



# Ascending subgraph decompositions of regular graphs

Hung-Lin Fu<sup>\*,1</sup>, Wei-Hsin Hu

*Department of Applied Mathematics, National Chaio Tung University, 1001 TA Hsueh, Hsinchu 30050, Taiwan*

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## Abstract

We prove that every regular graph with  $\binom{n+1}{2} + t$  edges,  $0 \leq t < n + 1$ , can be decomposed into  $n$  subgraphs  $G_1, G_2, \dots, G_n$  such that  $|E(G_i)| = i$  and  $G_i \leq G_{i+1}$  for  $i = 1, 2, \dots, n - 1$  and  $|E(G_n)| = n + t$ . © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The following conjecture about decomposing a graph  $G$  with  $\binom{n+1}{2} \leq |E(G)| < \binom{n+2}{2}$  into  $n$  ascending subgraphs has been one of the most fascinating problems regularly mentioned by P. Erdős in his talks on “Unsolved Problems” over the years before his decease. One of the reasons that this decomposition problem is interesting can be seen from the decomposition of a star forest into stars. This special occasion, in fact, corresponds to partition the set  $\{1, 2, \dots, n\}$  into subsets with prescribed sums, see [7,8] for details.

Ascending Subgraph Decomposition Conjecture (ASD Conjecture (Alavi et al. [1])). Let  $G$  be a graph of size  $\binom{n+1}{2} \leq |E(G)| < \binom{n+2}{2}$ . Then  $E(G)$  can be partitioned into  $n$  sets  $E_1, E_2, \dots, E_n$  which induce subgraphs  $G_1, G_2, \dots, G_n$  such that  $|E(G_i)| < |E(G_{i+1})|$  and  $G_i$  is isomorphic to a subgraph of  $G_{i+1}$  (denoted by  $G_i \leq G_{i+1}$ ) for  $i = 1, 2, \dots, n - 1$ .

\* Corresponding author.

*E-mail address:* hlfu@math.nctu.edu.tw (H.-L. Fu).

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A graph  $G$  is said to have an ascending subgraph decomposition  $G_1, G_2, G_3, \dots, G_n$  provided that the ASD conjecture holds for  $G$ .  $G_1, G_2, \dots, G_n$  are the members of the decomposition. In order to verify this conjecture, the following idea was utilized to obtain almost all the known results.

Let  $G$  be a graph with  $\binom{n+1}{2} + t$  edges, where  $0 \leq t \leq n$ . Then we decompose  $G$  into ascending subgraphs  $G_1, G_2, \dots, G_{n-1}, G_n \cup T$  such that  $|E(G_i)| = i$  for  $i = 1, 2, \dots, n$ ,  $|E(T)| = t$ , and  $G_i \leq G_{i+1}$  for  $i = 1, 2, 3, \dots, n-1$ . Most of the known results were obtained by induction, using edge deletion. For instance, forests and graphs with sufficiently small degree (with respect to the number of edges) all have ASDs, as shown in [1,3,6]. But, as we propose to prove that every regular graph has an ASD, the situation is quite different since the deletion of edges do not preserve regular graphs. This is the reason why we do not consider  $|E(G)| = \binom{n+1}{2}$  in this paper.

There are quite a few results in verifying the ASD Conjecture. In what follows, we mention several results which are important in the proof of our main result.

**Theorem 1.1** (Fu-Long Chen et al. [2]). *Let  $\langle m_1, m_2, \dots, m_k \rangle$  be a decreasing sequence of positive integers such that  $\sum_{i=1}^k m_i = \binom{n+1}{2}$  and  $m_{k-1} \geq n$ . Then the set  $\{1, 2, \dots, n\}$  can be partitioned into  $k$  sets  $S_1, S_2, \dots, S_k$  such that for each  $i = 1, 2, \dots, k$ ,  $\sum_{x \in S_i} x = m_i$ .*

**Theorem 1.2** (Ma et al. [8]). *Let  $G$  be a star forest with  $\binom{n+1}{2}$  edges and each component has at least  $n$  edges. Then  $G$  has an ASD with each member a star.*

Note that if we have one component with the number of edges less than  $n$  in Theorem 1.2, then we still have an ASD with each member a star (by Theorem 1.1).

