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Four-Cycle Systems with Two-Regular Leaves

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Abstract. In this paper we settle the existence problem for 4-cycle systems of $K_n - E(F)$ and of $2K_n - E(F)$ for all 2-regular subgraphs F.

1. Introduction

The *m-cycle* $(a_0, a_1, \ldots, a_{m-1})$ is the graph induced by the edges in $\{\{a_i, a_{i+1}\}, \{a_0, a_{m-1}\}\} \mid i \in \mathbb{Z}_{m-1}\}$. An *m-cycle system* of a graph G is an ordered pair $(V(G), C)$ where C is a set of m-cycles whose edges partition $E(G)$. There have been many results considering the existence of m-cycle systems of G; see [3] for a survey. In particular, necessary and sufficient conditions have been found for the existence of a 3-cycle system of $K_n - E(F)$ for any 2-regular subgraph F[2], and have been found for the existence of n-cycle systems (that is, hamilton decompositions) of $K_n - E(F)$ for any 2-factor F[1]. In this paper we solve the existence problem for 4-cycle systems of both $K_n - E(F)$ and of $2K_n - E(F)$ for any 2-regular subgraph F.

For any 2-regular subgraph F of G, let $G - F$ denote the graph formed from G by removing the edges in F (we use $G - F$ and $G - E(F)$ interchangeably).

2. The Results

We begin settling the existence of 4-cycle systems of $K_n - E(F)$ and of $2K_n - E(F)$, for any 2-regular subgraph F, by finding some solutions for small values of n.

Let $K(A, B)$ and $2K(A, B)$ be the set of 4-cycles in a 4-cycle system of $K_{|A|,|B|}$ and $2K_{|A|,|B|}$ respectively with bipartition $\{A, B\}$ of the vertex set. The following is easy to prove, and also follows from a more general result of Sotteau [4].

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Lemma 2.1. *There exists a* 4*-cycle system of* $K_{a,b}$ *and of* $2K_{a,b}$ *if and only if each vertex has even degree, the number of edges is divisible by* 4, *and* $a, b \geq 2$.

Lemma 2.2. *There exists a* 4*-cycle system of* $K_n - E(F)$

(a) if $n = 7$ *and* $F = C_5$,
(b) if $n = 19$ *and F is a* 2*-factor consisting of one* 4*-cycle and three* 5*-cycles, and*
(c) if $n = 35$ *and F is a* 2*-factor consisting of seven* 5*-cycles.*

Proof. (a) $(\mathbb{Z}_7, \{(0,3,1,5),(1,4,2,6),(0,6,5,2),(3,5,4,6)\})$ is a 4-cycle system of $K_7 - (0,1,2,3,4)$.

(b) Let $(\{a_i \mid i \in \mathbb{Z}_7\},\ P(a_0,a_1,a_2,a_3,a_4;a_5,a_6))$ be a 4-cycle system of $K_7 - (a_0,a_1,a_2,a_3,a_4)$ (see (a) above). Form a 4-cycle system $(\mathbb{Z}_{19}, C)$ of $K_{19} - (\{(i,i+1,i+2,i+3,i+4) \mid i \in \{0,5,10\}\} \cup \{(15,17,16,18)\})$ as follows. Let $C = P(0,1,2,3,4;15,16) \cup P(5,6,7,8,9;17,18) \cup P(10,11,12,13,14;0,5) \cup K(\{15,16\}, \{5i,5i+1,5i+2,5i+3,5i+4 \mid i \in \{1,2\}\}) \cup K(\{17,18\}, \{5i,5i+1, 5i+2,5i+3,5i+4 \mid i \in \{0,2\}\}) \cup K(\{i+1,i+2,i+3,i+4\}, \{j+1,j+2,j+3,j+4) \mid (i,j) \in \{(0,5),(0,10),(5,10)\} \cup \{(0,6,10,7),(0,8,10, 9),(5,1,10,2),(5,3,10,4)\}$.

