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TWO–COLORABLE $\{C_4, C_k\}$–DESIGNS

CHIN–MEI FU, HUNG–LIN FU AND ANNE PENFOLD STREET

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Abstract. In this paper, we show that there exists a 2–colorable $\{C_4, C_k\}$–design of order n for each $k \geq 3$ and for each admissible order n of a $\{C_4, C_k\}$–design.

1. Introduction

Let K_n be the complete graph on the set of n vertices $V_n = \{1, 2, \cdots, n\}$ with the set of $\binom{n}{2}$ edges, E_n, which join all possible pairs of vertices in V_n. A G–design of order n is an edge–disjoint decomposition of K_n into copies of the graph $G = \big(V(G), E(G)\big)$. The number n is called an *admissible* order of such a G–design. For example, if $G = K_k$ then the G–design is a 2–$(n, k, 1)$ balanced incomplete block design (BIBD) in the usual notation, and in particular if $k = 3$ then the G–design is a Steiner triple system of order n. Again if $G = C_k$, a cycle of k edges, then the G–design is called a *cycle design* or *cycle system*. Since K_3 and C_3 are the same graph, a Steiner triple system of order n is also a cycle system.

Here we consider a slightly more general situation where cycles of lengths 4 and k are both allowed. In the process of constructing a $\{C_4, C_k\}$–design with the properties we want, we use a more general structure again, namely a $\{C_4, K_k\}$–design.

The isomorphic copies of the graphs that occur in the partition are called *blocks* of the design. In an unfortunate clash of well–established terminology, a proper subset S of V_n is said to be a *blocking set* of the G-design provided that the vertex set of each of its blocks contains at least one element of S, but is not contained in S. Thus if we color S and $V_n \setminus S$ with two distinct colors, the vertex set of every block contains at least one vertex of each color. For convenience, we call a design with a blocking set a 2–*colorable design*. Quite a number of 2–colorable designs have been obtained so far; for example, see [**2**].

Since we use the following result several times, we state it here; for completeness, we include a proof.

Theorem 1.1. *Let $n \equiv 1 \pmod 8$. Then there exists a 2–colorable 4–cycle system of order n with a blocking set of size $\frac{n-1}{2}$.*

Proof. Let $V(K_n) = \mathcal{Z}_n$. We prove the theorem by induction on n.

For $n = 9$, let $S = \{1, 3, 5, 7\}$ be the blocking set, let (a, b, c, d) denote the 4–cycle with edges ab, bc, cd, da, and let

$$T = \{(0+i, 1+i, 5+i, 3+i) \mid i \in \mathcal{Z}_9\}.$$

Then $(\mathcal{Z}_9, T)$ is the system we need.

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Now assume the assertion is true for $n = 8k+1$, $k \geq 1$. Let $S = \{1, 3, 5, \cdots, 8k-1\}$ be the blocking set of the 2–colorable 4–cycle system $(\mathcal{Z}_n, T_1)$. Next let $X = \{0, 8k+1, 8k+2, \cdots, 8k+8\}$, where (X, T_2) is a 2–colorable 4–cycle system with blocking set $\{8k+1, 8k+3, 8k+5, 8k+7\}$. Finally let

$$T_3 = \{(8k+i, j, 8k+i+1, j+1) \mid i = 1, 3, 5, 7; j = 1, 3, 5, \cdots, 8k-1\},$$

so that a 2–colorable 4–cycle system of order $n+8$, (X, T), can be obtained by taking $X = \mathcal{Z}_{8k+9}$ and $T = T_1 \cup T_2 \cup T_3$. Note here that $\{1, 3, 5, \cdots, 8k+7\}$ is a blocking set of size $4k+4$.

This concludes the proof. □

In this paper, we use the idea of *packing* to obtain our construction; see [**4**] for instance. A *packing* of K_n with 4–cycles is an ordered triple (V_n, P, L), where P is a collection of edge–disjoint 4–cycles of the edge–set E_n and $L \subseteq E_n$ is the set of edges not belonging to any 4–cycle in P. The number n is called the *order* of the packing and the set of edges L is called the *leave*.