**Corollary 1.3.** *Let  $G$  be a graph with  $\binom{n+1}{2}$  edges and  $E(G)$  be a disjoint union of matchings  $M_1, M_2, \dots, M_k$  such that all the matchings except possibly one of them have at least  $n$  edges. Then  $G$  has an ASD with each member a matching.*

Other than the above results, the following theorems are worthy of mention.

**Theorem 1.4** (Fu [7]). *Let  $G$  be a graph with  $\binom{n+1}{2}$  edges. Then if  $\Delta(G) \leq (n-1)/2$ , then  $G$  has an ASD with each member a matching.*

**Theorem 1.5** (Chen and Ma [3]). *Let  $G$  be an  $r$ -regular graph with  $\binom{n+1}{2}$  edges. If  $r \leq 2n/3$ , then  $G$  has an ASD.*

In this paper, we complete this result and show that every regular graph has an ASD.

## 2. The main result

Let  $G$  be a regular graph with valency  $r$  and  $|E(G)| = \binom{n+1}{2} + t$  for some  $n \geq 1$  and  $0 \leq t \leq n$ . In what follows, we shall decompose  $G$  into ascending subgraphs

$G_1, G_2, \dots, G_n$  such that each member is a linear forest, this is, a forest with each component a path.

As mentioned in [6], if the maximum degree  $\Delta(G)$  of a graph  $G$  is not too big, then  $G$  can be decomposed into ascending subgraphs such that each member is a matching. Clearly, if  $\Delta(G)$  gets bigger compared to  $n$ , then we may not be able to do so. Thus, in [4], Faudree et al. started to use linear forests for their decompositions. Their linear forests consist of paths of length at most 2. Not before long, Chen and Ma used paths of length one and three to tackle a special class of regular graphs, see [2]. Now, we go a step further to prove that ASD Conjecture holds for all regular graphs.

The idea of the decomposition comes from rearranging matchings obtained by an equalized edge-coloring. First, we use Vizing's Theorem and then the work of de Werra to decompose  $G$  into  $r + 1$  matchings  $M_1, M_2, M_3, \dots, M_{r+1}$  such that their sizes differ by at most 1.

**Lemma 2.1.** *Let  $M_i$  be a matching of size  $m_i$  in  $G$  for  $i = 1, 2, \dots, k$  where  $\langle m_1, m_2, \dots, m_{k-1}, m_k \rangle$  is a decreasing sequence of positive integers such that  $\sum_{i=1}^k m_i = \binom{m+1}{2} + t', t' \geq 0$  and  $m_{k-1} \geq m$ . Then the edge induced subgraph  $\langle \bigcup_{i=1}^k M_i \rangle$  of  $G$  can be decomposed into ascending subgraphs  $G_1, G_2, \dots, G_m$  and  $T$  such that  $G_i$  is a matching of size  $i$  for  $i = 1, 2, \dots, m$ , and  $T$  is a subgraph of  $\langle \bigcup_{i=1}^k M_i \rangle$  with  $t'$  edges.*

**Proof.** Directly from Corollary 1.3.  $\square$

**Lemma 2.2.** *Let  $M$  and  $N$  be two disjoint matchings in  $G$  such that  $V(N) \subseteq V(M)$ . Then for any  $t \leq \lceil \frac{1}{3}|N| \rceil$ , there exists a linear forest whose paths are of length one or three obtained by adding  $t$  edges of  $N$  to  $M$ .*

**Proof.** We start with the matching  $M$ , then add the edges of  $N$  to  $M$  one by one to make sure the new graph obtained contains no cycles or paths of length larger than three. Clearly, this can be done if we add at most  $\lceil \frac{1}{3}|E(N)| \rceil$  edges of  $N$  to  $M$  to obtain  $M'$ . By letting  $N' = N \setminus M'$ , we have the proof.  $\square$

