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4-Cycle Group-Divisible Designs with Two Associate Classes

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In this paper we investigate a partitioning problem, setting the existence problem for all group-divisible designs with first and second associates in which the blocks are 4-cycles.

1. Introduction

A group-divisible design GDD$(n, m; k; \lambda_1, \lambda_2)$ is an ordered triple (V, G, B), where V is a set of mn elements, G is a partition of V into m groups of size n, and B is a collection of subsets of V called blocks, each of size k, such that:

(1) each pair of elements that occur in the same group, occur together in exactly λ_1 blocks, and
(2) each pair of elements that occur in different groups, occur together in exactly λ_2 blocks.

Two elements in the same group are called first associates and two elements in different groups are called second associates. We say that the GDD is defined on V.

The existence of GDDs has been studied for many years. GDDs have been of great use and interest to statisticians. Partially balanced designs with two associate classes were originally classified in 1952 by Bose and Shimamoto into five types [2].

In [7], Hanani considered GDDs with $\lambda_1 = 0$ and $k = 3$, and a complete solution was given. Later, Brouwer, Schrijver and Hanani [3] obtained an answer for $k = 4$ and $\lambda_1 = 0$. Recently, Fu, Rodger and Sarvate [6] and Fu and Rodger [4] explored the existence of

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GDD$(n, m; 3; \lambda_1, \lambda_2)$, and their combined results solved the problem, proving the following theorem.

Theorem 1.1 ([4, 6, 7]). *Let $n, m, \lambda_2 \geq 1$, $\lambda_1 \geq 0$. Then there is a* GDD$(n, m; 3; \lambda_1, \lambda_2)$ *if and only if:*

(1) 2 *divides* $\lambda_1(n-1) + \lambda_2(m-1)n$,
(2) 3 *divides* $\lambda_1 mn(n-1) + \lambda_2 m(m-1)n^2$,
(3) *if* $m = 2$ *then* $\lambda_1 \geq \lambda_2 n/2(n-1)$, *and*
(4) *if* $n = 2$ *then* $\lambda_1 \leq (m-1)\lambda_2$.

Alternatively, a block of size 3 can also be thought of as a cycle of length 3 (3-cycle). Let $(\lambda_1, \lambda_2)K_{(n;m)}$ be the graph with vertex set partitioned into m parts, say $V(G) = V_1 \cup V_2 \cup \cdots \cup V_m$ with $|V_i| = n$ for $1 < i \leq m$, and with two vertices being joined by λ_1 or λ_2 edges if they are on the same or different parts respectively. Then Theorem 1.1 determines when there exists an edge-disjoint decomposition of $(\lambda_1, \lambda_2)K_{(n;m)}$ into 3-cycles. Once described in this way, it is then natural to ask when there exists an edge-disjoint decomposition of $(\lambda_1, \lambda_2)K_{(n;m)}$ into cycles of length t; we call such a decomposition a GDDtC$(n; m)$ of index (λ_1, λ_2). These are also of interest to statisticians in the guise of neighbour designs [10, 13].

In this paper we consider the case when $t = 4$. Throughout the rest of the paper $t = 4$, so, for convenience, a GDD4C may also be referred to as a GDD$(n; m)$ of index (λ_1, λ_2) without mentioning 4-cycles. The main result of this paper is the following theorem.

After some preliminary results in Section 2, Theorem 1.2 is proved for $n \geq 4$ in Section 3, for $n = 2$ in Section 4, and for $n = 3$ in Section 5. Solving the case when $n = 3$ is particularly interesting, and by far the most difficult case, for we must find a way to deal with condition (4) in Theorem 1.2 when $\delta = 1$. Clearly we can assume that $\lambda_2 > 0$ (if $\lambda_2 = 0$ then consider each component in turn with $m = 1$).

Theorem 1.2. *Let n, $m \geq 1$ and $\lambda_1, \lambda_2 \geq 0$ be integers. There exists a* GDD4C$(n; m)$ *of index (λ_1, λ_2) if and only if:*

(1) 2 *divides* $\lambda_1(n-1) + \lambda_2 n(m-1)$,
(2) 8 *divides* $\lambda_1 mn(n-1) + \lambda_2 n^2 m(m-1)$, *and if* $\lambda_2 = 0$ *then* 8 *divides* $\lambda_1 n(n-1)$,
(3) *if* $n = 2$ *then* $\lambda_2 > 0$ *and* $\lambda_1 \leq 2(m-1)\lambda_2$, *and*
(4) *if* $n = 3$ *then* $\lambda_2 > 0$ *and* $\lambda_1 \leq 3(m-1)\lambda_2/2 - \delta(m-1)/9$, *where* $\delta = 0$ *or* 1 *if* λ_2 *is even or odd, respectively.*

2. Preliminary results

The following results are essential for the proof of Theorem 1.2. A GDD$(n; 1)$ of index $(\lambda, 0)$ is clearly an edge-disjoint decomposition of λK_n into 4-cycles; these are also known as *λ-fold 4-cycle systems* of *order n*.

Lemma 2.1 ([9, 11]). *A* GDD$(n;1)$ *of index* $(\lambda, 0)$ *exists if and only if* (a) 2 *divides* $\lambda(n-1)$, (b) 8 *divides* $\lambda n(n-1)$, *and* (c) $n \geq 4$.

By direct counting, the relationship between λ and n for such GDDs can be obtained easily: see Table 1.

Table 1 Values of n, λ for which GDD$(n;1)$s of index $(\lambda, 0)$ exist

λ (mod 4)	1	2	3	4
n (mod 8)	1	0, 1, 4, 5	1	any

Lemma 2.2 ([1]). *A* GDD$(n;m)$ *of index* $(0, \lambda)$ *exists if and only if* (a) 2 *divides* $\lambda n(m-1)$, *and* (b) 8 *divides* $\lambda n^2 m(m-1)$.

The relationship between $(n;m)$ and λ for such GDDs is found in Table 2.

Table 2 Values of n, m, λ for which GDD$(n;m)$s of index $(0, \lambda)$ exist

λ (mod 4)	1	2	3	4
$(n;m)$ (mod 8)	(even; any)	(even; any)	(even; any)	(any; any)
	(odd; 1)	(odd; 0, 1, 4, 5)	(odd; 1)	

A *packing* of λK_n with 4-cycles is a partition of the edges of a subgraph G of λK_n into 4-cycles; the *leave* of this packing is $\lambda K_n - E(G)$. A packing of λK_n with 4-cycles is also known as a *partial* 4-cycle system of λK_n, and is denoted more briefly by p4CS of λK_n. The following result is concerned with maximum packings of λK_n (*i.e.*, packings of λK_n with $|E(G)|$ as large as possible).

Lemma 2.3 ([14]). *The leaves for maximum packings of* λK_n *with* 4*-cycles are described in Table 3.*

If $\lambda_1 = 0$ then the graph $(\lambda_1, \lambda_2)K_{(n;m)}$ is the balanced complete m-partite graph with each partite set of size n and multiplicity λ_2; this is usually denoted by $\lambda K_{m(n)}$. So a GDD$(n;m)$ of index $(0, \lambda)$ corresponds to an edge-disjoint decomposition of $\lambda K_{m(n)}$ into 4-cycles. For those λ, m and n that do not satisfy the conditions in Table 2, we can instead consider maximum packings of $\lambda K_{m(n)}$. Clearly we need only consider the case when n is odd.

Lemma 2.4 ([1]). *For odd n, the maximum packing of* $\lambda K_{m(n)}$ *with* 4*-cycles has leave as described in Table 4.*

Table 3 Leaves of maximum packings of λK_n with 4-cycles. F is a 1-factor; C_n is a cycle of length n; E_6 is the set of graphs on n vertices with 6 edges in which each vertex has even degree (or is a doubled edge if the leave is allowed to be a multigraph); and F_2 is the set of at most two graphs on n vertices in which all vertices have degree 1 except for either one vertex that has degree 5, or two vertices that have degree 3

λ (mod 4) \ n (mod 8)	0	1	2	3	4	5	6	7
1	F	ϕ	F	C_3	F	E_6	F	C_5
2	ϕ	ϕ	E_6 if $n > 3$	E_6 if $n > 3$	ϕ	ϕ	E_6	E_6
3	F	ϕ	F_2 if $n > 2$	C_5 if $n > 3$	F	E_6	F_2	C_3
4	ϕ	ϕ	ϕ if $n > 2$	ϕ if $n > 3$	ϕ	ϕ	ϕ	ϕ

Table 4 Leaves of maximum packings of $\lambda K_{m(n)}$, n odd (F, C_n, E_6, F_2 defined in Table 3)

λ (mod 4) \ m (mod 8)	0	1	2	3	4	5	6	7
1	F	ϕ	F if $n=4k+1$ F_2 if $n=4k+3$	C_3	F	E_6	F if $n=4k+1$ F_2 if $n=4k+3$	C_5
2	ϕ	ϕ	E_6	E_6	ϕ	ϕ	E_6	E_6
3	F	ϕ	F_2 if $n=4k+1$ F if $n=4k+3$	C_5	F	E_6	F_2 if $n=4k+1$ F if $n=4k+3$	C_3
4	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ

Now, we are ready to consider the existence of a GDD$(n;m)$ of index (λ_1, λ_2). We refer to edges joining vertices in the same or different parts of $(\lambda_1, \lambda_2)K_{(n;m)}$ as *pure edges* and *crossed edges*, respectively. Lemma 2.5 proves the necessity of Theorem 1.2.

Lemma 2.5. *If a* GDD$(n;m)$ *of index* (λ_1, λ_2) *exists, then:*

(a) 2 *divides* $\lambda_1(n-1) + \lambda_2(m-1)n$,

(b) 8 *divides* $\lambda_1 mn(n-1) + \lambda_2 n^2 m(m-1)$, *and if* $\lambda_2 = 0$ *then* 8 *divides* $\lambda_1 n(n-1)$,

(c) *if* $n = 2$ *then* $\lambda_2 > 0$ *and* $\lambda_1 \leq 2(m-1)\lambda_2$, *and*

(d) *if* $n = 3$ *then* $\lambda_2 > 0$ *and* $\lambda_1 \leq 3(m-1)\lambda_2/2 - \delta(m-1)/9$, *where* $\delta = 0$ *or* 1 *if* λ_2 *is even or odd, respectively.*

Proof. (a) and (b) are true, since in $(\lambda_1, \lambda_2)K_{(n;m)}$ the degree of each vertex must be even, and the number of edges in each component (the graph is disconnected if $\lambda_2 = 0$) must be divisible by 4. If $n = 2$, then, in order to use up (*i.e.*, place in 4-cycles) the $m\lambda_1$ pure edges, we have to use at least $m\lambda_1$ of the $4\lambda_2\binom{m}{2}$ crossed edges. This implies that $2\lambda_2 m(m-1) \geq m\lambda_1$, and $\lambda_2 > 0$ since each component with edges must have at least 4 vertices, so we have (c). Finally, we consider $n = 3$. As in case (c) we have $\lambda_2 > 0$, and in order to use up all the pure edges we need at least $3\lambda_1 m$ crossed edges, so $\lambda_1 \leq 3(m-1)\lambda_2/2$. However, if λ_2 is odd then the number of edges between each pair of groups (*i.e.*, $9\lambda_2$) is odd; so there must be at least $\binom{m}{2}$ crossed edges that occur in 4-cycles containing at most one pure edge. Therefore, if λ_2 is odd then $9\lambda_2\binom{m}{2} - \binom{m}{2} \geq 3\lambda_1 m - \binom{m}{2}/3$, which implies that $\lambda_1 \leq 3(m-1)\lambda_2/2 - (m-1)/9$. Thus (d) is proved. □

Table 5 depicts the relationship between $(n; m)$ and (λ_1, λ_2) required for the necessary conditions (a) and (b) to be satisfied for $0 \leq \lambda_1,\ \lambda_2 \leq 4$.

Table 5 Necessary conditions on (n (mod 8); m (mod 8)) for each $\lambda_1, \lambda_2 \in \{0, 1, 2, 3, 4\}$

$\lambda_1 \backslash \lambda_2$	0	1	2	3	4
0 or 4	(any; any)	(even; any) (odd; 1)	(even; any) (odd; 0, 1, 4, 5) (1; 0, 1, 4, 5)	(even; any) (odd; 1) (1; 1), (5, 5)	(any; any) (1; any)
1	(1; any)	$n \cdot m \equiv 1$ (mod 8)	(3, 7; even) (5; 0, 3, 4, 7)	(3; 7), (7, 3)	(3, 7; 4, 0) (5; even)
2	(0, 1, 4, 5; any)	(0, 4; any) (1, 5; 1) (2, 6; even) (3, 7; 5) (1; 1), (5, 5)	$n \cdot m \equiv 0, 1, 4, 5$ (mod 8) (1; 0, 1, 4, 5)	(0, 4; any) (1, 5; 1) (2, 6; even) (3, 7; 5)	(0, 1, 4, 5; any) (2, 3, 6, 7; even) (1; any)
3	(1; any)	(3; 7), (7, 3)	(3, 7; even) (5; 0, 3, 4, 7)	$n \cdot m \equiv 1$ (mod 8)	(3, 7; 4, 0) (5; even)

The following lemmas are important in constructing GDD$(n; m)$s with larger λ_1 or λ_2; they are easy to prove, so this is left to the reader.

