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A STUDY OF TYPENUMBER IN BOOK-EMBEDDING

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Abstract

The type of a vertex v in a p -page book-embedding is the $p \times 2$ matrix of nonnegative integers

$$\tau(v) = \begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix},$$

where $l_{v,i}$ (respectively, $r_{v,i}$) is the number of edges incident to v that connect on page i to vertices lying to the left (respectively, to the right) of v . The typenumber of a graph G , $T(G)$, is the minimum number of different types among all the book-embeddings of G . In this paper, we disprove the conjecture by J. Buss et. al. which says for $n \geq 4$, $T(L_n)$ is not less than 5 and prove that $T(L_n) = 4$ for $n \geq 3$.

1 Introduction and Basic Properties

A book is a set of half-planes (the pages of the book) that share a common boundary line (the spine of the book). An embedding of a simple undirected graph of G (a pair of vertices are connected by at most one edge) in a book consists of an ordering of the vertices of G along the spine (horizontal line) of the book, together with an assignment of each edge of G to a page of the book, in which edges assigned to the same page do not cross.

There are three germane measure of the quality of a book-embedding: the thickness (number of pages) of the book, the individual and cumulative

widths of the pages, and the number of distinct vertex types. Throughout of this paper we shall focus on the study of the third measure.

Given a p -page book-embedding of a graph G , each vertex v of G has an associated $p \times 2$ matrix of nonnegative integers, called its *type*,

$$\tau(v) = \begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix},$$

where $l_{v,i}$ (respectively, $r_{v,i}$) is the number of edges incident to v that connect on page i to vertices lying to the left (respectively, to the right) of v . Thus for each graph of order n and each p -page book-embedding, there are n types, one for each vertex and two types are different provided that the two matrices are not equal. It's interesting to know : among all the book-embeddings of G what is the minimum number of different *vertex types*, $T(G)$. For its application, the number of types in a book-embedding relates to the amount of logic necessary to realize fault-tolerant arrays of processors using one specific design methodology. The methodology views the desired array as an undirected graph, with vertices representing processing elements and edges representing communication links; the design process operates in two stages: first, the graph representing the desired array is embedded in a book; then, the book-embedding is converted to an efficient fault-tolerant layout of the array. The significance of the notation of vertex type is that the type of a vertex "tell" it what role to play in the fault-free processor array. Thus, the base-two logarithm of the number of vertex types is the number of control bits per processing element needed to configure the array to its fault-free format. [2] [3]

In any book-embedding of G , there are two specific vertex types in the embedding : *source* and *sink*. A nonzero vertex type for a vertex v is a *source* if all $l_{v,i} = 0$ and is a *sink* if all $r_{v,i} = 0$. Clearly, every book-embedding of a graph has at least one source and one sink.

Let $N_i = \{v \in V(G) \mid \text{deg}_G(v) = i\}$ and T_i be the different types counting all the vertices in N_i , i.e., $T_i = |\{\tau(v) \mid v \in N_i\}|$. By considering the relation between the degree of v and the $p \times 2$ matrix, $\tau(v)$, we have the followings.

Lemma 1.1. [1] *For each $v \in V(G)$, $\text{deg}_G(v) = \sum_{i=1}^p (l_{v,i} + r_{v,i})$, hence $\tau(u) \neq \tau(v)$ provided that $\text{deg}_G(u) \neq \text{deg}_G(v)$. Thus $T(G) = \sum_{i \in D_G} T_i$.*

Lemma 1.2. $\sum_{v \in V(G)} \sum_{i=1}^p l_{v,i} = \sum_{v \in V(G)} \sum_{i=1}^p r_{v,i}$

Lemma 1.3. [1] *A connected graph G with at least two vertices has typenumber two if and only if G is a star.*

A lattice graph $L_{m,n}$ is a graph with vertex set $\{(a,b) | a \in Z_m \text{ and } b \in Z_n\}$ and edge set $\{(a,b), (c,d) | a = c \text{ and } |b - d| = 1 \text{ or } b = d \text{ and } |a - c| = 1\}$. While $m = 2$ or $n = 2$, it's called a ladder, denoted by $L_n(L_m)$.

Proposition 1.4. [1] *Restricting to 1-page book-embedding of L_n we have : $T(L_1) = 2, T(L_2) = 3, T(L_3) = 4,$ and $T(L_n) = 5,$ for $n \geq 4.$*

In [1], the authors conjectured that additional pages for $L_n, n \geq 4,$ will not lower its typenumber below 5. Unfortunately, this conjecture is not true as we shall see in section 2.

2 Main Result

A book - embedding graph G_K corresponding to a book-embedding K of G is a colored (not necessarily proper) digraph such that (i) an arc $\overrightarrow{uv} \in A(G_K)$ if and only if $uv \in E(G)$ and u is to the left of v along the spine in the embedding where the two sides are fixed, and (ii) two arcs have the same color if and only if the corresponding edges are embedded by K on the same page.