**Lemma 2.3.** *Let  $M$  and  $N$  be two disjoint matchings in  $G$  such that  $V(N) \subseteq V(M)$ . Let  $H$  be a subset of  $N$  with maximum size  $h$  such that  $M \cup H$  is a disjoint union of paths of length one or three. Then, for any integers  $i, j$ ,  $0 \leq i \leq k = \lceil |N|/2 \rceil - h$  and  $0 \leq j \leq h - k$ , there exists a subset  $N'$  of  $N$  such that  $M \cup N'$  is a disjoint union of  $i$  paths of length five,  $j$  paths of length three and the others are independent edges.*

**Proof.** First, we notice that if  $h > \lceil |N|/2 \rceil$ , then we discard the existence of paths of length five. Observe that  $M \cup N$  is a disjoint union of even cycles and paths of odd length. Now  $M \cup H$  is a disjoint union of paths of length one or three and the addition of any edge from  $N \setminus H$  will create an even cycle or a path of length five. ( $H$  is a set of maximum size which has this constraint.) This implies that in  $M \cup H$ , there are exactly  $|H| = h$  paths of length three. First, if  $M \cup N$  contains only cycles of length a multiple of 4, then  $|H| = |N|/2$  and thus no paths of length five exist and there are exactly  $h$

paths of length three. Clearly,  $N'$  can be chosen as a subset of  $H$  to obtain  $0 \leq j \leq h$  paths of length three. Now, we consider the case either  $M \cup N$  contains a cycle of length  $4s + 2$ ,  $s \geq 1$ , or paths of length larger than three. Let  $C = (v_1, v_2, \dots, v_{4s+1}, v_{4s+2})$  be a cycle in  $M \cup N$ ,  $s \geq 1$ . In order to obtain  $M \cup H$ , some edges of  $C$  must be deleted. It is easy to see that there are two edges  $e_1$  and  $e_2$  on  $C \setminus H$  which are incident to a common edge and the addition of  $e_1$  gives a path of length five, i.e.,  $M \cup H \cup \{e_1\}$  contains a path of length five. On the other hand, if  $M \cup N$  contains a path of length  $2t + 1$ ,  $t \geq 2$ , then the path contains exactly  $\lceil t/2 \rceil$  edges from  $N$ . This implies that we can obtain a path of length five if and only if that path is of length  $4x + 1$ ,  $x \geq 1$ . In this case,  $H$  contains exactly  $x$  edges in a path of length  $4x + 1$ . Here, exactly  $2x$  edges of this path are in  $N$ . Therefore, when we count  $k$ , it comes from the number of cycles of length  $4s + 2$  only, since in other cases the edges in  $H$  are a half of the edges in  $N$ . Hence, the number of paths of length five obtained is exactly the same as the number of paths of length three disappear in  $M \cup H$ .

Now, we are ready to choose  $N'$ . Let  $K$  be the set of edges in  $N \setminus H$  such that  $K \cup (M \cup H)$  has  $k$  paths of length five and let  $i$  be the number of edges chosen from  $K$  such that we have  $0 \leq i \leq k$  paths of length five. Also, let  $K'$  be the set of the above  $i$  edges. Then  $M \cup H \cup K'$  contains  $i$  paths of length five,  $h - i$  paths of length three and the other paths are independent edges. Let  $H'$  be the set of edges which are length five paths and  $H''$  be a set of edges in  $H \setminus H'$  such that  $|H''| = h - i - j$ ,  $0 \leq j \leq h - k$ . Then  $M \cup (H \setminus H'') \cup K'$  contains exactly  $i$  paths of length five,  $j$  paths of length three and all the others are independent edges. We conclude the proof by letting  $N' = (H \setminus H'') \cup K'$ .  $\square$

The following result is crucial in the proof of our main result.