Lemma 2.6. *If a* GDD$(n; 1)$ *of index* $(\lambda_1', 0)$ *exists and a* GDD$(n; m)$ *of index* $(\lambda_1 - \lambda_1', \lambda_2)$ *exists, then a* GDD$(n; m)$ *of index* (λ_1, λ_2) *exists.*

Lemma 2.7. *Let* $\lambda_1' \leq \lambda_1, \lambda_2$. *If a* 4*-cycle system of order mn and index* λ_1' *exists, and a* GDD$(n; m)$ *of index* $(\lambda_1 - \lambda_1', \lambda_2 - \lambda_1')$ *exists, then a* GDD$(n; m)$ *of index* (λ_1, λ_2) *exists.*

Lemma 2.8. *Let* $\lambda_2' \leq \lambda_2$. *If a* GDD$(n; m)$ *of index* $(0, \lambda_2')$ *exists, and a* GDD$(n; m)$ *of index* $(\lambda_1, \lambda_2 - \lambda_2')$ *exists, then a* GDD$(n; m)$ *of index* (λ_1, λ_2) *exists.*

Lemma 2.6, 2.7, and 2.8 are so-called 'reduction lemmas'. It is worth mentioning that, if $n = 2$ or 3, then these lemmas must be used with special care, since the extra conditions (c) and (d) on λ_1 and λ_2 in Lemma 2.5 must be satisfied. Before the proof of the main theorem, we also need the following series of results.

Theorem 2.9 ([5]). *Let F (or F_1) be a 2-regular subgraph of K_n (or $2K_n$, respectively). There exists a* 4*-cycle system:*

(a) *of $K_n - E(F)$ if and only if n is odd and* 4 *divides $|E(K_n)\backslash E(F)|$, and*
(b) *of $2K_n - E(F_1)$ if and only if* 4 *divides $|E(2K_n) \setminus E(F_1)|$.*

We will use particular applications of Theorem 2.9.

Corollary 2.10. *Let $\ell \in \{3, 5, 6\}$ and $s, t \geq 0$. There exists a partial* 4*-cycle system of K_n whose leave consists of $s + t$ vertex-disjoint cycles, s of length ℓ and t of length* 4*, providing $\ell s + 4t \leq n$, and*

(a) *if $\ell = 3$ then $n \equiv 3 \pmod 8$ and $s \equiv 1 \pmod 4$,*
(b) *if $\ell = 5$ then $n \equiv 7 \pmod 8$ and $s \equiv 1 \pmod 4$, and*
(c) *if $\ell = 6$ then $n \equiv 5 \pmod 8$ and s is odd.*

Proof. This follows immediately from Theorem 2.9(a). □

Sotteau [17] proved a more general result than the following, but this is all we need.

Lemma 2.11. *There exists a* 4*-cycle system of $\lambda K_{a,b}$ if and only if $a, b \geq 2$; if λ is odd then a and b are even, and* 4 *divides λab.*

Lemma 2.12. *There exists a* 4*-cycle system of K_9 that contains two vertex-disjoint* 4*-cycles.*

Proof. The following set of 4-cycles forms a 4-cycle system of K_9 that contains two vertex-disjoint 4-cycles:

$$\{(1,2,3,4),(5,6,7,8),(1,5,7,3),(2,6,8,4),$$
$$(2,5,9,7),(3,6,9,8),(1,6,4,7),(1,8,2,9),(3,5,4,9)\}.$$ □

A *bow-tie* is a graph isomorphic to the graph formed by the union of the two cycles $(1, 2, 3)$ and $(3, 4, 5)$; denote this graph by $(1, 2, 3; 3, 4, 5)$.

The following is used to prove Lemma 2.14.

Lemma 2.13. *There exists a* p4CS *of $2K_{20}$ with leave consisting of* 4 *vertex-disjoint bow-ties.*

Proof. By Lemma 2.4 there exists a p4CS C_1 of $K_{4(5)}$ with leave a 1-factor F_1 and vertex parts V_i $(1 \leq i \leq 4)$, and by Lemma 2.3 there exists a p4CS C_2 of K_{20} with leave a 1-factor F_2 on the vertex set $\bigcup_{i=1}^{4} V_i$; it is easy to name these so that $F_1 \cup F_2$ is the union of five 4-cycles, which we place in C_3. Finally, for $1 \leq i \leq 4$ let c_i be a 4-cycle defined on 4 vertices in V_i. Then $C_1 \cup C_2 \cup C_3 \cup \{c_i \mid 1 \leq i \leq 4\}$ is the required p4CS of $2K_{20}$. □

The next three lemmas will be used in Section 3.

Lemma 2.14. *Let n be even, $n > 2$. Then there exists a* p4CS *of $2K_n$ whose leave is the vertex-disjoint union of s bow-ties and t 4-cycles if and only if $5s+4t \le n$, $s \equiv n/2 \pmod 2$ and $s, t \ge 0$.*

Proof. The necessity is clear, and the sufficiency can be proved as follows. If $n = 4$, 6 or 8, then the result follows from Lemma 2.1, Lemma 2.3 and Theorem 2.1(b), respectively.

The remainder of the proof is by induction on n, with $n \ge 10$. If $t \ge 2$, then by Lemma 2.12 there exists a p4CS $(V_1 = \{\infty\} \cup \mathbb{Z}_8, C_1)$ of $2K_9$ with leave consisting of two 4-cycles $(0, 1, 2, 3)$ and $(4, 5, 6, 7)$. By induction there exists a p4CS $(V_2 = \{\infty\} \cup (\mathbb{Z}_{n-1} \backslash \mathbb{Z}_8), C_2)$ of $2K_{n-8}$ with leave consisting of the vertex-disjoint union of s bow-ties and $t-2$ 4-cycles. The result follows by applying Lemma 2.11 to $2K_{n-9,8}$ with bipartition V_1 and V_2 of the vertex set, then taking the union of the three sets of cycles.

If $5s+4t \le n-1$ then the result follows similarly by using a p4CS $(V_1 = \mathbb{Z}_6, C_1)$ of $2K_6$ with leave being a bow-tie, a p4CS $(V_2 = (\mathbb{Z}_n \backslash \mathbb{Z}_6), C_2)$ of $2K_{n-6}$ with leave $s-1$ bow-ties and t 4-cycles, and a 4-cycle system of $2K_{n-6,6}$ with bipartition V_1 and V_2 of the vertex set.

Finally, if $5s + 4t = n$ and $t \le 1$ then $s \equiv n \equiv 0 \pmod 4$. So the result follows by combining a p4CS of $2K_{20}$ with leave 4 vertex-disjoint bow-ties (see Lemma 2.13), then, if $n > 20$, by adding a p4CS of $2K_{n-20}$ with leave $s-4$ bow-ties and t 4-cycles, and a 4CS of $2K_{n-20,20}$ (see Lemma 2.11). □

Lemma 2.15. *Let $n \equiv 3$ or $7 \pmod 8$ and $m = 2$. Then there exists a* GDD$(n; m)$ *of index* $(1, 2)$.

Proof. Let $n \equiv 3 \pmod 8$. Let the vertex set of the two parts be $\mathbb{Z}_n \times \{0\}$ and $\mathbb{Z}_n \times \{1\}$. For $0 \le i \le 1$, let B_i be the set of 4-cycles in a p4CS of K_n on the vertex set $\mathbb{Z}_n \times \{i\}$ with leave the 3-cycle $((0, i), (1, i), (2, i))$ (see Lemma 2.3), and let B_2 be the set of 4-cycles in a p4CS of $2K_{2(n)}$ with leave the 6-cycle $((0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1))$ (see Lemma 2.4). Finally, let

$$B_4 = \{\big((0,0),(1,0),(1,1),(0,1)\big),\\ \big((0,0),(2,0),(1,1),(2,1)\big),\ \big((1,0),(2,0),(2,1),(0,1)\big)\}.$$

Then $\bigcup_{i=0}^{3} B_i$ forms the required GDD$(n; 2)$ of index $(1, 2)$.

Let $n \equiv 7 \pmod 8$. The proof is similar to the above case: let B_i have leave

$$\big((0,1),(1,i),(2,i),(3,i),(4,i)\big)$$

for $1 \le i \le 2$ (see Lemma 2.3); let B_2 have leave the bow-tie

$$\big((0,0),(0,1),(2,0);(2,0),(2,1),(4,1)\big);$$

and let

$$B_4 = \{\big((0,0),(1,0),(2,0),(0,1)\big),\ \big((0,0),(2,0),(3,0),(4,0)\big),\\ \big((0,1),(1,1),(2,1),(4,1)\big),\ \big((2,0),(2,1),(3,1),(4,1)\big)\}.$$

Then proceed as above. □

For any cycle $C_y = (0, 1, \ldots, y-1)$ with vertex set $\mathbb{Z}_y$, let $K_x \circ C_y$ be the graph with vertex set $\mathbb{Z}_x \times \mathbb{Z}_y$, and edge set $\{\{(b,a),(d,a+1)\} \mid a \in \mathbb{Z}_y, \{b,d\} \subseteq \mathbb{Z}_x \text{ (possibly } b = d)\}$, reducing the sum $a+1$ modulo y.

Lemma 2.16. *There exists a* 4CS *of* $K_n \circ C_4$.

Proof. $B = \{((i,0),(j,1),(i,2),(j,3)) \mid i,j \in \mathbb{Z}_n\}$ forms the required 4CS. □

3. The case $n \geq 4$

Since a GDD$(n;m)$ of index $(\lambda_1, 0)$ is simply m disjoint copies of a 4-cycle system of $\lambda_1 K_n$, their existence is settled by Lemma 2.3. Therefore we can assume that $\lambda_2 > 0$ throughout the rest of the paper.

In this section we provide a proof of Theorem 1.2 in the case where $n \geq 4$; so assume throughout Section 3 that $n \geq 4$.

If $\lambda_1' \geq 4$ or $\lambda_2' > 4$, then let $\lambda_1' = \lambda_1 + 4t_1$ or $\lambda_2' = \lambda_2 + 4t_2$ where $0 \leq \lambda_1 \leq 3$ and $1 \leq \lambda_2 \leq 4$. By Lemmas 2.1, 2.2, 2.6 and 2.8, the existence of GDD$(n;m)$ of index (λ_1, λ_2) implies the existence of GDD$(n;m)$ of index (λ_1', λ_2'). Therefore it suffices to construct a GDD$(n;m)$ of index (λ_1, λ_2) with $0 \leq \lambda_1 \leq 3$ and $1 \leq \lambda_2 \leq 4$. We shall consider several cases.

Case 1: $\lambda_1 = 0$.
This is a direct result of Lemma 2.2.

Case 2: $\lambda_1 = \lambda_2 \neq 0$.
This is a direct result of Lemma 2.1.

Case 3: $\lambda_1 \lambda_2 = 3$.
If $(n;m) \equiv (1;1)$ or $(5;5)$, then $nm \equiv 1 \pmod 8$. Therefore a GDD$(mn;1)$ of index $(1,0)$ exists. By the fact that a GDD$(n;m)$ of index $(0,2)$ and a GDD$(n;1)$ of index $(2,0)$ exist (by Lemmas 2.2 and 2.1), we conclude by Lemma 2.7 that GDD$(n;m)$s of index $(1,3)$ and $(3,1)$ exist respectively.

If $(n;m) \equiv (3;7)$ or $(7;3)$, then $mn \equiv 5 \pmod 8$. By Lemma 2.3, K_{mn} can be packed with 4-cycles such that the leave is two 3-cycles: $(1,2,3)$ and $(4,5,6)$. By Lemma 2.4, $2K_{m(n)}$ can be packed with 4-cycles such that the minimum leave is a C_6: $(1,4,2,5,3,6)$. By combining the two leaves together we obtain three 4-cycles: $(1,4,5,2)$, $(1,3,5,6)$ and $(2,3,6,4)$. Now, the two packings together with these three 4-cycles give the GDD$(n;m)$ of index $(1,3)$. On the other hand, by Lemma 2.3, $2K_n$ can be packed with 4-cycles where the leave is a 6-cycle; place such a packing on each group on the vertex set $\mathbb{Z}_n \times \{i\}$ with leave

$$\big((0,i),(1,i),(2,i),(3,i),(4,i),(5,i)\big),$$

for each $i \in \mathbb{Z}_m$. Also, by Corollary 2.10 (c(i)), K_{mn} can be packed with 4-cycles with leave a disjoint union of m 6-cycles; place such a packing on the vertex set $\mathbb{Z}_n \times \mathbb{Z}_m$, with leave the 6-cycles

$$\big((0,i),(3,i),(5,i),(1,i),(4,i),(2,i)\big),$$

for each $i \in \mathbb{Z}_m$. Then a GDD$(n;m)$ of index $(3,1)$ can be obtained by adding the 4-cycles

$$\big((0,i),(1,i),(2,i),(3,i)\big), \quad \big((0,i),(2,i),(4,i),(5,i)\big) \quad \text{and} \quad \big((1,i),(4,i),(3,i),(5,i)\big),$$

for each $i \in \mathbb{Z}_m$, to these two packings of 4-cycles.