First, in Section 2, we show that there exists a 2–colorable maximum packing of K_n with 4–cycles. Then in Section 3, for any odd integer $k \geq 3$ and for each admissible order n of a $\{C_4, K_k\}$–design, we construct a 2–colorable $\{C_4, K_k\}$–design of order n. Clearly, this implies the existence of a 2–colorable $\{C_4, C_k\}$–design with odd k. Finally, for each $k \geq 3$ and for each admissible order n of a $\{C_4, C_k\}$–design, we construct a 2–colorable $\{C_4, C_k\}$–design of order n.

2. 2–Colorable Maximum Packings with C_4

It is well–known (see for example [**4**]) that any maximum packing of K_n with copies of C_4 has leave a 1–factor for n even, and leave as shown in Table 1 for n odd. For convenience, such a packing will be denoted by MP4CS(n).

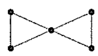
Order (mod 8)	1	3	5	7
Minimum Leave	$\varnothing$	C_3	Bow–tie 	C_5

Table 1

Often we need the idea of a *balanced* 2–coloring. This is one in which the number of vertices colored 1 and the number of vertices colored 2 differ by at most one.

Theorem 2.1. *For each $n \geq 1$, there exists a 2–colorable* MP4CS(n). *If $n = 2m$ or if $n = 2m+1$, the blocking set has size m.*

Proof. If $n \equiv 1 \pmod 8$, then the proof follows by Theorem 1.1.

If n is even, let $n = 2m$, let $V(K_n) = \{a_i, b_i \mid i \in \mathcal{Z}_m\}$ and let $F = \{a_ib_i \mid i \in \mathcal{Z}_m\}$ be the leave. If we color a_i and b_i with 1 and 2 respectively for each $i \in \mathcal{Z}_m$ and let (a_i, a_j, b_i, b_j) be a 4–cycle for each unordered pair $\{i, j\}$, $i \neq j$ and $i, j \in \mathcal{Z}_m$, we obtain a 2–colorable MP4CS(n). This leaves three cases.

(i) $n \equiv 3 \pmod 8$. Let $V(K_n) = \mathcal{Z}_{8k+1} \cup \{\infty_1, \infty_2\}$. By Theorem 1.1, we have a 2–colorable 4CS$(8k+1)$, $(\mathcal{Z}_{8k+1}, T)$, with 2–coloring ϕ such that

$$\phi(i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

The 4–cycles in T, together with $\{(\infty_1, 2j+1, \infty_2, 2j+2) \mid j = 0, 1, \cdots, 4k-1\}$, form a MP4CS$(n)$ with leave $C_3 = (0, \infty_1, \infty_2)$. The colors for ∞_1 and ∞_2 may be chosen arbitrarily, but we may as well color ∞_i with i, $i = 1, 2$, in order to obtain a balanced 2–colouring for a 2–colorable MP4CS(n).

(ii) $n \equiv 5 \pmod 8$. Let $V(K_n) = \mathcal{Z}_{8k+1} \cup \{a, b, c, d\}$, and let the 2–colorable 4CS$(8k+1)$ be defined as in case (i). By a similar technique, we find that the 4–cycles in T, together with $\{(a, 2j+1, b, 2j+2), (c, 2j+1, d, 2j+2) | j = 0, 1, \cdots, 4k-1\}$ and (a, b, c, d), give an MP4CS(n) with leave $(0, a, c) \cup (0, b, d)$. Now coloring a, c with 2 and b, d with 1 gives a 2–colorable MP4CS(n), as required.