**Lemma 2.4.** *Let  $M_1, M_2, \dots, M_k$  and  $L_1, L_2, \dots, L_k$  be matchings of size  $m$  and  $l$ , respectively, such that  $k \leq \lceil l/2 \rceil$  and  $|V(M_i) \setminus V(L_i)| \leq 2$  for  $i = 1, 2, \dots, k$ . Then for each  $i = 1, 2, \dots, k$ ,  $M_i \cup L_i$  can be decomposed into two subgraphs  $G_{m+i}$  and  $G_{m-i}$  such that  $G_{m+i}$  is a linear forest containing a matching of size  $m$  and  $G_{m-i}$  is a matching of size  $l - i$ , furthermore,  $G_{m+i} \leq G_{m+i+1}$  for  $i = 1, 2, \dots, k - 1$ .*

**Proof.** Let  $L'_i$  be a subset of  $L_i$  such that  $V(L'_i) \subseteq V(M_i)$  and  $|L'_i| = l - 2$ . Note that we shall choose the two edges in  $L_i \setminus L'_i$  which are either incident to one vertex of  $V(M_i) \setminus V(L'_i)$ , respectively, or contained in an even cycle of  $M_i \cup L_i$ , or contained in a path of length larger than one. Now, let  $H_i$  be a subset of  $L'_i$  with maximum size such that  $M_i \cup H_i$  is a linear forest with paths of length one or three. Without loss of generality, let  $g_i = |H_i|$  and  $g_1 \leq g_2 \leq \dots \leq g_k$ . First, if  $g_i \geq i$  for each  $i = 1, 2, \dots, k$ , then let  $G_{m+i}$  be the disjoint union of  $i$  paths of length three and  $(m - 2i)$  independent edges, and  $G_{m-i}$  a matching of size  $l - i$  from  $L_i \setminus G_{m+i}$  for  $i = 1, 2, \dots, k$ . Since  $G_{m+i+1}$  has one more length three path and  $G_{m+i+1}$  has  $3(i + 1) + m - 2i - 2 = m + i + 1$  edges which is one larger than the number of edges in  $G_{m+i}$ ,  $G_{m+i} \leq G_{m+i+1}$ . This concludes the proof of this case.

On the other hand, if there exists a  $j$ ,  $g_j < j \leq k$ , let  $j$  be the smallest one such that  $g_j = g_{j-1} = j - 1$ . Note that by Lemma 2.2,  $j \geq \lceil (l - 2)/3 \rceil + 1$ . Then, by Lemma 2.3,

we construct  $G_{m+i}$  and  $G_{m-i}$  as follows:

- (i) For  $i = 1, 2, \dots, k - j$ , let  $G_{m+i}$  be the union of  $i$  paths of length three and  $(m - 2i)$  independent edges obtained from  $M_{j+i} \cup L'_{j+i}$ . Put  $G_{m-i} = L_{j+i} \setminus G_{m+i}$ .
- (ii) For  $i = k - j + 1, k - j + 2, \dots, \lceil (l - 2)/3 \rceil$ , let  $G_{m+i}$  be the union of  $i$  paths of length three and  $(m - 2i)$  independent edges obtained from  $M_{j-k+i} \cup L'_{j-k+i}$ . Put  $G_{m-i} = L_{j-k+i} \setminus G_{m+i}$ .
- (iii) For  $i = \lceil (l - 2)/3 \rceil + 1, \lceil (l - 2)/3 \rceil + 2, \dots, \lceil (l - 2)/3 \rceil + (k - j)$ , let  $G_{m+i}$  be the union of  $i - \lceil (l - 2)/3 \rceil$  paths of length five,  $2\lceil (l - 2)/3 \rceil - i$  paths of length three and  $m - \lceil (l - 2)/3 \rceil - i$  independent edges from  $M_{j-k+i} \cup L'_{j-k+i}$ . Put  $G_{m-i} = L_{j-k+i} \setminus G_{m+i}$ .
- (iv) For  $i = \lceil (l - 2)/3 \rceil + (k - j + 1), \dots, k - 1$ , let  $G_{m+i}$  be the union of  $(k - j)$  paths of length five,  $i - 2(k - j)$  paths of length three and  $m + (k - j) - 2i$  independent edge from  $M_{j-k+i} \cup L'_{j-k+i}$ . Put  $G_{m-i} = L_{j-k+i} \setminus G_{m+i}$ .
- (v) Let  $G_{m+k}$  be the union of  $G'_{m+k} \cong G_{m+k-1}$  from  $M_j \cup L'_j$  and an edge from  $L_j \setminus L'_j$ . Put  $G_{m-k} = L_j \setminus G_{m+k}$ .