Case 4: $\lambda_1 = 1$ and $\lambda_2 = 2$.
First, if $n \equiv 1 \pmod 8$, then the construction follows by Lemma 2.6, by putting a copy of a GDD$(n;1)$ of index $(1,0)$ on each group together with a GDD$(n;m)$ of index $(0,2)$ (see Tables 1 and 2).

For $(n;m) \equiv (3,7;\text{ even})$, let $m = 2k$, $k \geq 1$, and $V = \mathbb{Z}_n \times \mathbb{Z}_m$. Let $(\mathbb{Z}_m, C)$ be a maximum packing of K_m with 4-cycles (see Lemma 2.3); so the leave is a 1-factor, F. Now, for each 4-cycle $c \in C$, let $(V(c), B_c)$ be a 4-cycle system of the graph $K_n \circ C_4$ on the vertex set $\mathbb{Z}_n \times V(c)$ (see Lemma 2.16). For each edge in F, say $\{x,y\}$, use Lemma 2.15 to obtain a GDD$(n;2)$ $(\mathbb{Z}_n \times \{x,y\}, B)$ of index $(1,2)$. Then, by taking two copies of B_c for each $c \in C$, and one copy of B, we have a GDD$(n;m)$ of index $(1,2)$.

Finally, consider $(n;m) \equiv (5;0,3,4,7)$. First, suppose $(n;m) \equiv (5;0,4)$. Since $K_{m(n)}$ can be packed with 4-cycles such that the leave is a 1-factor (see Lemma 2.4) and K_{mn} can also be packed by 4-cycles with leave a 1-factor (see Lemma 2.3), the GDD$(n;m)$ of index $(1,2)$ can easily be obtained by choosing the two 1-factors so that together they form vertex-disjoint 4-cycles, and then combining these 4-cycles with the two maximum packings.

Now suppose $(n;m) \equiv (5;3,7)$. By Lemma 2.4, if $m \equiv 3 \pmod 8$, then $K_{m(n)}$ can be packed with 4-cycles such that the leave is a C_3, say (a,b,c). By Lemma 2.3 the maximum packing of K_{mn} has a leave C_5, say (a,d,e,b,f). Hence the proof follows by adding the two 4-cycles (a,b,e,d) and (a,c,b,f) to these two maximum packings. On the other hand, if $m \equiv 7 \pmod 8$ then $K_{m(n)}$ can be packed with 4-cycles such that the leave is a C_5 and K_{mn} has a maximum packing with leave a C_3, so the proof follows similarly.

Case 5: $\lambda_1 = 2$ and $\lambda_2 = 1$.
If $(n;m) \equiv (0,4;\text{ any})$ or $(1,5;1)$, then the result follows from the existence of a GDD$(n;m)$ of index $(2,0)$ (see Lemma 2.1) and one of index $(0,1)$ (see Lemma 2.2).

If $(n;m) \equiv (2,6;\text{ even})$ then, for each $j \in \mathbb{Z}_m$, let $(\mathbb{Z}_n \times \{j\}, B_j)$ be a 4-cycle packing of $2K_n$ with leave the two copies $((0,j),(1,j),(2,j))$ and $((0,j),(1,j),(3,j))$ of C_3 (see Table 2). Also, for $0 \leq j_1 < j_2 < m$, let $(\mathbb{Z}_n \times \{j_1,j_2\}, B_{\{j_1,j_2\}})$ be a 4-cycle system of $K_{2(n)}$ containing the 4-cycle

$$c(j_1,j_2) = \big((0,j_1),(0,j_2),(1,j_1),(1,j_2)\big).$$

Replacing $c(2j, 2j+1)$ with the 4-cycles

$$\big((0,2j),(1,2j),(1,2j+1),(0,2j+1)\big) \quad \text{and} \quad \big((0,2j),(1,2j),(0,2j+1),(1,2j+1)\big),$$

for each $j \in \mathbb{Z}_{m/2}$, and then adding the 4-cycle $((0,j),(2,j),(1,j),(3,j))$, for each $j \in \mathbb{Z}_m$, completes the required GDD$(n;m)$ of index $(2,1)$.

Finally, let $(n;m) = (3,7;5)$. If $(n;m) = (3;5)$, then by Corollary 2.10(b) K_{nm} can be packed with 4-cycles such that the leave is a disjoint union of m C_5s; let these copies of C_5 be $((0,j),(1,j),(2,j),(3,j),(4,j))$, for each $j \in \mathbb{Z}_m$. Furthermore, K_n can be packed with 4-cycles such that the leave is a C_3; for each $j \in \mathbb{Z}_m$, place such a packing on the vertices $\mathbb{Z}_n \times \{j\}$ and let the copy of C_3 be $((0,j),(3,j),(5,j))$. Then, adding the 4-cycles $((0,j),(1,j),(2,j),(3,j))$ and $((0,j),(4,j),(3,j),(5,j))$ to these packings provides the required GDD.

If $(n;m) \equiv (7;5)$, the proof is essentially the same as the previous case except that Corollary 2.10(a) is used instead of Corollary 2.10(b), and the leave in each 4-cycle packing of K_n is a C_5.

Case 6: $\lambda_1\lambda_2 = 6$.
Note that our necessary conditions for a GDD$(n;m)$ of index $(2,3)$ or $(3,2)$ are exactly the same as for one of index $(2,1)$ or $(1,2)$, respectively, and the existence of these latter GDDs has been established in Cases 4 and 5, respectively. Also, if there exists a GDD$(n;m)$ of index $(2,1)$, then one exists of index $(0,2)$ (see Lemma 2.2), so together these form the required GDD$(n;m)$ of index $(2,3)$. Similarly, unless $(n;m) \equiv (3,7;\ \text{even})$, a GDD$(n;m)$ of index $(3,2)$ can be obtained by combining one of index $(1,2)$ with one of index $(2,0)$ (see Lemma 2.1). Finally, consider $(n;m) \equiv (3,7;\text{even})$. By Theorem 2.9(b) and Lemma 2.14, $2K_{mn}$ can be packed with 4-cycles such that the leaves are $m/2$ disjoint C_6s and bow-ties when $n \equiv 3$ and 7 (mod 8), respectively. Also, K_n can be packed with 4-cycles such that the leaves are a C_3 and a C_5 when $n \equiv 3$ and 7 (mod 8), respectively. Combining two C_3s and one C_6, and two C_5s and one bow-tie (in the latter case, take the 4-cycles

$$\begin{aligned}&\big((0,2i),(1,2i),(2,2i),(0,2i+1)\big),\\&\big((0,2i),(2,2i),(3,2i),(4,2i)\big),\\&\big((0,2i+1),(1,2i+1),(2,2i+1),(4,2i+1)\big),\quad \text{and}\\&\big((2,2i),(2,2i+1),(3,2i+1),(4,2i+1)\big)\end{aligned}$$

for each $i \in \mathbb{Z}_{m/2}$), respectively, we have the desired constructions.

Case 7: $\lambda_2 = 4$.
The result follows from Lemmas 2.1 and 2.2 if $\lambda_1 = 0$.

Suppose $\lambda_1 = 2$. If $(n;m) \equiv (1,4,5,0;\ \text{any})$, then m copies of a GDD$(n;1)$ of index $(2,0)$, one on each group, together with a GDD$(n;m)$ of index $(0,4)$ give the construction. If $(n;m) \equiv (3,7;\ \text{even})$, then two GDD$(n;m)$s of index $(1,2)$ give the construction. If $(n;m) \equiv (2,6;\ \text{even})$, then $nm \equiv 0$ (mod 8), so we can take a GDD$(n;m)$ of index $(2,2)$ together with a GDD$(n;m)$ of index $(0,2)$ to obtain the required GDD.

Next, let $\lambda_1 = 1$. The case $(n;m) \equiv (1;\ \text{any})$ follows from GDD$(n;m)$s of index $(1,0)$ and index $(0,4)$. The cases $(n;m) \equiv (3,7;4,0)$ and $(n;m) \equiv (5;0,4)$ follow from the existence of GDD$(n;m)$s of index $(1,2)$ and index $(0,2)$. Hence, it remains to consider $(n;m) \equiv (5;2,6)$.

If $m = 2$ then we can obtain a solution by using a partial 4-cycle system of K_{mn} with leave a 1-factor (see Lemma 2.3), and three partial 4-cycle systems of $K_{m(n)}$ each with leave a 1-factor (see Lemma 2.4), then easily combining the leaves into 4-cycles. The result then follows for all m by using $m/2$ copies of the above solution together with a 4-cycle system of $4K_{(m/2)(2n)}$ (since $4K_{(m/2)(2n)}$ is the edge-disjoint union of $\binom{m/2}{2}$ copies of $4K_{2n,2n}$, we can simply use Lemma 2.11).

Finally, the case $\lambda_1 = 3$ is similar and is left to the reader.

This completes the proof of Theorem 1.2 for the case $n \geq 4$.

4. The case $n = 2$

From Lemma 2.5 we have that, if there exists a GDD$(2; m)$ of index (λ_1, λ_2), then:

- if m is even then $\lambda_1 \equiv 0 \pmod 2$, and
- if m is odd then $\lambda_1 \equiv 0 \pmod 4$.

From Lemma 2.5(c) we also have that $\lambda_1/2 \leq (m-1)\lambda_2$.

Partition the edges of $\lambda_2 K_m$ into two sets E_1 and E_2 so that E_1 induces a regular graph of degree $\lambda_1/2$ (take $\lambda_1/2 \leq \lambda_2(m-1)$ 1-factors in a 1-factorization of $\lambda_2 K_m$ if m is even, and $\lambda_1/4 \leq \lambda_2(m-1)/2$ Hamilton cycles in a Hamilton decomposition of $\lambda_2 K_m$ if m is odd).

The required GDD$(2, m)$ of index (λ_1, λ_2) on the vertex set $\mathbb{Z}_2 \times \mathbb{Z}_m$ is formed by the 4-cycles in

$$\{((0,i),(1,i),(1,j),(0,j)),\ ((0,i),(1,i),(0,j),(1,j)),\\ ((0,k),(0,\ell),(1,k),(1,\ell)) \mid \{i,j\} \in E_1, \{k,\ell\} \in E_2\}. \qquad \square$$

5. The case $n = 3$

By Table 5 we have the smaller Table 6, listing necessary conditions on m derived from Lemma 2.5$(a-b)$ for the existence of a GDD$(3; m)$ of index (λ_1, λ_2).

Table 6 Necessary conditions on m (mod 8) and λ_1, λ_2 with $0 \leq \lambda_1 \leq 4$ and $1 \leq \lambda_2 \leq 4$ for the existence of a GDD$(3, m)$ of index (λ_1, λ_2)

$\lambda_1 \backslash \lambda_2$	1	2	3	4
0	1	0, 1, 4, 5	1	any
1	3	even	7	4, 0
2	5	0, 3, 4, 7	5	even
3	7	even	3	4, 0
4	1	0, 1, 4, 5	1	any

In view of Lemma 2.5(d), it certainly helps to consider separately the cases where λ_2 is even or odd.

5.1. λ_2 is even

Let $G \sim H$ be the graph with vertex set $V(G) \times V(H)$ and $E(G \sim H) = \{\{(a,b),(c,d)\} \mid \{a,c\} \in E(G)$ and $b = d$, or $\{b,d\} \in E(H)\}$. We begin with some building blocks.