(iii) $n \equiv 7 \pmod 8$. Let $V(K_n) = \mathcal{Z}_{8k+1} \cup \{a, b, c, d, e, f\}$. Again using a similar argument, we find that the 4–cycles in T, together with

$$\{(a, 2j+1, b, 2j+2), (c, 2j+1, d, 2j+2), (e, 2j+1, f, 2j+2) \mid j = 0, 1, \cdots, 4k-1\}$$

and $\{(0, a, d, c) \cup (a, c, f, e) \cup (0, b, a, f) \cup (b, e, 0, d)\}$, decompose $K_n \setminus C_5$ where the leave $C_5 = (b, c, e, d, f)$. Now the 2–colorable MP4CS(n) is obtained by coloring a, c, e with 1 and b, d, f with 2.

□

3. 2–Colorable $\{C_4, K_{2h+1}\}$–Designs

First, we need a lemma. Note that we use balanced colorings here.

Lemma 3.1. *If $n - m \equiv 1 \pmod 8$ and m is even, then there exists a 2–colorable $\{C_4, K_{m+1}\}$–design of order n.*

Proof. Let $(\mathcal{Z}_{n-m}, T)$ be a 2–colorable 4CS$(n-m)$ with 2–coloring ϕ such that

$$\phi(i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

Since m is even, let $m = 2s$ and let $\{c_1, d_1, c_2, d_2, \cdots, c_s, d_s\}$ be a set of m points. Now the cycles of T, together with the cycles in

$$\{(c_i, 2j+1, d_i, 2j+2) \mid j = 0, 1, \cdots, (n-m-3)/2, i = 1, 2, \cdots, s\}$$

and the complete graph based on $\{0, c_1, d_1, \cdots, c_s, d_s\}$, decompose K_n into 4–cycles and a K_{m+1}. If we color each c_i with color 1 and each d_i with color 2, then we have a 2–colorable $\{C_4, K_{m+1}\}$–design. □

Lemma 3.2. *Let n be an admissible order of a $\{C_4, K_{2h+1}\}$–design. Then $n \equiv 1$ or $5 \pmod 8$ if $h \equiv 2 \pmod 4$, $n \equiv 1 \pmod 8$ if $h \equiv 0 \pmod 4$, and n is odd if h is odd.*

Proof. Since C_4 is a 2–regular graph and K_{2h+1} is $2h$–regular, each vertex of K_n must have even degree and thus n is odd. If h is even, then the number of edges in K_{2h+1} is also even, implying that K_n must have an even number of edges and hence that $n \equiv 1$ or $5 \pmod 8$. Next, if $4 \mid h$, then the number of edges in K_{2h+1} is also a multiple of 4, and so is the number of edges in K_n. Thus $n \equiv 1 \pmod 8$. □

Lemma 3.3. *There exists a 2–colorable $\{C_4, K_{4l+1}\}$–design of order $n = 8k+5$ for all k and all odd l such that $n \geq 4l+1$.*

Proof. Since l is odd, $n-4l \equiv 1 \pmod 8$. By Lemma 3.1, a 2–colorable $\{C_4, K_{4l+1}\}$–design of order n exists. □

We note here that a 4CS(n) can be considered as a $\{C_4, K_{2h+1}\}$–design of order n with no blocks of size $2h+1$.

Next, we consider the 2–colorable $\{C_4, K_{4l+3}\}$–designs.

Lemma 3.4. *There exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n = 8k+3$ for all k, provided that n is an admissible order of a $\{C_4, K_{4l+3}\}$–design.*

Proof. First, if l is even, then $n-(4l+2) \equiv 1 \pmod 8$. The proof follows by Lemma 3.1.

Next, if l is odd, then $4l+3 \equiv 7 \pmod 8$. Let $j = 4l+3$. Then $\binom{8k+3}{2} - i\binom{8j+7}{2} = \frac{1}{2}[(8k+3)(8k+2) - i(8j+7)(8j+6)]$ which is not a multiple of 4 for $i = 0, 1$ and 2. So if there exists a $\{C_4, K_{4l+3}\}$–design of order n, then the design must contain at least three blocks of size $4l+3$. This implies that $n \geq 3(4l+3)-2$. Let $l = 2l'+1$. By direct counting, $[n-2(4l+2)]-(4l+2) = n-3(4l+2) = (8k+3)-3(8l'+6) \equiv 1 \pmod 8$. Since $4l+2$ is even, there exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n-2(4l+2)$ by Lemma 3.1.

