result.

For the blocks of the third type, it is not difficult to see, two blocks $\{(x, a), (y, b), (z, c), (w, d)\}$ $(a < b < c < d) \in q$, $\{(x', a'), (y', b'), (z', c'), (w', d')\} \in$

q' $(a' < b' < c' < d')$ are in common, if and only if $\{a, b, c, d\} = \{a', b', c', d'\}$ (a block in SQS(u)) W.L.O.G. $(a, b, c, d) = (a', b', c', d')$, and $(x, y, z, w) = (x', y', z', w')$, i.e., the (x, y)-entry in L_z of the corresponding latin cubes are the same. Hence we have the following lemma.

LEMMA 3.3. $J[uv] \supseteq q_u \cdot K[v]$, u, v, are admissible orders of SQS.

LEMMA 3.4. $K[v] \supseteq T[v] \setminus \{v^3-21, v^3-14\}$ for every even $v \geqslant 20$. [2]

THEOREM 3.5. $J[uv] = I[uv]$ for every u and $v \equiv 2$ or 4 (mod 6) with max $\{u, v\} \geqslant 20$.

PROOF. Without loss of generality, we let $v \geqslant 20$. Let (Q, q), and (Q, q′) be two SQS(uv) obtained from 2v to uv construction. Moreover, we use the same SQS(u) in the construction, and also if in (Q, q) construction latin cube corresponding to the block $\{a, b, c, d\}$ is defined as the (x, y) entry in L_z is w if and only if $\{(x, a), (y, b), (z, c), (w, d)\} \in q$, then the latin cube in (Q, q′) construction shall have the same property, i.e., the (x', y') entry in $L_{z'}$ is w' if and only if $\{(x', a), (y', b), (z', c), (w', d)\} \in q'$.

It is clear now $J[uv] \supseteq \frac{u}{2}\{0, q_{2v}-25, q_{2v}-21, q_{2v}-15, q_{2v}-14, q_{2v}\} + \frac{u(u-2)}{2} \cdot \{0, \frac{v}{2}, 2\cdot\frac{v}{2}, \cdots, (t_v-6)\frac{v}{2}, (t_v-4)\frac{v}{2}, t_v\frac{v}{2}\} + q_u \cdot (T[v] \setminus \{v^3-21, v^3-14\})$ which is the set $I[uv]$. (We omit the detailed counting here.) By $J[uv] \subseteq I[uv]$ [3], we conclude the proof for $v \geqslant 20$.

It has been shown in [3, 4, 8], $J[2v] \supseteq I[2v] \setminus \{q_{2v}-17\}$, $v = 10, 14, 16$. Since $v^3-17 \in K[v]$ for every $v \geqslant 10$, [2] and $K[v] \supseteq \{0, v^2, 2v^2, \cdots, (v-2)v^2, v^3\}$, we have the main theorem.

THEOREM 3.6. $J[uv] = I[uv]$ for every u and $v \equiv 2$ or 4 (mod 6) with max $\{u, v\} \geqslant 10$.

PROOF. For $v = 10$, 14, and 16,

$$J[uv] \supseteq \frac{u}{2} \cdot (I[2v] \setminus \{q_{2v}-17\})$$
$$+ u(u-2)/2 \cdot \{0, \frac{v}{2}, \cdots, (t_v-6)\frac{v}{2}, (t_v-4)\frac{v}{2}, t_v(\frac{v}{2})\}$$
$$+ q_u\{0, v^2, 2v^2, \cdots, (v-2)v^2, v^3-17, v^3\} = I[uv],$$

this completes the proof with $J[uv] \subseteq I[uv]$ and Theorem 3.5.

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