Lemma 5.1. *For any $m \equiv 0 \pmod 4$ there exists a 1-factorization $\{F_1, \ldots, F_{m-1}\}$ of K_m such that $F_1 \cup F_2$ is the union of vertex-disjoint 4-cycles.*

Proof. Let

$$\begin{aligned} F_1 &= \{\{4i, 4i+1\}, \{4i+2, 4i+3\} \mid i \in \mathbb{Z}_{m/4}\}, \\ F_2 &= \{\{4i, 4i+3\}, \{4i+1, 4i+2\} \mid i \in \mathbb{Z}_{m/4}\}, \\ F_3 &= \{\{4i, 4i+2\}, \{4i+1, 4i+3\} \mid i \in \mathbb{Z}_{m/4}\}, \end{aligned}$$

and then add a 1-factorization of $K_{(m/4)(4)}$ (this is easy to find, or use [8]). □

Lemma 5.2. *For any cycle c there exists a 4-cycle system of $2K_3 \sim 2c$.*

Proof. Let $c = (v_0, v_1, \ldots, v_x)$. For each $i \in \mathbb{Z}_x$, let

$$\begin{aligned} B_i = \{&\big((j,i), (j+1, i+1), (j+1, i), (j, i+1)\big), \\ &\big((j,i), (j+1, i), (j+2, i), (j+1, i+1)\big) \mid j \in \mathbb{Z}_3\}, \end{aligned}$$

reducing the first component modulo 3, the second modulo x. Then

$$\left(\mathbb{Z}_3 \times \mathbb{Z}_x, \bigcup_{i \in \mathbb{Z}_x} B_i\right)$$

is a 4-cycle system of $2K_3 \sim 2c$. □

Lemma 5.3. *If c is a cycle of length 4, then there exists a 4-cycle system of $K_3 \sim c, 2K_3 \sim c$ $3K_3 \sim 2c$, and of $5K_3 \sim 2c$.*

Proof.

$$\begin{aligned} \big(\mathbb{Z}_3 \times \mathbb{Z}_4, \{&\big((i,0), (i+1,3), (i,3), (i+1,0)\big),\ \big((0,1), (1,1), (2,1), (1,0)\big), \\ &\big((0,1), (2,1), (0,2), (2,2)\big),\ \big((0,2), (1,2), (2,2), (1,3)\big), \\ &\big((0,0), (0,1), (1,2), (0,3)\big),\ \big((2,0), (2,1), (1,2), (2,3)\big), \\ &\big((1,0), (1,1), (1,2), (1,3)\big),\ \big((0,0), (1,1), (2,2), (2,1)\big), \\ &\big((2,0), (1,1), (0,2), (0,1)\big),\ \big((0,2), (2,3), (2,2), (0,3)\big) \mid i \in \mathbb{Z}_3\}\big) \end{aligned}$$

is a 4-cycle system of $K_3 \sim c$. Further,

$$\begin{aligned} \big(\mathbb{Z}_3 \times \mathbb{Z}_4, \{&\big((i,j), (i+1,j), (i+2,j), (i+1,j+1)\big), \\ &\big((i,0), (i,1), (i,2), (i,3)\big) \mid i \in \mathbb{Z}_3, j \in \mathbb{Z}_4\}\big), \end{aligned}$$

reducing the first component modulo 3 and the second modulo 4, is a 4-cycle system of $2K_3 \sim c$.

Combining 4-cycle systems of $K_3 \sim c$ and $2K_3 \sim c$ produces a 4-cycle system of $3K_3 \sim 2c$.

Let

$$\begin{aligned} B_1(a,b) &= \{\big((i,a), (i+1,a), (i+2,a), (i+2,b)\big) \mid i \in \mathbb{Z}_3\}, \quad \text{and} \\ B_2(a,b) &= \{\big((i,a), (i+1,a), (i+2,a), (i+1,b)\big) \mid i \in \mathbb{Z}_3\}, \end{aligned}$$

reducing the sums modulo 3. Let $(\mathbb{Z}_3 \times \{0,3\}, B_3)$ be a GDD(3; 2) of index (3, 2). Then

$$\big(\mathbb{Z}_3 \times \mathbb{Z}_4, B_1(0,1) \cup B_1(1,0) \cup B_1(2,3) \cup B_1(3,2) \cup B_2(1,0) \cup B_2(2,1) \cup B_3 \cup \\ \{((i,1),(i+1,1),(i+1,2),(i,2)),\ ((i,1),(i+1,2),(i,3),(i+2,2)) \mid i \in \mathbb{Z}_3\}\big)$$

is a 4-cycle system of $5K_3 \sim 2c$. □

Lemma 5.4. *Let G be a connected graph with an even number of edges. Then there exists a 4-cycle system of $K_3^c \sim 2G$ (where K_3^c denotes the complement of K_3).*

Proof. Since G is connected with an even number of edges, there exists a partition P of $E(G)$ into paths of length 2. For each $p = (v_0, v_1, v_2) \in P$, let

$$B(p) = \{((j,v_i),(j+1,v_{i+1}),(j+1,v_i),(j,v_{i+1})), \\ ((j+1,v_0),(j,v_1),(j+1,v_2),(j+2,v_1)) \mid j \in \mathbb{Z}_3, i \in \{0,1\}\},$$

reducing the sum in the first coordinate modulo 3. Then

$$\left(\mathbb{Z}_3 \times V(G), \bigcup_{p \in P} B(p)\right)$$

is a 4-cycle system of $K_3^c \sim 2G$. □

Lemma 5.5. *There exists a* GDD(3; 2) *of index* $(\lambda_1, 2)$ *for each* $\lambda_1 \in \{1,3\}$.

Proof.

$$\big(\mathbb{Z}_3 \times \mathbb{Z}_2, \{((i,j),(i+1,j),(i+2,j),(i+1,j+1)), \\ ((i,0),(i+1,0),(i+1,1),(i,1)) \mid i \in \mathbb{Z}_3, j \in \mathbb{Z}_2\}\big),$$

reducing the first and second components modulo 3 and 2, respectively, is a GDD(3; 2) of index (3, 2). Further,

$$\big(\mathbb{Z}_3 \times \mathbb{Z}_2, \{((i,0),(i+1,0),(i,1),(i+1,1)), \\ ((i,0),(i,1),(i+1,0),(i+1,1)) \mid i \in \mathbb{Z}_3\}\big)$$

is a GDD(3; 2) of index (1, 2). □

Proposition 5.6. *The necessary conditions in Lemma 2.5 for the existence of a* GDD(3; *m*) *of index* (λ_1, λ_2) *are sufficient if* λ_2 *is even.*

Proof. The proof consists of three cases.

Case 1: $m \equiv 0 \pmod 4$; either $\lambda_2 \equiv 0 \pmod 4$ and $\lambda_1 \equiv 1 \pmod 2$ or $\lambda_2 \equiv 2 \pmod 4$ and $\lambda_1 \equiv 0 \pmod 2$.
Let $F_1, \ldots, F_z$ with $z = \lambda_2(m-1)/2 \geq 3$ be a 1-factorization of $(\lambda_2/2)K_m$ in which $F_1 \cup F_2$ is the union of vertex-disjoint 4-cycles (see Lemma 5.1). Choose k so that $\lambda_1 - 3k = \epsilon \in \{0,1,2\}$. If $\lambda_2 \equiv 2 \pmod 4$ then z is odd and $k \equiv \epsilon \pmod 2$. In either case, $z - (k + \epsilon + 1)$ is even. By Lemma 2.5(d), $k + \epsilon/3 \leq z$, so by the parity of k, ϵ and z, we have that if ϵ is even then $k + 1 \leq z$ and if $\epsilon = 1$ then $k + 2 \leq z$.

If $\lambda_1 = 2$ then the result follows by combining a 4-cycle system of $2K_{3m}$ (see Lemma 2.1) and a GDD$(3;m)$ of index $(0, \lambda_2 - 2)$ (see Lemma 2.2); so we can assume $\lambda_1 \geq 4$ and so $k \geq 1$. For each 4-cycle c in $F_1 \cup F_2$, let $(\mathbb{Z}_3 \times V(c), B(c))$ be a 4-cycle system of $3K_3 \sim 2c$ if $\epsilon \in \{0,1\}$ and of $5K_3 \sim 2c$ if $\epsilon = 2$ (see Lemma 5.3), and let $B(c) \subseteq B$. For $3 \leq i \leq k+1$ and for each $e \in F_i$, let $(\mathbb{Z}_3 \times e, B(e))$ be a GDD$(3;2)$ of index $(3,2)$ (see Lemma 5.5), and let $B(e) \subseteq B$. If $\epsilon = 1$, then for each $e \in F_{k+2}$ let $(\mathbb{Z}_3 \times e, B(e))$ be a GDD$(3;2)$ of index $(1,2)$ (see Lemma 5.5), and let $B(e) \subseteq B$. Finally, the number of remaining 1-factors $F_{k+2+\alpha}, \ldots, F_z$ with $\alpha = 0$ or 1, if $\epsilon \in \{0,2\}$ or $\epsilon = 1$, respectively, is even, so arbitrarily pair off these 1-factors. Then, for each cycle c in the union of each pair, let $(\mathbb{Z}_3 \times V(c), B(c))$ be a 4-cycle system of $K_3^c \sim 2c$ (see Lemma 5.4), and let $B(c) \subseteq B$. Then $(\mathbb{Z}_3 \times \mathbb{Z}_m, B)$ is a GDD$(3;m)$ of index (λ_1, λ_2).

Case 2: $m \equiv 0 \pmod 2$; either $\lambda_2 \equiv 0 \pmod 4$ and $\lambda_1 \equiv 0 \pmod 2$, or $\lambda_2 \equiv 2 \pmod 4$ and $\lambda_1 \equiv 1 \pmod 2$.
Let $F_1, \ldots, F_z$ with $z = \lambda_2(m-1)/2$ be a 1-factorization of $(\lambda_2/2)K_m$. Choose k so that $\lambda_1 - 3k = \epsilon \in \{0,1,2\}$. If $\lambda_2 \equiv 2 \pmod 4$, then z is odd and $k \equiv \epsilon + 1 \pmod 2$; if $\lambda_2 \equiv 0 \pmod 4$, then z is even and $k \equiv \epsilon \pmod 2$. In either case $z - k + \epsilon$ is even. By Lemma 2.5(d) $k + \epsilon/3 \leq z$, so by the parity of k, ϵ and z we have that $z \geq k + \epsilon$.

For $1 \leq i \leq k$ and for each $e = \{u,v\} \in F_i$, let $(\mathbb{Z}_3 \times \{u,v\}, B(e))$ be a GDD$(3,2)$ of index $(3,2)$ (see Lemma 5.5) and let $B(e) \subseteq B$. If $\epsilon = 1$, then for each $e \in F_{k+1}$ let $(\mathbb{Z}_3 \times e, B(e))$ be a GDD$(3,2)$ of index $(1,2)$ (see Lemma 5.5), and let $B(e) \subseteq B$. If $\epsilon = 2$, then for each cycle c in $F_{k+1} \cup F_{k+2}$ let $(\mathbb{Z}_3 \times V(c), B(c))$ be a 4-cycle system of $2K_3 \sim 2c$ (see Lemma 5.2), and let $B(c) \subseteq B$. Finally, since $z - k + \epsilon$ is even, $F_{k+\epsilon+1}, \ldots, F_z$ can be arbitrarily paired, and for each cycle c in the union of each pair let $(\mathbb{Z}_3 \times V(c), B(c))$ be a 4-cycle system of $K_3^c \sim 2c$ (see Lemma 5.4), and let $B(c) \subseteq B$. Then $(\mathbb{Z}_3 \times \mathbb{Z}_m, B)$ is a GDD$(3;m)$ of index (λ_1, λ_2).

Case 3: either $\lambda_2 \equiv 2 \pmod 4$, $\lambda_1 \equiv 0 \pmod 2$ and $m \equiv \lambda_1 + 1 \pmod 4$, or $\lambda_2 \equiv 0 \pmod 4$, $\lambda_1 \equiv 0 \pmod 4$ and $m \equiv 1 \pmod 2$.
Let $H_1, \ldots, H_z$ with $z = \lambda_2(m-1)/4$ be a Hamilton decomposition of $(\lambda_2/2)K_m$. Choose k so that $\lambda_1 - 6k = \epsilon \in \{0,2,4\}$. Then $z - k \equiv \epsilon/2 \pmod 2$ in both cases. By Lemma 2.5(d), $k + \epsilon/6 \leq z$, so by the parity of k, ϵ and z we have $z \geq k + \epsilon/2$.

For $1 \leq i \leq k$ and for each $e \in E(H_i)$ let $(\mathbb{Z}_3 \times e, B(e))$ be a GDD$(3;2)$ of index $(3,2)$, and let $B(e) \subseteq B$. For $k+1 \leq i \leq k + \epsilon/2$ let $(\mathbb{Z}_3 \times \mathbb{Z}_m, B(H_i))$ be a 4-cycle system of $2K_3 \sim 2H_i$, and let $B(H_i) \subseteq B$. Finally, since $z - (k + \epsilon/2)$ is even, if G is the union of $H_{k+\epsilon/2+1}, \ldots, H_z$ then G is connected and $|E(G)|$ is even, so let $(\mathbb{Z}_3 \times \mathbb{Z}_m, B(G))$ be a 4-cycle system of $K_3^c \sim 2G$ (see Lemma 5.4), and let $B(G) \subseteq B$. Then $(\mathbb{Z}_3 \times \mathbb{Z}_m, B)$ is a GDD$(3;m)$ of index (λ_1, λ_2). □

5.2. λ_2 is odd

In view of the following result, it will suffice to let $\lambda_2 = 1$.

Proposition 5.7. *If the necessary conditions in Lemma 2.5 are sufficient for the existence of a* GDD$(3;m)$ *of index* $(\lambda_1, 1)$*, then they are sufficient for the existence of a* GDD$(3;m)$ *of index* (λ_1, λ_2) *for all odd* λ_2.