Now let the design of order $n-8l-4$ that we have just described be (X_1, T_1), where $X_1 = \{0, c_1, c_2, \cdots, c_{n-8l-5}\}$. In addition, let $X_2 = \{0, a_1, a_2, \cdots, a_{4l+2}\}$, $X_3 = \{0, b_1, b_2, \cdots, b_{4l+2}\}$, where X_1, X_2 and X_3 have exactly one element in common, namely 0; see Figure 3.1. Since the design (X, T_1) is 2–colorable, let the vertices of X_1 be colored with 1 and 2 respectively, and let the colors of the vertices of $X_2 \cup X_3 \setminus \{0\}$ be defined as follows:

$$\phi(a_i) = \phi(b_i) = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 2, & \text{otherwise.} \end{cases}$$

Therefore a 2–colorable $\{C_4, K_{4l+3}\}$–design of order n can be obtained by letting $T = T_1 \cup T_2$ where $T_2 = \{(a_i, c_j, a_{i+1}, c_{j+1}), (b_i, c_j, b_{i+1}, c_{j+1}) \mid i = 1, 3, 5, \cdots, 4l+1; j = 1, 3, 5, \cdots, n-8l-6\} \cup \{(a_i, b_h, a_{i+1}, b_{h+1}) \mid i, h = 1, 3, 5, \cdots, 4l+1\}$. The 4–cycles in T_2 are depicted in Figure 3.1. □

Lemma 3.5. *There exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n = 8k+5$ for all k, provided that n is an admissible order of a $\{C_4, K_{4l+3}\}$–design.*

Proof. Since $\binom{8k+5}{2} - i\binom{4l+3}{2}$ is not a multiple of 4 for $i = 0, 1$, a $\{C_4, K_{4l+3}\}$–design must contain at least two blocks of size $4l+3$. Therefore $n \geq 2(4l+3)-1$. Direct counting shows that $[n-(4l+2)]-(4l+2) \equiv 1 \pmod 8$. By Lemma 3.1, there exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n-(4l+2)$. By a construction similar to that shown in Figure 3.1, but adding only one block of size $4l+3$ this time, we obtain a 2–colorable $\{C_4, K_{4l+3}\}$–design of order n. □

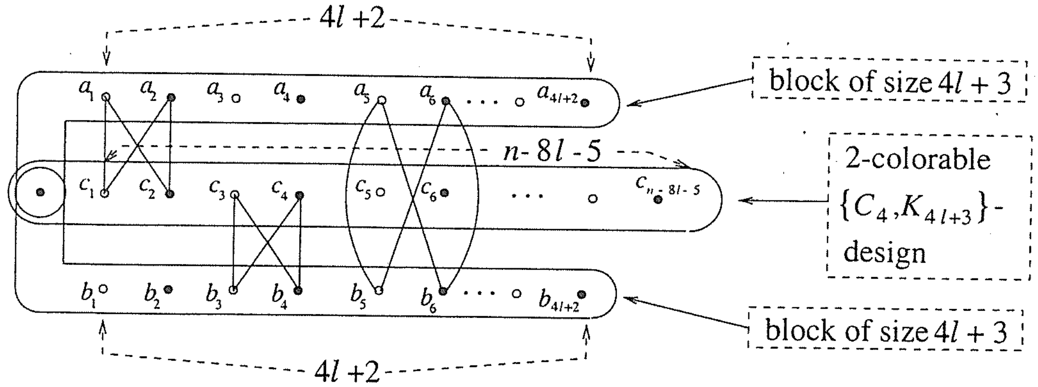


Figure 3.1: The construction of Lemma 3.4.