For convenience, we use $G_K(\Pi)$ to denote the subgraph of G_K which is induced (edge) by the set of edges whose colors are in Π . If $\Pi = \{c_1, c_2, \dots, c_k\}$, we also use $G_K(c_1, c_2, \dots, c_k)$ to denote the subgraph. Now, we have the following results.

Lemma 2.1. *Let G_K be a book-embedding graph of G corresponding to a book-embedding K . Then, for each color c , each cycle in $G_K(c)$ is almost a directed cycle except exactly one arc.*

Proof. First, we observe that a directed path (a_0, a_1, \dots, a_t) in $G_K(c)$ gives the relation that a_i is to the left of a_j if and only if $0 \leq i < j \leq t$. Furthermore, any vertex v lies between a_i and a_{i+1} can be incident to neither a_0 nor a_t for any $i \geq 0$ and $i < j$ because they are at the same page in the book-embedding. Now, given a cycle C in $G_K(c)$ and choose $P = (a_0, a_1, \dots, a_t)$ to be the longest directed path in C . If

(1). $t = 1$.

So $P = \overleftarrow{a_1 a_2 a_3} \dots \overleftarrow{a_t a_1}$. Hence, the type number of this cycle C is 2, a contradiction to Lemma 1.3.

(2). $t \geq 2$ and the length of C is $t + 1$.

Obviously, C is almost a directed cycle except exactly one arc.

(3). $t \geq 2$ and the length of C is large than $t + 1$.

Let $v (\neq a_1)$ be incident to a_0 in C and v must be to the right of a_0 because P is the longest directed path. By the observation above, v must be to the right of a_t too. Similarly, let $v' (\neq a_{t-1})$ be incident to a_t in the cycle, we can obtain the relation that v' is to the left of a_1 . So $v \neq v'$ and a cross happens between a_0v and $v'a_t$, a contradiction.

Then the only possibility is that the cycle C is almost a directed cycle except exactly one arc.

Now, we consider the graph induced by two colors.

Lemma 2.2. *Let G_K be a book-embedding graph of G corresponding to a book-embedding K . Then there doesn't exist a cycle C in $G_K(c_1, c_2)$ such that the arcs of the same color have the same orientation and the number of vertices which are incident with 2 colors edges are at least 4.*

Proof. Suppose not. Let C be such a cycle in $G_K(c_1, c_2)$ and C can be partitioned into

$$P_1(v_1, v_2)P_2(v_2, v_3)P_3(v_3, v_4) \cdots P_{k-1}(v_{k-1}, v_k)P_k(v_k, v_1)$$

such that P_i and P_{i+1} are two directed paths with the opposite orientations and colors. And obviously k is an even number and $k \geq 4$. By contracting each P_i to an arc $\overrightarrow{v_i, v_{i+1}}$ (or $\overleftarrow{v_i, v_{i+1}}$), the cycle C becomes to a new cycle C'

$$\overrightarrow{v_1v_2} \overleftarrow{v_2v_3} \overrightarrow{v_3v_4} \cdots \overleftarrow{v_{k-1}v_k} \overrightarrow{v_kv_1}.$$

Since C is a subgraph of the book-embedding graph G_K , we can obtain a 2 page book-embedding of a cycle which is the underlying graph of C . And it's easy to see that the underlying graph of the contracted cycle C' can be embedded in this 2 page book and each vertex is either a source $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ or a sink $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. So the typenumber of this cycle is two. But it's not a star, a contradiction.

Theorem 2.3. $T(L_3) = 4$

Proof. By Figure 2, $T(L_3) \leq 4$. We suppose $T(L_3) < 4$. Since L_3 isn't a star, by lemma 1.3, $T(L_3) \geq 3$. Assume that $T(L_3) = 3$ and we have two cases to consider now.

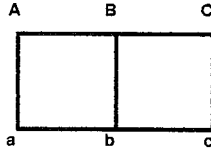


Figure 1: L_3

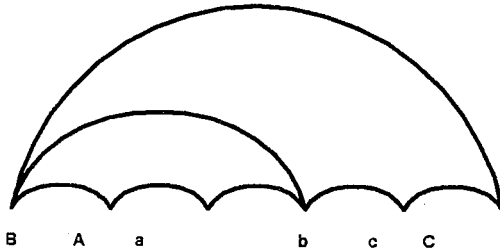


Figure 2: An book-embedding of $T(L_3)$

(1). $T_2 = 2, T_3 = 1$:

Since $Aa, Cc \in E(L_3)$, one of $\{A, a\}$ is a source and the other is a sink. So is C, c . Since $Bb \in E(L_3)$, W.L.O.G., we may let $\sum l_{B,i} = \sum l_{b,i} = 2$. Hence $\sum \sum l_{v,i} \neq \sum \sum r_{v,i}$ and we have a contradiction.

(2). $T_2 = 1, T_3 = 2$:

The only vertex type of vertices of N_2 is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then W.L.O.G.