Proof. Suppose $\lambda_2 \geq 3$ is odd. We construct a GDD$(3;m)$ of index (λ_1, λ_2) by induction, assuming conditions $(a-d)$ of Lemma 2.5 are sufficient for the existence of a GDD$(3;m)$ of index (λ_1', λ_2') for all odd $\lambda_2' < \lambda_2$. In each case we form the required GDD by combining two GDD$(3;m)$s, one of index (λ_1', λ_2'), the other of index $(\lambda_1'', \lambda_2'')$.

(1) Suppose $\lambda_1 \geq 3(m-1)$. Let

$$(\lambda_1', \lambda_2') = (\lambda_1 - 3(m-1), \lambda_2 - 2) \quad \text{and} \quad (\lambda_1'', \lambda_2'') = (3(m-1), 2).$$

The first GDD exists by induction since we know $\lambda_1 \leq 3(m-1)\lambda_2/2 - (m-1)/9$, so $\lambda - 3(m-1) \leq 3(m-1)(\lambda_2 - 2)/2 - (m-1)/9$, and the necessary conditions $(a-b)$ of Lemma 2.5 are easily seen to be satisfied (use Table 6).

(2) Suppose $\lambda_1 < 3(m-1)$ and is odd. Let $x \in \{1,3\}$ with $x \equiv \lambda_1\lambda_2 \pmod 4$. If $\lambda_1 > 1$, then let $(\lambda_1', \lambda_2') = (x, 1)$ and $(\lambda_1'', \lambda_2'') = (\lambda_1 - x, \lambda_2 - 1)$. If $\lambda_1 = 1$, then let $(\lambda_1', \lambda_2') = (1, x)$ and $(\lambda_1'', \lambda_2'') = (0, 4\lfloor \lambda_2/4 \rfloor)$.

(3) Suppose $\lambda_1 < 3(m-1)$ and is even. If $\lambda_2 \geq 5$, then let $(\lambda_1', \lambda_2') = (0, 2)$ and $(\lambda_1'', \lambda_2'') = (\lambda_1, \lambda_2 - 2)$. If $\lambda_2 = 3$, then let $x \in \{2,4\}$ with $x \equiv \lambda_1 \pmod 4$, and define $(\lambda_1', \lambda_2') = (x, 1)$ and $(\lambda_1'', \lambda_2'') = (\lambda_1 - x, 2)$ (notice that if $\lambda_1' = 4$ and $\lambda_2' = 1$ then Lemma 2.5(d) is satisfied, since then $m \equiv 1 \pmod 8$, so $m \geq 9$). □

We begin by first considering $\lambda_1 \equiv 0 \pmod 4$, which is then used to solve the remaining three cases.

5.2.1. $\lambda_1 \equiv 0 \pmod 4$ and $\lambda_2 = 1$

For any $S \subseteq \{1, 2, \ldots, \lfloor (m-1)/2 \rfloor\}$, let $D_m(S)$ be the graph

$$\big(\mathbb{Z}_m, \{\{u, v\} \mid u - v \equiv s \text{ or } -s \pmod m \text{ for some } s \in S\}\big).$$

Then clearly $D_m(\{i\})$ is a 2-factor of K_m, and if $\gcd\{i, \ell, m\} = 1$ then $D_m(\{i, \ell\})$ is connected with an even number of edges (see Lemma 7.3.3 of [12], for example). We now begin with some more building blocks.

Lemma 5.8. *For any cycle c there exists a* 4*-cycle system of*

$$(2K_3 \sim c) - \{\{(j,u),(j,v)\} \mid j \in \mathbb{Z}_3, \{u,v\} \in E(c)\}.$$

Proof. Let $c = (u_0, u_1, \ldots, u_{\ell-1})$. Then

$$\big(\mathbb{Z}_3 \times V(c), \{\big((j,u_i),(j+1,u_i),(j+2,u_i),(j+1,u_{i+1})\big) \mid j \in \mathbb{Z}_3, i \in \mathbb{Z}_\ell\}\big)$$

is the required 4-cycle system. □

Corollary 5.9. *Let* $i \in \{1, 2, \ldots, \lfloor (m-1)/2 \rfloor\}$. *There exists a* 4*-cycle system of*

$$2K_3 \sim D_m(i) - \{\{(j,u),(j,v)\} \mid j \in \mathbb{Z}_3, \{u,v\} \in E(D_m(\{i\}))\}.$$

Lemma 5.10. *Let G be a connected graph with an even number of edges. Then there exists a* 4*-cycle system of*

$$(K_3^c \sim G) - \{\{(j,u),(j,v)\} \mid j \in \mathbb{Z}_3, \{u,v\} \in E(G)\}.$$

Proof. Since G is connected with an even number of edges, there exists a partition P of $E(G)$ into paths of length 2. For each $p = (v_0, v_1, v_2) \in P$, let

$$B(p) = \{((j, v_0), (j+1, v_1), (j, v_2), (j+2, v_1)) \mid j \in \mathbb{Z}_3\},$$

reducing the sum modulo 3. Then

$$\left(\mathbb{Z}_3 \times V(G), \bigcup_{p \in P} B(p)\right)$$

is the required 4-cycle system. □

Corollary 5.11. *Let* $\gcd\{i, \ell, m\} = 1$ *and let* $\{i, \ell\} \subseteq \{1, 2, \ldots, (m-1)/2\}$. *There exists a* 4*-cycle system of*

$$K_3^c \sim D_m(\{i, \ell\}) - \{((j,u),(j,v)) \mid j \in \mathbb{Z}_3, \{u,v\} \in E(D_m(\{i,\ell\}))\}.$$

Proof. Since $D_m(\{i, \ell\})$ is connected with an even number of edges, the result follows from Lemma 5.10. □

Lemma 5.12. *Let* $\gcd\{i_1, \ell, m\} = \gcd\{i_2, \ell, m\} = 1$. *Let*

$$S = \{i_1, i_1 + \ell, i_2, i_2 + \ell\} \subseteq \{1, 2, \ldots, (m-1)/2\}.$$

There exists a 4*-cycle system of* (a) $K_3^c \sim D_m(S)$, (b) $4K_3 \sim D_m(S)$, *and* (c) $8K_3 \sim D_m(S)$.

Proof. Let

$$B = \{((j,k), (j, k+i_1), (j, k+i_1+i_2+\ell), (j, k+i_2)) \mid j \in \mathbb{Z}_3, k \in \mathbb{Z}_m\},$$

reducing the sums modulo m. Add to the 4-cycles in B the 4-cycles in a 4-cycle system of

$$K_3^c \sim G - \{\{(j,u),(j,v)\} \mid j \in \mathbb{Z}_3, \{u,v\} \in E(G)\},$$
$$\text{for each } G \in \{D_m(\{i_1, i_1+\ell\}), D_m(\{i_2, i_2+\ell\})\}$$

(see Corollary 5.11) to prove (a);

$$2K_3 \sim D_m(i_1), 2K_3 \sim D_m(i_1+\ell) \quad \text{and} \quad K_3^c \sim D_m(\{i_2, i_2+\ell\})$$

(see Corollaries 5.9 and 5.11) to prove (b); and $2K_3 \sim D(j)$, for each $j \in S$, to prove (c). □

Lemma 5.13. *For some* i, *let* S *be the set of*

(a) 8 *consecutive integers* $\{i, i+1, \ldots, i+7\} \subseteq \{1, 2, \ldots, (m-1)/2\}$, *or*
(b) 8 *integers* $\{i, i+1, \ldots, i+9\} \backslash \{i+1, i+8\} \subseteq \{1, 2, \ldots, (m+1)/2\}$.

There exists a 4*-cycle system of* $20K_3 \sim D_m(S)$.

Proof. By applying Lemma 5.8 to each cycle in $D_m(k)$ for each $k \in S$, we obtain a 4-cycle system $(\mathbb{Z}_3 \times \mathbb{Z}_m, B_1)$ of

$$16K_3 \sim D_m(S) - \{\{(j,u),(j,v)\} \mid j \in \mathbb{Z}_3, \{u,v\} \in E(D_m(S))\}.$$

For each $j \in \mathbb{Z}_3$, in case (a) let $D(j) = \{i+2j, i+2j+1\}$, and in case (b) let $D(0) = \{i, i+9\}$, $D(1) = \{i+2, i+7\}$, and $D(2) = \{i+3, i+6\}$. In either case, for each $j \in \mathbb{Z}_3$, let

$$B_1(j) = \{\big((j+1,u),(j+2,u),(j+2,v),(j+1,v)\big) \mid \{u,v\} \in E(D_m(D(j))\}$$

(reducing the first coordinate modulo 3); notice that since $D(j)$ is 4-regular, $B_1(j)$ contains four 4-cycles that contain the edge $\{(j+1,u),(j+2,u)\}$.

Finally, in case (a), for each $j \in \mathbb{Z}_3$, let

$$B_2(j) = \{\big((j,k),(j,k+i+2j),(j,k+2i+2j+7),(j,k+i+2j+1)\big) \mid k \in \mathbb{Z}_m\};$$

these 4-cycles cover the edges $\{(j,u),(j,v)\}$ for each $\{u,v\} \in E(D_m(\{i+2j, i+2j+1, i+6, i+7\}))$. Together these form a 4-cycle system of $20K_3 \sim D_m(S)$. Similarly, in case (b) let

$$B_2(j) = \{\big((j,k),(j,k+i+b_j),(j,k+2i+9),(j,k+i+4)\big) \mid k \in \mathbb{Z}_m\},$$

where $b_j = 0, 2$ or 3 if $j = 0, 1$ or 2, respectively; these 4-cycles cover the edges $\{(j,u),(j,v)\}$ for each $\{u,v\} \in E(D_m(\{i+b_j, i+9-b_j, i+4, i+5\}))$. □

Lemma 5.14. *For some i, let S be the set of:*

(a) 12 *consecutive integers* $\{i, i+1, \ldots, i+11\} \subseteq \{1, 2, \ldots, (m-1)/2\}$, *or*
(b) 12 *integers* $\{i, i+1, \ldots, i+13\} \backslash \{i+1, i+12\} \subseteq \{1, 2, \ldots, (m+1)/2\}$.

There exists a 4-*cycle system of* $32K_3 \sim D_m(S)$.

Proof. By applying Lemma 5.8 to each cycle in $D_m(j)$ for each $j \in S$, we obtain a 4-cycle system $(\mathbb{Z}_3 \times \mathbb{Z}_m, B)$ of

$$24K_3 \sim D(S) - \{\{(j,u),(j,v)\} \mid j \in \mathbb{Z}_3, \{u,v\} \in E(D_m(S))\}.$$

For each $j \in \mathbb{Z}_3$, in case (a) let $D(j) = \{i+4j, i+4j+1, i+4j+2, i+4j+3\}$, and in case (b) let $D(j) = \{i+4j, i+4j+2, i+4j+3, i+4j+5\}$. For each $j \in \mathbb{Z}_3$, let

$$B_1(j) = \{\big((j+1,u),(j+2,u),(j+2,v),(j+1,v)\big) \mid \{u,v\} \in E(D_m(D(j))\};$$

then the edge $\{(j+1,u),(j+2,u)\}$ occurs in eight 4-cycles in $B_1(j)$. For each $j \in \mathbb{Z}_3$, in case (a) let

$$B_2(j) = \{\big((j,k),(j,k+4j+i),(j,k+8j+2i+3),(j,k+4j+i+1)\big) \mid k \in \mathbb{Z}_m\},$$

and in case (b) let

$$B_2(j) = \{\big((j,k),(j,k+i+4j),(j,k+2i+8j+5),(j,k+i+4j+3)\big) \mid k \in \mathbb{Z}_m\}.$$

The 4-cycles in $\bigcup_{j\in\mathbb{Z}_3}(B_1(j) \cup B_2(j)) \cup B$ form the required 4-cycle system. □

Lemma 5.15. *There exists a* 4-*cycle system* $3K_3 \sim K_3 - \{\{(1,0),(2,0)\},\{(1,0),(2,0)\}\}$ (*i.e., a doubled edge*).