(c) Let $c_i = (i,i+1,i+2,i+3,i+4)$. Let $(\mathbb{Z}_{16} \cup \{32,33,34\}, C_1)$ be a 4-cycle system of $K_{19} - (c_0 \cup c_5 \cup c_{10} \cup (15,34,32,33))$, and let $(\{i \mid 16 \leq i \leq 34\}, C_2)$ be a 4-cycle system of $K_{19} - (c_{16} \cup c_{21} \cup c_{26} \cup (31,32,34,33))$ (these exist by (b)). Then a 4-cycle system $(\mathbb{Z}_{35},\ C)$ of $K_{35} - (c_0 \cup c_5 \cup c_{10} \cup c_{16} \cup c_{21} \cup c_{26} \cup (15,33, 31,32,34))$ can be formed by defining $C = C_1 \cup C_2 \cup K(\mathbb{Z}_{16},\ \mathbb{Z}_{32} \backslash \mathbb{Z}_{16})$. □

We now turn to settling the existence of 4-cycle systems of $K_n - E(F)$. Let $\epsilon(G) = |E(G)|$.

Theorem 2.1. *Let F be a* 2*-regular subgraph of* K_n*. There exists a* 4*-cycle system of* $K_n - E(F)$ *if and only if n is odd and* 4 *divides* $\epsilon(K_n) - \epsilon(F)$.

Proof. The necessity follows since each vertex in each 4-cycle has even degree, and since each 4-cycle contains 4 edges.

The sufficiency is proved by induction. If $n = 3$ then $F = C_3$, if $n = 5$ then no 2-regular graph F satisfies the necessary conditions, and if $n = 7$ then the result follows from Lemma 2.2(a).

Now suppose that for some $n = 2x + 1 \geq 9$, for all odd $z < 2x + 1$ and for any 2-regular subgraph F' of K_z for which 4 divides $\binom{z}{2} - \epsilon(F')$, there exists a 4-cycle system of $K_z - E(F')$. Let F be a 2-regular subgraph of K_{2x+1} such that 4 divides $\binom{n}{2} - \epsilon(F)$. Notice that if $|E(F')| = |E(F)| - 3$, then $\binom{2x-1}{2} - \epsilon(F') = \binom{2x+1}{2} - \epsilon(F) - 4(x-1)$, so by induction there exists a 4-cycle system of $K_{2x-1} - E(F')$. Notice also that it suffices to assume that $|V(F)| \geq 2x - 2$, since otherwise cycles of length 4 can be added to F to form a larger 2-regular graph that also satisfies the necessary conditions. We consider four cases in turn.

Case 1. Suppose that F contains a cycle c of length 3, say $c = (2x-2, 2x-1, 2x)$. Let $F' = F - c$. By induction there exists a 4-cycle system $(\mathbb{Z}_{2x-1}, C)$ of $K_{2x-1} - E(F')$. Then $(\mathbb{Z}_{2x-1}, C) \cup K(\{2x-1, 2x\}, \mathbb{Z}_{2x-2})$ is a 4-cycle system of $K_{2x+1} - F$.

Case 2. Suppose that F contains a cycle of length $k \geq 6$, say $c = (2x-k+1, 2x-k+2, \ldots, 2x)$. Let $c_1 = (2x-k+1,\ 2x-k+2, \ldots, 2x-3)$ and let $F' = (F-c) \cup c_1$. Then by induction there exists a 4-cycle system $(\mathbb{Z}_{2x-1}, C)$ of $K_{2x-1} - E(F')$. Let the 4-cycle in C containing the edge $\{2x-3, 2x-2\}$ be $c_2 = (2x-3, 2x-2, z, y)$; clearly $y \neq 2x-k+1$ since $\{2x-k+1, 2x-3\}$ is in c_1. Let $C_1 = (C \backslash \{c_2\}) \cup \{(2x-k+1, 2x-3, y, 2x-1),$ $(y, z, 2x-2, 2x)\} \cup K(\{2x-1, 2x\}, \mathbb{Z}_{2x-1} \backslash \{2x-k+1, 2x-2, y\})$. Then $(\mathbb{Z}_{2x+1}, C_1)$ is easily seen to be a 4-cycle system of $K_{2x+1} - E(F)$.