Figure 3 show the possible book-embedding graph only can be the following two types:

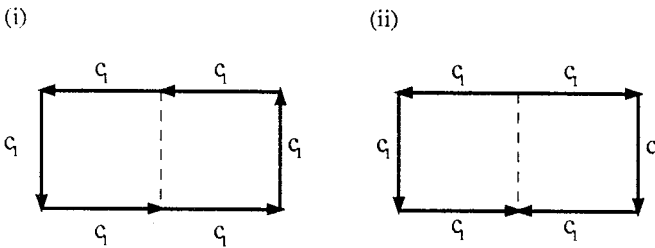


Figure 3: The six arcs must have the same color.

For each type, there exists a one-color cycle which is forbidden in Lemma 2.1, a contradiction. So we conclude that $T(L_3) = 4$.

Theorem 2.4. $T(L_n) = 4$, $n \geq 4$.

Proof. By Figure 4, $T(L_n) \leq 4$, since the vertex types are $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

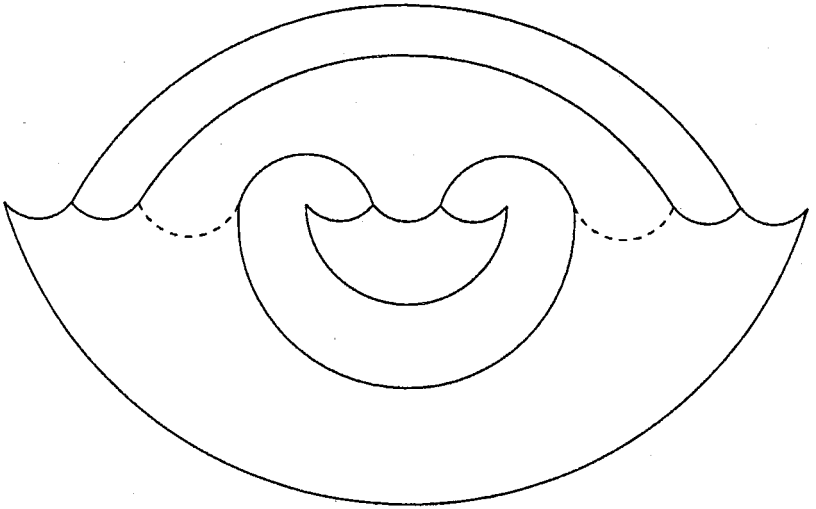


Figure 4: A book-embedding of $T(L_n)$

Assume that $T(L_n) < 4$. There are Three cases to consider:

- (1). $T_2 = 1$, $T_3 = 1$ (i.e. $T(L_n) = 2$)
It's trivially impossible because L_n isn't a star.
- (2). $T_2 = 2$, $T_3 = 1$ (i.e. $T(L_n) = 3$)
By the same reason as (1) in Theorem 2.3. $\sum \sum l_{v,i} \neq \sum \sum r_{v,i}$, a contradiction.
- (3). $T_2 = 1$, $T_3 = 2$ (i.e. $T(L_n) = 3$)
Obviously, the only vertextype of vertices of N_2 is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and the only two vertex types of vertices of N_3 are a source and a sink.

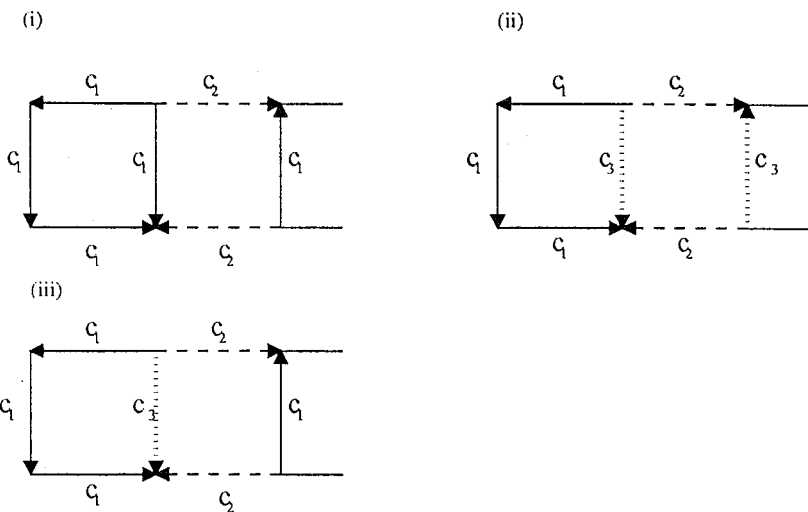


Figure 5: Each subgraph contains a two-color cycle forbidden in Lemma 2.2.

W.L.O.G., the possible book-embedding graph must contain one of the following three induced subgraphs :

Since, there exists a two-color cycle for each subgraph which is forbidden in Lemma 2.2, a contradiction. So $T(L_n) = 4$.

■

References

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