Now, it is a routine matter to check that  $G_{m+i} \leq G_{m+i+1}$  for  $i = 1, 2, \dots, k - 1$ . (In (i)–(v),  $G_{m+i}$  has  $m + i$  edges for  $i = 1, 2, \dots, k$ ,  $G_{m+i} \leq G_{m+i+1}$  follows.)  $\square$

**Theorem 2.5.** *Let  $G$  be an  $r$ -regular graph of size  $\binom{n+1}{2} + t$  edges where  $0 \leq t \leq n$ . Then  $G$  has an ascending subgraph decomposition.*

**Proof.** Let  $|V(G)| = v$ . There are three cases to consider. Since the first case can be proved by using the technique in [2] and also the proof is similar to the second case, we omit the proof here.

Case 1:  $r \leq 2n/3$ . See [2] for details.

Case 2:  $2n/3 < r < v/2$ .

By Vizing's Theorem and the adjustment of colors,  $G$  has an equalized  $(r + 1)$ -edge coloring which induces  $r + 1$  matchings  $M_1, M_2, \dots, M_{r+1}$  where  $m = |M_1| \geq |M_2| \geq \dots \geq |M_{r+1}| \geq m - 1$ . Now, consider two subcases.

Case 2(a):  $v$  is even. First, we claim  $m = |M_1| = |M_2| = \dots = |M_{2(n-m)+1}| = (v - 2)/2$ . By direct counting, we see that  $m \leq (v - 2)/2$ . Assume that  $m \leq (v - 4)/2$ . Then  $rv/2 \leq (v - 4)/2(r + 1)$  which implies  $v \geq 4r + 4$ . Since  $|E(G)| = \binom{n+1}{2} + t = vr/2$ ,  $(n^2 + 3n)/2 \geq vr/2 \geq r(2r + 2)$ ,  $n^2 + 3n \geq 4r^2 + 4r$ . Now, if  $r > \frac{2}{3}n$ , then  $n^2 + 3n \geq 4(2n/3)^2 + 4(2n/3)$  which is not possible. Hence, we conclude that  $m = (v - 2)/2$ . In order to prove the claim, suppose that  $|M_{2(n-m)+1}| = m - 1$ . Then  $rv/2 \leq 2(n - m)((v - 2)/2) + [(r + 1) - 2(n - m)]((v - 4)/2)$ , and we have  $v - 4r - 4 + 4n - 4m \geq 0$ . By the assumption that  $r > 2n/3$ ,  $v + 2r - 4m - 4 > 0$ . Since  $r < v/2$ ,  $2v - 4m - 4 > 0$ , i.e.,  $m < (v - 2)/2$  which is a contradiction. Thus, we have the claim.

Now, we are ready for the decomposition. Since for any two matchings  $M_{i_1}$  and  $M_{i_2}$ ,  $1 \leq i_1, i_2 \leq 2(n - m) + 1$ ,  $|V(M_{i_1}) \setminus V(M_{i_2})| \leq 2$ , and  $|M_{i_1}| + \lceil |M_{i_2}|/2 \rceil \geq n$ . Therefore, we can pair off  $M_1, M_2, \dots, M_{2(n-m)+1}$  into  $n - m$  pairs and one matching  $M_1$  of size  $m$ . By Lemma 2.4, we obtain  $G_{m+i}$ 's and  $G_{m-i}$ 's for  $i = 1, 2, \dots, (n - m)$ , such that  $G_{m+i} \leq G_{m+i+1}$  and  $G_{m-i} \leq G_{m-i+1}$ . Also, we let  $G_m$  be  $M_1$ .