Lemma 3.6. *There exists a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n = 8k+7$ for all k, provided that n is an admissible order of a $\{C_4, K_{4l+3}\}$–design.*

Proof. If l is odd, then $n-(4l+2) \equiv 1 \pmod 8$ and the proof follows by Lemma 3.1. On the other hand, if l is even, then $\binom{8k+7}{2} - i\binom{4l+3}{2}$ is not a multiple of 4 for $i = 0, 1$ and 2. Therefore $n \geq 3(4l+3) - 2$. Since $[n - 2(4l+2)] - (4l+2) \equiv 1 \pmod 8$, a 2–colorable $\{C_4, K_{4l+3}\}$–design of order $n - 2(4l+2)$ exists, by Lemma 3.1. Again by a construction similar to that shown in Figure 3.1, we can construct a 2–colorable $\{C_4, K_{4l+3}\}$–design of order n directly. □

Combining Lemmas 3.2–3.6, we have the following result.

Theorem 3.7. *For each n and h, if n is an admissible order of a $\{C_4, K_{2h+1}\}$–design, then there exists a 2–colorable $\{C_4, K_{2h+1}\}$–design of order n.*

Corollary 3.8. *If there exists a 2–colorable $\{C_4, K_{2h+1}\}$–design of order n, then there exists a 2–colorable $\{C_4, C_{2h+1}\}$–design of order n.*

Proof. It is well–known that a complete graph of order $2h+1$ can be decomposed into h hamiltonian cycles. Therefore, by replacing each K_{2h+1} with h copies of C_{2h+1}, we have the desired 2–colorable $\{C_4, C_{2h+1}\}$–design. □

Corollary 3.9. *If there exists a 2–colorable $\{C_4, K_{4l+1}\}$–design with balanced coloring, then there also exists a 2–colorable $\{C_4, C_{4l+1}\}$–design with balanced coloring.*

4. 2–Colorable $\{C_4, C_k\}$–Designs

By Corollary 3.8, we have dealt with the case of odd k. If k is a multiple of 4, a straightforward counting argument shows that a $\{C_4, C_k\}$–design must be of order $n \equiv 1 \pmod 8$. Therefore we can handle this case with a 2–colorable 4CS(n), without using any C_k.

If we insist on having a C_k in the design, then we can use the construction indicated in Figure 4.1 to obtain such a design, where we replace the 4–cycles in the shaded area with $k/4$ copies of C_k. This construction depends on Sotteau's theorem [**5**].

Theorem 4.1. [**5**] *$K_{m,n}$ can be decomposed into copies of C_{2t} if and only if m, n are even, $m, n \geq 2t$ and $2t$ divides mn.*

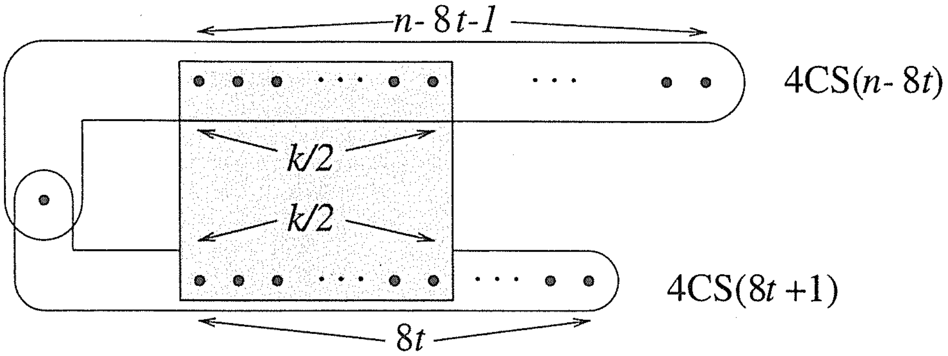


Figure 4.1: Construction of a $\{C_4, C_k\}$–design of order $n \equiv 1 \pmod 8$.

Thus $K_{k/2,k/2}$ (represented by the shaded area in Figure 4.1) can be decomposed into cycles of length k. Since each k–cycle is a hamiltonian cycle in $K_{k/2,k/2}$, it is 2–colored in accordance with the 2–colorings of the 4CS$(n-8t)$ and 4CS$(8t+1)$ in Figure 4.1.