Proof. Let

$$B = \big(\{((i,j),(i+1,j),(i+2,j),(i+1,j+1)) \mid i \in \mathbb{Z}_3, j \in \mathbb{Z}_3\} \setminus \{((0,0),(1,0),(2,0),(1,1)),\ ((0,0),(2,0),(1,0),(2,1))\}\big) \cup \{((1,0),(2,0),(2,2),(1,2)),\ ((0,1),(2,1),(2,2),(0,2)),\ ((0,0),(0,2),(1,2),(1,1)),\ ((0,0),(2,0),(1,1),(0,1)),\ ((0,0),(2,0),(2,1),(1,0)),\ ((0,0),(1,0),(1,1),(2,1))\}.$$

□

Corollary 5.16. *For each $j \in \mathbb{Z}_3$, let $\{z_{0,j}, z_{1,j}, z_{2,j}\} \subseteq \{1,2,\ldots,(m-1)/2\}$ such that $z_{0,j} + z_{1,j} = \pm z_{2,j} \pmod m$ and let $S = \{z_{\ell,j} \mid \ell, j \in \mathbb{Z}_3\}$. There exists a 4-cycle system of $25K_3 \sim D_m(S)$.*

Proof. For each $j \in \mathbb{Z}_3$, let $B(j)$ be a set of 4-cycles of a 4-cycle system of

$$3K_3 \sim K_3 - \{\{(j,0),(j+1,0)\},\{(j,0),(j+1,0)\}\}$$

(reducing the sum modulo 3) on the vertex set $\mathbb{Z}_3 \times \{0, z_{0,j}, z_{0,j}+z_{1,j}\}$ (see Lemma 5.15). Let $B(j)+(0,k)$ be formed by adding k modulo m to the second component of each vertex in $B(j)$. Then

$$\Bigg(\mathbb{Z}_3 \times \mathbb{Z}_m, \bigcup_{\substack{j\in\mathbb{Z}_3\\ k\in\mathbb{Z}_m}} (B(j)+(0,k))\Bigg)$$

is a 4-cycle system of $25K_3 \sim D(S)$ (the edge $\{(x,0),(x+1,0)\}$ occurs in seven 4-cycles in $\bigcup_{k\in\mathbb{Z}_m}(B(j)+(0,k))$ if $x=j$ and nine 4-cycles otherwise). □

Lemma 5.17. *Let $S' \subseteq \{1,2,\ldots,(m-1)/2\}$, be a set for which there exists a partition P into 4 difference triples, and for some i let:*

(a) $S'' = \{i, i+1, i+2, i+3\} \subseteq \{1,2,\ldots,(m-1)/2\}$, *or*
(b) $S'' = \{i, i+2, i+3, i+5\} \subseteq \{1,2,\ldots,(m+1)/2\}$,

such that $S' \cap S'' = \phi$. Let $S = S' \cup S''$. Then there exists a 4-cycle system of $44K_3 \sim D(S)$.

Proof. For each difference triple $t = \{z_0, z_1, z_2\} \in P$ with $z_0 + z_1 = \pm z_2 \pmod m$, let $(\mathbb{Z}_3 \times \mathbb{Z}_m, B(t))$ be defined by

$$B(t) = \bigcup_{k\in\mathbb{Z}_m} B(t,k),$$

where $B(t,k)$ is a 4-cycle system of $3K_3 \sim K_3 - \{\{(1,k),(2,k)\},\{(1,k),(2,k)\}\}$ on the vertex set $\mathbb{Z}_3 \times \{k, z_0+k, z_0+z_1+k\}$ (see Lemma 5.15); so the edge $\{(j,k),(j+1,k)\}$ is in 36 4-cycles in $\bigcup_{t\in P} B(t)$ if $j \in \{0,2\}$, and in 28 4-cycles if $j=1$. Let

$$B_1(k) = \{((s,k),(s+1,k),(s+2,k),(s+1,k+\ell)) \mid s \in \mathbb{Z}_3, k \in \mathbb{Z}_m, \ell \in S''\} \cup \{((0,k),(0,k+i,(0,k+2i+\alpha),(0,k+i+2)) \mid k \in \mathbb{Z}_3\} \cup \{((1,k),(2,k),(2,k+\ell),(1,k+\ell)) \mid \ell \in S''\},$$

where $\alpha = 3$ in case (a) and $\alpha = 5$ in case (b). □

Proposition 5.18. *Let $a, x \geq 1$, let $m \geq 72x$, and let λ be divisible by 4 with $0 \leq \lambda \leq 100x$. Let $\beta = 36a$ if $\lambda = 96x+4a$ and let $\beta = 0$ otherwise. For any $\alpha \geq \beta$, let S' be a set of size β that has a partition P into $12a$ difference triples, and with $S' \cap \{\alpha+1, \alpha+2, \ldots, \alpha+36x-\beta\} = \emptyset$. Let*

$$S = S' \cup \{\alpha+1, \alpha+2, \ldots, \alpha+36x-\beta\} \subseteq \{1, 2, \ldots, \lfloor (m-1)/2 \rfloor\}.$$

Then there exists a 4-cycle system of $\lambda K_3 \sim D_m(S)$.

Proof. Let

$$\begin{aligned} S_{i,1} &= \{\alpha + 4i + j + 1 \mid j \in \mathbb{Z}_4\}, \\ S_{i,2} &= \{\alpha + 8i + j + 1 \mid j \in \mathbb{Z}_8\}, \\ S_{i,3} &= \{\alpha + 12i + j + 1 \mid j \in \mathbb{Z}_{12}\}, \quad \text{and} \\ S_{i,4} &= \{36i + j + 1 \mid j \in \mathbb{Z}_{36}\}. \end{aligned}$$

If $0 \leq \lambda = 4a \leq 72x$, then partition S into sets $S_{i,1}$ for $i \in \mathbb{Z}_{9x}$. Apply Lemma 5.12(c) to $\lfloor a/2 \rfloor$ of these sets, Lemma 5.12(b) to one set if a is odd, and Lemma 5.12(a) to the remaining sets.

If $72x + 4 \leq \lambda = 72x + 4a \leq 88x$, then partition S into sets $S_{i,2}$ for $i \in \mathbb{Z}_a$ and $S_{i,1}$ for $2a \leq i \leq 9x - 1$. Apply Lemma 5.13 to each set $S_{i,2}$ and Lemma 5.12(c) to each $S_{i,1}$. Then, as required, each pure edge appears in $20a + 8(9x - 2a) = \lambda$ 4-cycles, and $8a + 4(9x - 2a) = 36x$ differences are used.

If $88x + 4 \leq \lambda = 88x + 4a \leq 92x$, then partition S into sets $S_{i,3}$ for $i \in \mathbb{Z}_{2a}$, $S_{i,2}$ for $3a \leq i \leq 4x - 1$, and $S_{i,1}$ for $8x \leq i \leq 9x - 1$. Apply Lemma 5.14 to each $S_{i,3}$, Lemma 5.13 to each $S_{i,2}$, and Lemma 5.12(c) to each $S_{i,1}$. Then, as required, each pure edge appears in $32(2a) + 20(4x - 3a) + 8(x) = \lambda$ 4-cycles, and $12(2a) + 8(4x - 3a) + 4(x) = 36x$ differences are used.

If $92x+4 \leq \lambda = 92x+4a \leq 96x$, then partition S into sets $S_{i,3}$ for $i \in \mathbb{Z}_{2x+a}$, $(24x+12a)+S_{i,2}$ for $i \in \mathbb{Z}_{x-a}$ (where $\gamma + S_{i,2} = \{\gamma + k \mid k \in S_{i,2}\}$), and $S_{i,1}$ for $8x + a \leq i \leq 9x - 1$. Apply Lemma 5.14 to each $S_{i,3}$, Lemma 5.13 to each $(24x + 12a) + S_{i,2}$ and Lemma 5.12(c) to each $S_{i,1}$. Then, as required, each pure edge appears in $32(2x+a)+20(x-a)+8(x-a) = \lambda$ 4-cycles, and $12(2x + a) + 8(x - a) + 4(x - a) = 36x$ differences are used.

If $96x + 4 \leq \lambda = 96x + 4a \leq 100x$, then arbitrarily form a partition P' of P into $4a$ sets, each containing three difference triples, and partition $S \backslash S'$ into sets $S_{i,3}$ for $i \in \mathbb{Z}_{3x-3a}$. Apply Corollary 5.16 to each set in P' and apply Lemma 5.14 to each $S_{i,3}$. Then, as required, each pure edge appears in $25(4a) + 32(3x - 3a) = \lambda$ 4-cycles, and $9(4a) + 12(3x - 3a) = 36x$ differences are used. □

In order to find the partition P in Proposition 5.18, we will make use of the following theorem of Skolem [15] and Simpson [16].

Theorem 5.19 ([15, 16]). *Let $S = \{1, 2, \ldots, 3s\}$ if $s \equiv 0$ or 1 (mod 4) and let $S = \{1, 2, \ldots, 3s+1\} \backslash \{3s\}$ if $s \equiv 2$ or 3 (mod 4). There exists a partition P of S into sets of size 3 such that if $\{z_0, z_1, z_2\} \in P$ then $z_0 + z_1 = z_2$.*

Proposition 5.20. *The necessary conditions in Lemma 2.5 are sufficient for the existence of a* GDD$(3;m)$ *of index* $(\lambda_1, 1)$ *if* $\lambda_1 \equiv 0 \pmod 4$.

Proof. Since $\lambda_1 \equiv 0 \pmod 4$ and $\lambda_2 = 1$, let $m = 72x + 8b + 1$, with $b \in \mathbb{Z}_9$.

If $x \geq 1$ and $\lambda_1 \leq 100x$ then let $(\mathbb{Z}_3 \times \mathbb{Z}_m, B_1)$ be a 4-cycle system of $\lambda_1 K_3 \sim D_m(\{1, \dots, 36x\})$ (use Proposition 5.18 and Theorem 5.19), and for each $i \in \mathbb{Z}_b$ let $(\mathbb{Z}_3 \times \mathbb{Z}_m, B_{2,i})$ be a 4-cycle system of

$$K_3^c \sim D_m(\{36x + 4i + j + 1 \mid j \in \mathbb{Z}_4\})$$

(use Lemma 5.12(a)). Together these form the required 4-cycle system.

Otherwise let $\lambda_1 = 100x + 4a$ for some $x \geq 0$ and $a \geq 1$ (so $b \geq 1$). We inductively consider the cases $b = 1, 2, \dots, 8$. Let $\lambda_{(m)} = 4\lfloor (3(m-1)/2 - (m-1)/9)/4 \rfloor$ be the largest possible value of λ_1. Suppose we know there exists a GDD$(3; m-8)$ of index $(\lambda, 1)$ for any $\lambda \leq \lambda_{(m-8)}$ formed by partitioning $\{1, \dots, (m-9)/2\}$ into sets S_i and then finding 4-cycle systems of $\lambda(i)K_3 \sim D_{m-8}(S_i)$, for some $\lambda(i)$. Then a GDD$(3;m)$ of index $(\lambda + \lambda', 1)$, where $\lambda \leq \lambda_{(m-8)}$ and $\lambda' \in \{0, 4, 8\}$, can be formed by taking exactly the same partition P of $\{1, \dots, (m-9)/2\}$, and for each $S_i \in P$ obtain a 4-cycle system of $\lambda(i)K_3 \sim D_m(S_i)$ (this is clearly always possible), and using Lemma 5.12 add the 4-cycles in a 4-cycle system of $\lambda' K_3^c \sim D(\{m' - 3, m' - 2, m' - 1, m'\})$, where $0K_3$ denotes K_3^c, and $m' = (m-1)/2$. So it remains to consider the case where $\lambda_1 > \lambda_{(m-8)} + 8$. That is, we need to find a 4-cycle system when $(b, a) \in \{(2, 20), (3, 32), (4, 44), (6, 64), (7, 76), (8, 88)\}$. If $b \in \{4, 8\}$ then partition $\{1, \dots, 36x + 3b\}$ into difference triples: apply Corollary 5.16 to $12x$ of the difference triples, and apply Lemma 5.17 $b/4$ times to the remaining difference triples and the remaining differences (namely $36x + 3b + 1, \dots, 36x + 4b$) to obtain a GDD$(3;m)$ of index $100x + 44(b/4) = \lambda_{(m)}$. If $b \in \{2, 3\}$ or $\{6, 7\}$, begin with the same partition of the differences used to form a GDD$(3; 100x)$ or GDD$(3; 100x + 33)$ of index $(1, 100x)$ or $(1, 100x + 44)$, respectively, and apply the same construction to each set in the partition (with a new value of m, of course). To the remaining 8 or 12 differences, if $b \in \{2, 6\}$ or $\{3, 7\}$, apply Lemma 5.13 or Lemma 5.14, respectively, to obtain the required GDD. □

5.2.2. $\lambda_1 \not\equiv 0 \pmod 4$ and $\lambda_1 = 1$

Lemma 5.21. *Let* $S(u, v, w)$ *be the graph with vertex set* $\mathbb{Z}_3 \times \{u, v, w\}$, *in which two vertices* (i_1, j_1) *and* (i_2, j_2) *are joined by* 5 *edges if* $j_1 = j_2 = u$, *and by* 1 *edge otherwise. There exists a* 4-*cycle system of* $S(u, v, w)$.

Proof. Let

$$\begin{aligned} B = \{ &\big((i,u),(i+1,u),(i+2,u),(i+1,v)\big),\ \big((i,u),(i+1,u),(i+2,u),(i+2,w)\big), \\ &\big((i,u),(i+2,u),(i,w),(i,v)\big),\ \big((i,v),(i+1,v),(i,w),(i+1,w)\big) \mid i \in \mathbb{Z}_3\}, \end{aligned}$$

reducing the first component modulo 3. Then $(\mathbb{Z}_3 \times \{u, v, w\}, B)$ is the required 4-cycle system. □

Lemma 5.22. *There exists a* GDD(3; 11) *of index* $(\lambda_1, 1)$ *for each* $\lambda_1 \in \{1, 5, 9, 13\}$*, and a* GDD(3, 3) *of index* (1, 1).