Finally, we consider the decomposition of  $M = \bigcup_{i=2(n-m)+2}^{r+1} M_i$ . Since  $|M_i| \geq m-1$  for each  $i=2(n-m)+2, \dots, r+1$ , and  $\sum_{i=2(n-m)+2}^{r+1} |M_i| = \binom{2m-n}{2} + t$ . By Lemma 2.1,  $M$  can be decomposed into matchings of size  $i$ ,  $i=1, 2, \dots, 2m-n-1$  and a graph  $T$  induced by  $t$  edges in  $M$ . Let these matchings be  $G_1, G_2, \dots, G_{2m-n-1}$ . Then  $G_1, G_2, \dots, G_n \cup T$  is the desired decomposition.

*Case 2(b):*  $v$  is odd. By a similar argument, we have  $m = |M_1| = |M_2| = \dots = |M_{n-m+1}| = (v-1)/2$ , and  $|M_{r+1}| = |M_r| = \dots = |M_{r-n+m+1}| = (v-3)/2$ . Now, let  $L_i = M_{r-n+m+i}$  for  $i=1, 2, \dots, n-m+1$  as in the proof of Lemma 2.4. Then the proof follows by the same argument as in Case 2(a).

*Case 3:*  $r \geq v/2$ . First, we assume that  $G$  is not a complete graph which has an ASD with each member a star without a doubt. Thus,  $|V(G)| \geq n+2$ . Since  $r \geq v/2$ ,  $G$  contains a hamiltonian cycle by basic property of graph theory. Now, let  $H = \{C_1, C_2, \dots, C_x\}$  be a collection of  $x$  hamiltonian cycles such that  $x$  is the smallest integer with  $r-2x < v/2$ . Let  $G' = G \setminus H$ . Clearly,  $G'$  is an  $(r-2x)$ -regular graph. For convenience, let  $r' = r-2x$ . Now, consider 4 cases on  $r'$ .

*Case 3(a):*  $r' = (v-2)/2$  ( $v$  is even). Again,  $G'$  has a  $v/2$  equalized edge-coloring which induces  $v/2$  matchings  $M_1, M_2, \dots, M_{v/2}$ , furthermore  $|M_i| = (v-2)/2$  for  $i=1, 2, \dots, v/2$ . First, if  $(v-2)/2 + \lceil (v-2)/4 \rceil \geq n$  then we can use the argument as in Case 2 to obtain the ASD. Note here that a hamiltonian cycle can be decomposed into two matchings each of size  $v/2$ .

On the other hand, let  $((v-2)/2) + \lceil (v-2)/4 \rceil < n$  and  $v/2 = 2k$  or  $2k-1$  as the case may be. Let  $L_i = M_{k+i}$  for  $i=1, 2, \dots, k-1$ . By Lemma 2.4, for  $i=1, 2, \dots, k-1$ ,  $M_i \cup L_i$  can be decomposed into a linear forest  $G_{(v-2)/2+i}$  with  $(v-2)/2+i$  edges which is the union of disjoint paths with length one or three (or five), and  $G_{(v-2)/2-i}$  a matching with  $(v-2)/2-i$  edges. Let  $G_{(v-2)/2}$  be the matching  $M_k$ . Then we have  $G_i \leq G_{i+1}$  for  $i = (v-2)/2 - (k-1), (v-2)/2 - (k-1) + 1, \dots, (v-2)/2 + (k-1)$ .