Thus we need only consider $k \equiv 2 \pmod 4$. Now $n \equiv 1$ or $5 \pmod 8$; again a straightforward counting argument shows that we need only consider $n \equiv 5 \pmod 8$. Since $k/2$ is odd, we have to modify Figure 4.1 to make sure that we have C_k in the design. First, we need a lemma.

Lemma 4.2. *Let h be an integer such that $8h+5 > k = 4t+2$. Then $K_{8h+5} \setminus C_k$ can be decomposed into 4–cycles, and colored so that each of the 4–cycles is 2–colored. Further, the C_k is also 2–colored.*

Proof. The proof is by induction on the order $8h+5$ and on k.

First, we claim that $K_{8h+5} \setminus C_6$ can be decomposed into 2–colored 4–cycles if $8h+5 \geq 6$. For we have a 2–colored MP4CS$(8h+5)$ with leave $(0,a,c) \cup (0,b,d)$ by (ii) of Theorem 2.1. Also in the construction $(a,1,b,2)$ and $(c,1,d,2)$ are two 2–colored 4–cycles in the maximum packing. Since $(0,a,c) \cup (0,b,d) \cup (a,1,b,2) \cup (c,1,d,2)$ contains the same set of edges as $(1,a,c,2,b,d) \cup (1,c,0,b) \cup (2,a,0,d)$, and since both of $(1,c,0,b)$ and $(2,a,0,d)$ are 2–colored, our first claim is proved. The fact that the C_6 is 2–colored follows from the fact that $V(C_6) \supseteq \{a,1,b,2\}$.

Secondly, we claim that $K_{8h+5} \setminus C_{10}$ can be decomposed into 2–colored 4–cycles. To see this, let $h = h'+h''+1$, so that $8h+5 = (8h'+6)+(8h''+6)+1$. We already have a 2–colored MP4CS$(8h'+7)$ and a MP4CS$(8h''+7)$, each with leave a C_5. (Note here that the proof of Theorem 2.1 shows that the leave C_5 is also 2–colored. Thus there are two adjacent vertices in C_5 which have different colours; let them be d and e (d' and e' respectively) in Figure 4.2. Also let (d,d',e,e') be one of the 4–cycles between A and B, just as we have assumed in the preceding lemmas.) Now the 10–cycle can be obtained from $(a,b,c,d,e) \cup (a',b',c',d',e') \cup (d,d',e,e')$ which contains the same set of edges as $(a,b,c,d,d',c',b',a',e',e) \cup (d,e,d',e')$.

Finally we assume as our induction hypothesis that $8h+5 > 4t'+2$ and that there is a 2–colored 4–cycle decomposition of $K_{8(h-1)+5} \setminus C_{4t'+2}$, where $C_{4t'+2}$ is itself 2–colored. We claim that there is also a 2–colored 4–cycle decomposition of $K_{8h+5} \setminus C_{4t'+10}$. Since $K_9 \setminus C_8$ has a 2–colored 4–cycle decomposition, as shown in Figure 4.3, we can use the same idea again, as shown in Figure 4.4, to obtain the required construction. Since $V(C_{4t'+10}i) \supseteq V(C_{4t'+2})$, $C_{4t'+10}$ is also 2–colored.