Proof. If $\lambda_1 = 1$, then the required GDD is a 4-cycle system of K_{33} or of K_9, which exists by Lemma 2.1.

Let

$$T_1 = \{(0,3,10),(1,5,10),(2,4,10),(3,1,9),(4,0,9),\\ (5,2,9),(6,4,5),(7,0,5),(8,3,5),(9,7,10),(10,6,8)\}$$

and

$$T_2 = \{(0,1,6),(2,3,6),(1,2,7),(3,4,7),(0,2,8),(1,4,8),(6,7,8,9)\}.$$

Then $(\mathbb{Z}_{11}, T_1 \cup T_2)$ is a partition of $E(K_{11})$ into seventeen 3-cycles and one 4-cycle.

If $\lambda_1 = 5$, then, for each cycle $c \in T_1 \cup T_2$ of length 3 or 4, let C contain the set of 4-cycles in a 4-cycle system of K_9 (see Lemma 2.1) or $K_3 \sim c$ (see Lemma 5.3), respectively, each with vertex set $\mathbb{Z}_3 \times V(c)$. Then, since each vertex is in exactly 5 cycles in $T_1 \cup T_2$, $(\mathbb{Z}_{11}, C)$ is a GDD(3, 11) of index (5, 1).

If $\lambda_1 = 9$, then, for each $(v_1, v_2, v_3) \in T_1$, let C contain the 4-cycles in a 4-cycle system of $S(v_1, v_2, v_3)$ (see Lemma 5.21), and, for each cycle $c \in T_2$ of length 3 or 4, let C contain the 4-cycles in a 4-cycle system of K_9 or of $K_3 \sim c$, respectively (as described above). Since each vertex is the first coordinate of exactly one 3-cycle in T_1, $(\mathbb{Z}_{11}, C)$ is GDD(3, 11) of index (9, 1).

If $\lambda_1 = 13$, then let H be the set of 4-cycles in a 4-cycle system of

$$3K_3 \sim K_3 - \{\{(1,0),(2,0)\},\{(1,0,(2,0)\}\}$$

on the vertex set $\mathbb{Z}_3 \times \{0, 2, 6\}$ (see Lemma 5.15). Let

$$B = H \cup \{((i,0),(i+1,0),(i+2,0),(i+1,3)) \mid i \in \mathbb{Z}_3\} \cup\\ \{((1,0),(2,0),(2,3),(1,3)),\ ((0,0),(0,1),(1,2),(0,3)),\\ ((0,1),(1,1),(2,1),(2,0)),\ ((0,0),(2,0),(1,0),(2,1)),\\ ((0,0),(1,0),(1,1),(2,0))\}.$$

Let $B + (0, j)$ be formed by adding j (modulo 11) to the second component of each vertex in each 4-cycle in B. Then

$$\left(\mathbb{Z}_3 \times \mathbb{Z}_{11}, \bigcup_{j \in \mathbb{Z}_{11}} B + (0, j)\right)$$

is a GDD(3; 11) of index (13, 1). □

Recall that an edge $\{(i_1, j_1), (i_2, j_2)\}$ is called *pure* if $j_1 = j_2$.

Lemma 5.23. *Let* $z \in \{5, 9, 13, 17, 21, 25\}$*. There exists a multigraph* $T_z(u, v, w)$ *on the vertex set* $\mathbb{Z}_3 \times \{u, v, w\}$ *which*

(a) *contains exactly z pure edges,*

(b) *contains exactly one edge joining* (i_1, j_1) *to* (i_2, j_2) *whenever* $j_1 \neq j_2$, *and*
(c) *has a partition of its edges into* 4-*cycles.*

Proof. Let

$$B_5 = \{\big((0,u),(1,u),(1,v),(1,w)\big),\ \big((0,u),(1,u),(0,w),(0,v)\big),\\ \big((2,w),(0,w),(1,w),(1,u)\big),\ \big((1,w),(2,w),(2,v),(2,u)\big),\\ \big((0,u),(0,w),(2,u),(2,w)\big),\ \big((i+1,u),(i,v),(i+1,w),(i+2,v)\big) \mid i \in \mathbb{Z}_3\}.$$

Then $(\mathbb{Z}_3 \times \{u,v,w\}, B_5)$ is the required 4-cycle system of $T_5(u,v,w)$.

Let

$$B_{13} = \{\big((i,u),(i+1,u),(i,v),(i+1,v)\big),\ \big((i,u),(i,v),(i,w),(i+2,w)\big),\\ \big((0,u),(0,w),(2,v),(1,w)\big),\ \big((1,u),(1,w),(0,v),(2,w)\big),\\ \big((0,w),(1,w),(2,w),(1,v)\big),\ \big((0,w),(1,w),(2,w),(2,u)\big) \mid i \in \mathbb{Z}_3\}.$$

Then $(\mathbb{Z}_3 \times \{u,v,w\}, B_{13})$ is the required 4-cycle system of $T_{13}(u,v,w)$.

Let

$$B_{17} = \big(B_{13} \backslash \{\big((1,u),(1,w),(0,v),(2,w)\big)\}\big) \cup\\ \{\big((1,u),(1,w),(0,w),(2,w)\big),\ \big((1,w),(0,v),(2,w),(0,w)\big)\}.$$

Then $(\mathbb{Z}_3 \times \{u,v,w\}, B_{17})$ is the required 4-cycle system of $T_{17}(u,v,w)$.

A 4-cycle system of K_9 (see Lemma 2.1), the graph $S(u,v,w)$ (see Lemma 5.21), and the graph defined in Lemma 5.15 provide the required 4-cycle system when λ is 9, 21 or 25, respectively. □

Lemma 5.24. *There exists a* GDD$(3,m)$ *of index* $(\lambda_1, 1)$ *whenever* $m \equiv 3 \pmod 8$ *with* $m \geq 19$ *and* $\lambda_1 \equiv 1 \pmod 4$ *with* $1 \leq \lambda_1 \leq 25$.

Proof. If $\lambda_1 = 1$ then the required GDD is a 4-cycle system of K_{3m}, which exists by Lemma 2.1 since $3m \equiv 1 \pmod 8$.

In each other case, let

$$m = 8x + 3,$$
$$D_1 = \{(1,5,6),(2,8,10),(3,4,7)\}, \quad \text{and}$$
$$D_2 = \{\{4i+1, 4i+3, 4i+4, 4i+6\} \mid 2 \leq i \leq x-1\}.$$

Then each element of $\{1,2,\ldots,(m+1)/2\} \backslash \{(m-1)/2\}$ occurs in exactly one element of $D_1 \cup D_2$.

For each permitted λ_1, each $i \in \mathbb{Z}_3$, and each $j \in \mathbb{Z}_m$, let $B_{\lambda_1}(u,v,w) + (i,j)$ be formed from a 4-cycle system of $T_{\lambda_1}(u,v,w)$ by adding (i,j) to each vertex in each 4-cycle (see Lemma 5.23), reducing the first and second components modulo 3 and m, respectively.

Let $S = \{1,2,\ldots,10\} \backslash \{9\}$. Then

$$B_1 = \bigcup_{\substack{i \in \mathbb{Z}_3, j \in \mathbb{Z}_m,\\ (u,v,w) \in D_1}} (B_{\lambda_1}(0,u,u+v) + (i,j))$$

forms a 4-cycle system of $\lambda_1 K_3 \sim D_m(S)$ (note that $D_m(\{m+1)/2\}) = D_m(\{m-1)/2\})$. For $2 \le i \le x-1$, let B_i be the set of 4-cycles of a 4-cycle system of $K_3^c \sim D_m(\{4i+1, 4i+3, 4i+4, 4i+6\})$ (see Lemma 5.12(a) with $\ell = 2$). Then

$$\left(\mathbb{Z}_3 \times \mathbb{Z}_m, \bigcup_{1 \le i \le x-1} B_i\right)$$

is a GDD$(3, m)$ of index $(\lambda_1, 1)$. □

Recall that $\lambda_{(m)}$ is the largest value of λ_1 for which the necessary conditions of Theorem 1.2 are satisfied when $\lambda_2 = 1$ and $n = 3$.

Lemma 5.25. $\lambda_{(m)} - 25 = \lambda_{(m-18)}$.

Proof. Since $\lambda_2 = 1$ we have that $\lambda_1 = 4a + b$ and $m \equiv 2b + 1 \pmod 8$ (see Table 6). So the only condition of Theorem 1.2 that needs to be checked is (4). Let $m = 8x + 2b + 1$. Then

$$3(m-1)/2 - (m-1)/9 = 12x + 4b - (8x + 11b)/9 = \ell_1,$$

and

$$3(m-19)/2 - (m-19)/9 = 12x + 4b - (8x + 11b)/9 - 25 = \ell_2.$$

Since $\lambda_{(m)} \equiv \lambda_{(m-18)} + 1 \pmod 4$, we have that $\ell_1 - \lambda_{(m)} = \ell_2 - \lambda_{(m-18)}$, so the result follows. □

Proposition 5.26. *The necessary conditions in Lemma 2.5 are sufficient for the existence of a* GDD$(3; m)$ *of index* $(\lambda_1, 1)$ *if* $\lambda_1 \equiv 1 \pmod 4$.

Proof. In view of Lemmas 5.22 and 5.24, we can assume that $m \ge 19$ and $\lambda_1 \ge 29$. From Lemma 5.25 we have that $\lambda_1 - 25 \le \lambda_{(m-18)}$, and clearly $\lambda_1 - 25 \equiv 0 \pmod 4$ and $m - 18 \equiv 1 \pmod 8$, so by Proposition 5.20 there exists a GDD$(3; m-18)$ of index $\lambda_1' = \lambda_1 - 25$. Furthermore, this GDD can be constructed by partitioning $\{1, 2, \ldots, (m-1)/2\}$ into $S_1' = \{1, 2, \ldots, 36a\}$ and $S_2' = \{36a+1, \ldots, (m-19)/2\}$, the first set being partitioned into $12a$ difference triples, the second into sets of size 4, 8 and 12. To obtain the required GDD, partition $\{1, 2, \ldots, (m-1)/2\}$ into $S_1 = \{1, 2, \ldots, 36a+10\} \backslash \{36a+9\}$, which can be partitioned into $12a+3$ difference triples by Theorem 5.19, and $S_2 = \{36a+9, \ldots, (m+1)/2\} \backslash \{36a+10, (m-1)/2\}$. Apply Corollary 5.16 to three of the difference triples in S_1 (thus increasing λ_1 by 25), and use the remaining $12a$ difference triples as described in the proof of Proposition 5.20. Finally, replace each set $s_i' = \{i, i+1, \ldots, i+z\}$ of size 4, 8 or 12 in the partition of S_2', with $s_i = \{i+9-1, i+9, \ldots, i+9+z+1\} \backslash \{i+9, i+9+z\}$ to form a partition of S_2. Apply the same one of Lemmas 5.12, 5.13, 5.14 and 5.17 to s_i as was applied to s_i' in the proof of Proposition 5.20, except that case (b) is used in the last three lemmas instead of case (a). The result follows because $D_m((m-1)/2) = D_m((m+1)/2)$. □

Lemma 5.27. *There exists a* GDD$(3; m)$ *of index* $(\lambda_1, 1)$ *for all* $m \equiv 5 \pmod 8$ *and* $\lambda_1 \equiv 2 \pmod 4$ *whenever* $m \le 29$ *and* $\lambda_1 \le 26$.

Proof.

$$\big(\mathbb{Z}_3 \times \mathbb{Z}_5, \{\big((j,k),(j+1,k),(j+2,k),(j+1,k+2)\big), \\ \big((j,k),(j,k+1),(j,k+3),(j+1,k+4)\big) \mid j \in \mathbb{Z}_3, k \in \mathbb{Z}_5\}\big)$$

is a GDD$(3;5)$ of index $(2,1)$.

If $\lambda_1 = 2$ and $m = 8x+5 \geq 13$, then let $((\mathbb{Z}_3 \times \mathbb{Z}_{8x}) \cup \{(0,8x)\}, B_1)$ be a partial 4-cycle system of K_{24x+1} with leave consisting of the $4x$ 6-cycles in

$$\{\big((0,2j),(1,2j+1),(2,2j),(0,2j+1),(1,2j),(2,2j+1)\big) \mid j \in 4x\}.$$

Also, let $(\mathbb{Z}_3 \times (\mathbb{Z}_{8x+5} \backslash \mathbb{Z}_{8x}), B_2)$ be a GDD$(3;5)$ of index $(2,1)$ (just constructed above), and let B_3 be a set of 4-cycles in a 4-cycle system of $K_{24x,14}$ with bipartition $(\mathbb{Z}_3 \times \mathbb{Z}_{8x})$ and $(\mathbb{Z}_3 \times (\mathbb{Z}_{8x+5} \backslash \mathbb{Z}_{8x}) \backslash \{(0,8x)\})$ (see Lemma 2.11). Then

$$\big(\mathbb{Z}_3 \times \mathbb{Z}_{8x+5}, B_1 \cup B_2 \cup B_3 \cup \\ \{\big((i,2j),(i+1,2j+1),(i+2,2j+1),(i+1,2j)\big) \mid i \in \mathbb{Z}_3, j \in \mathbb{Z}_{4x}\}\big)$$

is a GDD$(3;m)$ of index $(2,1)$.