Since we have  $x$  hamiltonian cycles,  $(v-2)/2 + k - 1 + x \geq n$ . For otherwise,

$$\begin{aligned} |E(G)| &\leq \frac{v-2}{2} \frac{v}{2} + \left( n - \frac{v-2}{2} - k \right) v \\ &= \frac{v^2}{4} - \frac{v}{2} + nv - \frac{v^2}{2} + v - kv \\ &= -\frac{v^2}{4} + \frac{v}{2} + nv - kv < n(n+1)/2 \quad (v \geq n+2). \end{aligned}$$

Since  $x \geq n - (v-2)/2 - k + 1$ , we can take  $n - (v-2)/2 - k + 1$  hamiltonian cycles and make the following decomposition:

- (1) For  $i = v/2 + k - 1, v/2 + k, \dots, n$ ,  $G_i$  is a subgraph of a hamiltonian cycle which has  $i$  edges such that  $G_i \leq G_{i+1}$ .
- (2) As a counterpart of  $G_i$ , we have  $M'_{v/2-k+1}, M'_{v/2-k}, G_{v/2-k-1}, G_{v/2-k-2}, \dots, G_{v-n}$  which are matchings with  $v/2 - k + 1, v/2 - k, \dots, v - n$  edges, respectively.

(\*) Finally, take  $x - (n - (v - 2)/2 - k + 1)$  remaining hamiltonian cycles which can be decomposed into two perfect matchings, respectively and  $M'_{v/2-k-1}, M'_{v/2-k-2}$ , to obtain the smaller members  $G_i$  and  $T$  a set of  $t$  edges where  $G_i$  is a matching with  $i$  edges,  $i = 1, 2, \dots, v - n - 1$ . (Note here that  $v/2 - k \geq v - n - 1$ .)

Case 3(b):  $r' = v/2 - 2$  ( $v$  is even). In this cases  $E(G')$  can be partitioned into  $(v-2)/2$  matchings such that  $|M_1| = |M_2| = \dots = |M_{(v-4)/2}| = (v-2)/2$  and  $|M_{(v-2)/2}| = (v-4)/2$ . Now, let  $G'_{(v-2)/2} = M_{(v-2)/2}$  and the other members be obtained by the other matchings and  $x$  hamiltonian cycles following the same idea as in the proof of Case 3(a). Now, there are  $t + 1$  edges in  $T'$  (corresponding to  $T$  in Case 3(a)). By letting  $e \in T'$ ,  $G_{(v-2)/2} = G'_{(v-2)/2} \cup \{e\}$  and  $T = T' \setminus \{e\}$  we have the desired ASD.

Case 3(c):  $r' = (v - 1)/2$  ( $v$  is odd). Again, by direct counting,  $E(G')$  can be partitioned into matchings  $M_1, M_2, \dots, M_{(v+1)/2}$  such that  $|M_1| = |M_2| = \dots = |M_{(v+3)/4}| = (v-1)/2$  and the others are of size  $(v-3)/2$ . Now, the proof follows by a similar argument as above, see Case 3(a). Note here that  $v$  is odd, so we decompose every unused hamiltonian cycles into a matching of size  $(v-1)/2$  and a subgraph of size  $(v+1)/2$  containing a matching of size  $(v-1)/2$ . Since every unused subgraph contains more edges than the largest smaller members in (\*) of Case 3(a). We have the proof.

Case 3(d):  $r' = (v - 3)/2$  ( $v$  is odd). The proof is similar to that of Case 3(c).  $\square$

### 3. Concluding remarks

In Case 3, we have spent a lot of time in dealing with the case when  $G$  is of class 2. In fact, it is conjectured that all  $r$ -regular graphs of even order not greater than  $2r$  is of class 1. Therefore, if the conjecture is true, then our proof will be much shorter for this case. On the other hand, since we mainly apply edge-coloring to obtain matchings and then combine them together, suitably before we decompose the graph. Hence, in case that we have a graph  $G$  which is almost regular and we can make sure for any two matchings  $M_i, M_j$ ,  $|V(M_i) \setminus V(M_j)| \leq 2$ , then we can prove that  $G$  has an ASD as well.

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