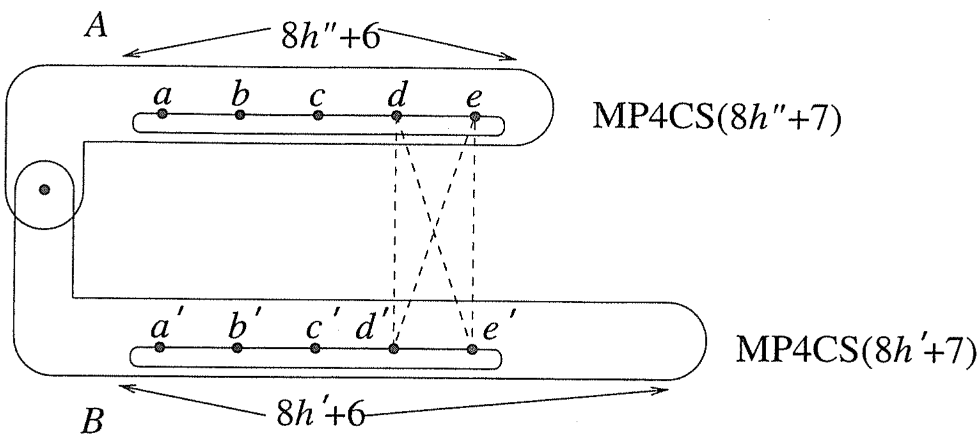


Figure 4.2: Construction of Lemma 4.1.

This completes the proof. □

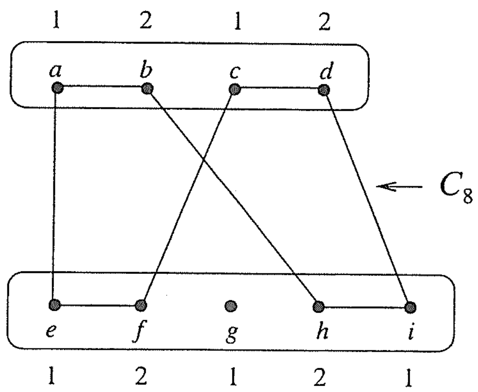


Figure 4.3: 4–cycles in $K_9 \setminus C_8$:
(a, c, b, d), (e, h, f, i), (b, e, g, f), (c, h, g, i), (a, f, d, h), (a, g, b, i), (c, e, d, g).

Theorem 4.3. *For each $k \geq 3$, if n is an admissible order of a $\{C_4, C_k\}$-design, then there exists a 2-colorable $\{C_4, C_k\}$-design of order n.*

Proof. First, if $k \equiv 0 \pmod 4$, then $n \equiv 1 \pmod 8$ and $n \geq k$. The proof then follows from the comment before Lemma 4.2.

Next, if $k \equiv 2 \pmod 4$, then $n \equiv 1$ or $5 \pmod 8$; we need only consider the case where $n \equiv 5 \pmod 8$, and the proof follows from Lemma 4.2.

Now if k is odd, then n can be any odd integer, but again we need only consider the case where $n \not\equiv 1 \pmod 8$.

1. If $k \equiv 3 \pmod 4$, the proof follows from Lemmas 3.4, 3.5, 3.6 and Corollary 3.8.

2. Finally, consider the case where $k \equiv 1 \pmod 4$.

   (a) If $n \equiv 5 \pmod 8$, the proof follows from Lemma 3.3 and Corollary 3.8.

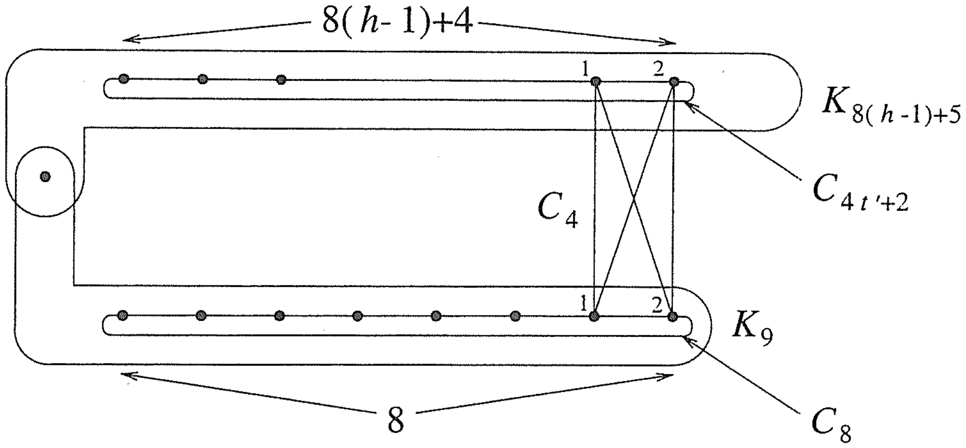


Figure 4.4: Replacing $C_{4t'+2} \cup C_4 \cup C_8$ with $C_{4t'+10} \cup C_4$.