If $m = 21$ and $\lambda = 26$, then apply Corollary 5.16 to $S = \{1,4,5\} \cup \{2,8,10\} \cup \{3,6,9\}$, and then note that $K_m \sim D_m(\{7\})$ consists of 7 vertex-disjoint copies of K_9, for each of which there exists a 4-cycle system (Lemma 2.1).

If $6 \leq \lambda_1 \leq 26$, then let $D_1 = \{(1,3,4),(2,5,7)\}$ and let $D_2 = \{\{4i+2, 4i+4, 4i+5, 4i+7\} \mid 1 \leq i \leq (m-13)/8\}$; so each element of $\{1,2,\dots,(m+1)/2\} \backslash \{(m-1)/2\}$ occurs in exactly one element of $D_1 \cup D_2$. Let $(\mathbb{Z}_3 \times \{u,v,w\}, B_3(u,v,w))$ be a 4-cycle system of K_9, and let $(\mathbb{Z}_3 \times \{u,v,w\}, B_7(u,v,w))$ be a 4-cycle system of $S(u,v,w)$ in Lemma 5.21. Then, for each $\lambda \in \{3,7\}$ and each $(u,v,w) \in D_1$,

$$\bigcup_{j \in \mathbb{Z}_m} (B_\lambda(0,u,u+v) + (0,j))$$

forms a 4-cycle system of $\lambda K_3 \sim D_m(\{u,v,w\})$. Form the required GDD as follows. If $\lambda_1 \in \{6,14\}$, then apply the above with $\lambda = \lambda_1/2$ for each element of D_1, then apply Lemma 5.12(a) to each element of D_2. If $\lambda_1 = 10$, then apply the above, using $\lambda = 3$ with one element of D_1 and $\lambda = 7$ with the other; then again use Lemma 5.12(a). If $\lambda = 18$ or 22, then use $\lambda = 7$ for both elements of D_1; use Lemma 5.12(b) or (c), respectively, for one element of D_2 and Lemma 5.12(a) for the remaining elements of D_2. If $\lambda_1 = 26$, then we have already dealt with $m = 21$, so $m \geq 29$; use $\lambda = 7$ for both elements of D_2, apply each of Lemma 5.12(b) and (c) to a different element of D_2 and Lemma 5.12(a) to the remaining elements of D_2.

Suppose $m = 29$. If $\lambda_1 = 30$, then use $\lambda = 3$ and $\lambda = 7$ for the first and second elements of D_1 and, if $\lambda_1 = 34$, then use $\lambda = 7$ with both elements of D_1; in either case apply Lemma 5.13(b) to the union of the two elements of D_2. If $\lambda_1 = 38$, then let $D = \{\{2,5,7\},\{4,11,15\},\{6,10,13\},\{8,9,12\}\}$ be a partition of $\{1,2,\dots,15\} \backslash \{1,3,14\}$ into difference triples (mod 29); modify the construction of a GDD$(3;11)$ of index $(5,1)$ by renaming H to be defined on the vertex set $\mathbb{Z}_3 \times \{0,2,7\}$ (thus using the differences 2, 5 and 7 instead of 2, 4 and 6), and then applying Corollary 5.16 to the three difference triples in $D \backslash \{\{2,5,7\}\}$. □

Proposition 5.28. *The necessary conditions in Lemma 2.5 are sufficient for the existence of a* GDD(3; *m*) *of index* $(\lambda_1, 1)$ *if* $\lambda_1 \equiv 2 \pmod 4$.

Proof. By Lemma 5.27 we can assume that $m \geq 37$ and $\lambda_1 \geq 30$. The proof is essentially the same as the proof of Proposition 5.26. Simply partition the set

$$\{1, 2, \ldots, 9\} \cup (S_1' + 9) = \{1, 2, \ldots, 36a + 19\} \backslash \{36a + 18\}$$

into difference triples (see Theorem 5.19), apply Corollary 5.16 to three of them (thus increasing λ_1 by 25) and use the remaining ones in the same way as described in the proof of Proposition 5.26. Complete the proof by using each set $s_i + 9$ in the same way that s_i is used in the proof of Proposition 5.26. □

Lemma 5.29. *There exists a* GDD(3; *m*) *of index* $(\lambda_1, 1)$ *for all* $m \equiv 7 \pmod 8$ *and* $\lambda_1 \equiv 3 \pmod 4$ *whenever* $\lambda_1 \leq 27$*, and when* $(m, \lambda_1) \in \{(39, 51), (47, 63)\}$.

Proof. As in the proof of Lemma 5.27, for any difference triple $\{u, v, w\}$ and for each $\lambda \in \{3, 7\}$, let $B_\lambda(u, v, w)$ be the set of 4-cycles in a 4-cycle system of $\lambda K_3 \sim D_m(\{u, v, w\})$.

If $\lambda_1 \in \{3, 11, 19\}$ then let $\lambda = 3$, and if $\lambda_1 \in \{7, 15, 23, 27\}$ then let $\lambda = 7$. Let $P = \{\{4i, 4i+1, 4i+2, 4i+3\} \mid 1 \leq i \leq (m-7)/8\}$ be a partition of $\{4, 5, \ldots, (m-1)/2\}$. If $\lambda_1 \leq 23$, then let $m = \lambda + 4\alpha$ where $\lambda \in \{3, 7\}$ and $\alpha \in \{0, 1, 2\}$. Providing $(m, \lambda_1) \neq (15, 19)$, the required GDD can be obtained by applying Lemma 5.12(c) to α sets in P, applying Lemma 5.12(a) to the remaining sets in P, then taking the union of all these 4-cycles with those in $B_\lambda(1, 2, 3)$. If $\lambda_1 = 27$, then take the union of the 4-cycles in $B_7(1, 2, 3)$ with those formed by applying Lemma 5.13 to the union of two sets in P, and applying Lemma 5.12(a) to the remaining sets in P.

To construct a GDD(3; 15) of index (19, 1), let B_1 be a set of 4-cycles of a 4-cycle system of $3K_3 \sim K_3 - \{\{(0,0),(1,0)\},\{(0,0),(1,0)\}\}$ on the vertex set $\mathbb{Z}_3 \times \{0, 1, 3\}$ (see Lemma 5.15). Let

$$\begin{aligned} B_2 = \{&\big((i,0),(i+1,0),(i,4),(i+1,4)\big),\\ &\big((0,0),(1,0),(1,4),(0,4)\big),\ \big((i,0),(i+1,0),(i+2,5),(i,5)\big),\\ &\big((0,0),(1,0),(0,5),(2,5)\big),\ \big((i,0),(i+1,0),(i+2,6),(i,6)\big),\\ &\big((0,0),(2,0),(1,6),(2,6)\big),\ \big((i,0),(i+1,0),(i+1,7),(i+2,7)\big),\\ &\big((0,0),(1,0),(2,7),(1,7)\big),\ \big((2,0),(2,4),(0,11),(1,5)\big) \mid i \in \mathbb{Z}_3\}. \end{aligned}$$

Then

$$\left(\mathbb{Z}_3 \times \mathbb{Z}_{15}, \bigcup_{j \in \mathbb{Z}_{15}} ((B_1 \cup B_2) + (0, j))\right)$$

is a GDD(3, 15) of index (19, 1) (reducing the sum in the first and second coordinates modulo 3 and 15, respectively).

To construct a GDD(3; 39) of index (51, 1), let B_1 and B_2 be defined as above, and let

B_3 be the set of 4-cycles formed by applying Lemma 5.14 to $S = \{8, 9, 10, \ldots, 19\}$. Then

$$\left(\bigcup_{j \in \mathbb{Z}_{39}} ((B_1 \cup B_2) + (0, j)\right) \cup B_3,$$

reducing the sum in the second coordinate modulo 39, forms the required GDD.

Finally, to produce a GDD(3; 47) of index (63, 1), proceed as follows. First note that, if B_1 above is defined on $\mathbb{Z}_3 \times \{0, u, u+v\}$ instead of specifically $\mathbb{Z}_3 \times \{0, 1, 3\}$, and if each 4, 5, 6, 7 and 11 in the second coordinate of each vertex in B_2 is replaced with $\alpha, \alpha+1, \alpha+2, \alpha+3$ and $2\alpha+3$, respectively, for some $\alpha \geq 4$, then $\bigcup_{j \in \mathbb{Z}_m} ((B_1 \cup B_2) + (0, j))$ form a 4-cycle system of $19K_m \sim D_m(S_1 = \{u, v, u+v, \alpha, \alpha+1, \alpha+2, \alpha+3\})$, whenever $m > 2 \max\{u+v, \alpha+3\}$. So, partition $\{1, 2, \ldots, 23\}$ into 5 difference triples in $\{1, 2, \ldots, 15\}$ (see Theorem 5.19), and $\{4i, 4i+1, 4i+2, 4i+3\}$ for each $i \in \{4, 5\}$. Let $\{u, v, w\}$ be one such difference triple and let $\alpha = 16$ to obtain the 4-cycles in a 4-cycle system of $19K_m \sim D_{47}(\{u, v, w, 16, 17, 18, 19\})$, and to this add the 4-cycles obtained by applying Lemma 5.17 to the remaining 4 difference triples and the set $\{20, 21, 22, 23\}$. □

Proposition 5.30. *The necessary conditions in Lemma 2.5 are sufficient for the existence of a* GDD(3; *m*) *of index* $(\lambda_1, 1)$ *if* $\lambda_1 \equiv 3 \pmod 4$.

Proof. In view of Lemma 5.29, we can assume that $\lambda_1 \geq 31$ and

$$(m, \lambda_1) \notin \{(39, 51), (47, 63)\}.$$

Therefore, from the proofs of Lemma 5.27 and Proposition 5.28 there exists a GDD(3; $m-$18) of index $(\lambda_1 - 25, 1)$ that can be constructed using $12a+\alpha$ difference triples where $\alpha = 2$ or 6 when $\lambda_1 - 25 \leq 26$ (since $(m, \lambda_1) \neq (21, 26)$) or $\lambda_1 - 25 > 26$ (since $(m, \lambda_1) \neq (29, 38)$), respectively.

Following the proofs of Propositions 5.26 and 5.28, partition $\{1, 2, \ldots, 36a + 3\alpha + 9\}$ into difference triples (see Theorem 5.19). Apply Corollary 5.16 to three of the difference triples and use the remaining difference triples as in forming the GDD(3; $m-18$). Similarly, for each set, say $\{x-1, x, \ldots, x+z+1\} \backslash \{x, x+z\}$, of size 4, 8 or 12 in the partition of $\{1, 2, \ldots, (m-19)/2\}$ used in constructing the GDD(3; $m-18$), apply the same construction to the set $\{x+9, x+10, \ldots, x+z+9\}$ instead. □

6. Conclusions

We can now gather the results of the previous sections to prove Theorem 1.2.

Theorem 1.2. *Let* n, $m \geq 1$ *and* $\lambda_1, \lambda_2 \geq 0$ *be integers. There exists a* GDD4C(n; m) *of index* (λ_1, λ_2) *if and only if:*

(1) 2 *divides* $\lambda_1(n-1) + \lambda_2 n(m-1)$,

(2) 8 *divides* $\lambda_1 mn(n-1) + \lambda_2 n^2 m(m-1)$, *and if* $\lambda_2 = 0$ *then* 8 *divides* $\lambda_1 n(n-1)$,

(3) *if* $n = 2$ *then* $\lambda_2 > 0$ *and* $\lambda_1 \leq 2(m-1)\lambda_2$, *and*

(4) *if* $n = 3$ *then* $\lambda_2 > 0$ *and* $\lambda_1 \leq 3(m-1)\lambda_2/2 - \delta(m-1)/9$, *where* $\delta = 0$ *or* 1 *if* λ_2 *is even or odd, respectively.*

Proof. The necessity follows from Lemma 2.5. The sufficiency follows from: Section 3 if $n \geq 4$; Section 4 if $n = 2$; Proposition 5.6 if $n = 3$ and λ_2 is even; Proposition 5.7 if $n = 3$, λ_2 is odd and $\lambda_2 > 1$; and Propositions 5.20, 5.26, 5.28 and 5.30 if $n = 3$, $\lambda_2 = 1$ and $\lambda_1 \equiv 0, 1, 2$ or 3 (mod 4), respectively. □

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