(b) This leaves the cases where $n \equiv 3$ or $7 \pmod 8$.
By Lemmas 3.4 and 3.6, we have a 2–colorable $\{C_4, K_{4l+3}\}$–design and, by Corollary 3.9, the vertices in $V(K_{4l+3})$ have a balanced coloring, with at least $2l+1$ vertices of each colour.
Now a 2–colored $\{C_4, C_{4l+1}\}$–design exists provided that there exists a 2–colored $\{C_4, C_{4l+1}\}$–design of order $4l+3$.
In the 2–colorable $\{C_4, K_{4l+3}\}$–design that we already have, let $V(K_{4l+3}) = \{\infty_1, \infty_2\} \cup \mathbb{Z}_{4l+1}$, where ∞_i is colored i, for $i = 1, 2$. Now K_{4l+1} can be decomposed into $2l$ hamiltonian cycles and one of these, say $(0, 1, 2, \cdots, 4l)$, can be matched with ∞_1, ∞_2 to form $3l+1$ 4–cycles, as follows:

$$(\infty_1, \infty_2, 0, 4l), (\infty_1, 0, 1, 2), (\infty_2, 2, 3, 4), \cdots, (\infty_2, 4l-2, 4l-1, 4l),$$

$$(\infty_1, 1, \infty_2, 3), (\infty_1, 5, \infty_2, 7), \cdots, (\infty_1, 4l-3, \infty_2, 4l-1).$$

Since we can arrange the colours of $0, 1, 2, \cdots, 4l$, to alternate between 1 and 2, all these 4–cycles are 2–colored, and K_{4l+3} is now decomposed into $2l-1$ cycles of length $4l+1$ and $3l+1$ 4-cycles, all of which are 2–colored.
This completes the proof. □

5. Concluding Remarks

We have constructed 2–colorable $\{C_4, K_{2h+1}\}$–designs in Section 3. This suggests that a 2–colorable $\{C_4, K_k\}$–design may well exist for each $k \geq 3$. The construction of such a design is easy for $k = 4$ and $k = 8$, but we have been unable to construct one for $k = 6$.

We recall that Alspach [**1**] asked the following question in 1981: Let n be a positive integer and let $a_1 + a_2 + \cdots + a_r$ be a partition of either $\binom{n}{2}$ if n is odd, or $\binom{n}{2} - n/2$ if n is even, such that $3 \leq a_i \leq n$ for $i = 1, 2, \cdots, r$. Does there exist a partition, into cycles of lengths $a_1, a_2, \cdots, a_r$, of the edge–set of K_n when n is odd, or of K_n with a 1–factor removed when n is even?

The existence of a $\{C_4, C_k\}$–design for all admissible orders certainly suggests that Alspach's conjecture may hold for two cycle sizes, 4 and k. To construct a 2–colorable $\{C_4, C_k\}$-design with a prescribed number of 4–cycles (and hence a prescribed number of k-cycles) sounds feasible and interesting. Note that for $k = 3$, some restriction must be made; for instance the 2-colorable $\{C_3, C_4\}$-design cannot have too many triangles [**3**].

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Chin-Mei Fu
Department of Mathematics
Tamkang University
Tamsui
Taipei Shien
TAIWAN, R.O.C.
cmfu@mail.tku.edu.tw

Hung-Lin Fu
Department of Applied Mathematics
National Chiao Thung University
Hsin Chu
TAIWAN, R.O.C.
hlfu@math.nctu.edu.tw

Anne Penfold Street
Centre for Discrete Mathematics and Computing
Department of Mathematics
The University of Queensland
Brisbane
AUSTRALIA
aps@maths.uq.edu.au