Case 3. Suppose that F contains at least two cycles of length 4, say $c_1 = (2x-7, 2x-6, 2x-5, 2x-4)$ and $c_2 = (2x-3, 2x-2, 2x-1, 2x)$. By induction there exists a 4-cycle system $(\mathbb{Z}_{2x-1}, C)$ of $K_{2x-1} - E(F')$, where $F' = (F - (c_1 \cup c_2)) \cup (2x-7, 2x-6, 2x-5, 2x-4, 2x-3)$. Let the 4-cycles in C that contain the edges $\{2x-2, 2x-3\}$ and $\{2x-4, 2x-7\}$ be $c_3 = (2x-2, u, w, 2x-3)$ and $c_4 = (2x-4, y, z, 2x-7)$ respectively ($c_3 \neq c_4$ since $2x-3$ is joined to $2x-4$ and to $2x-7$ in a cycle in F'). Then either $w \neq y$ or $w \neq z$.

If $w \neq y$ then let $C_1 = C \cup K(\{2x, 2x-1\},\ \mathbb{Z}_{2x-1} \backslash \{2x-7, 2x-3, 2x-2, w, y\}) \cup \{(2x-2, 2x, w, u), (w, 2x-3, 2x-7, 2x-1), (2x, 2x-7, z, y),$ $(y, 2x-4,\ 2x-3, 2x-1)\}$. Otherwise, since $w \neq z$, let $C_1 = C \cup K(\{2x, 2x-1\},$ $\mathbb{Z}_{2x-4} \backslash \{w, z\}) \cup \{(2x-2, 2x, w, u),\ (w, 2x-3, 2x-4, 2x-1),\ (2x-4, y, z, 2x),$ $(z, 2x-7, 2x-3, 2x-1)\}$. In either case, $(\mathbb{Z}_{2x+1}, C_1)$ is a 4-cycle system of $K_{2x+1} - E(F)$.

Case 4. Suppose that F consists of cycles of length 5, except possibly for one 4-cycle; since $n \geq 9$, F contains at least two cycles. We consider two possibilities in turn.

First suppose that F is not a 2-factor of K_n. Then let $c = (n-1, n-2, n-3, n-4, n-5) \in F$, and we can assume that $n-6$ is a vertex that is in no cycle of F. Let $(\mathbb{Z}_{n-6}, C_1)$ be a 4-cycle system of $K_{n-6} - E(F-c)$; this exists by induction because, since 4 divides $\binom{n}{2} - \epsilon(F)$, 4 divides $\binom{n}{2} - \epsilon(F) - \left(\binom{n-6}{2} - (\epsilon(F) - 5)\right) = 12x - 20$, so 4 divides $\binom{n-6}{2} - (\epsilon(F) - 5)$. Let $(\{i \mid n-7 \leq i \leq n-1\},\ C_2)$ be a 4-cycle system of $K_7 - c$; this exists by Lemma 2.2(a). Then $(\mathbb{Z}_n, C_1 \cup C_2 \cup K(\mathbb{Z}_{n-7}, \{i \mid n-6 \leq i \leq n-1\}))$ is a 4-cycle system of $K_n - E(F)$.

Finally, suppose that F is a 2-factor of K_n. If F consists of one 4-cycle and $(n-4)/5$ 5-cycles, then 4 divides $\binom{n}{2} - n$ and 5 divides $(n-4)$, so $n \equiv 19 \pmod{40}$. If F consists of $n/5$ 5-cycles, then similarly $n \equiv 35 \pmod{40}$. By Lemma 2.2(b) and (c) there exists a 4-cycle system $(\mathbb{Z}_{n_1}, C_1)$ of $K_{n_1} - E(F_1)$ if $n_1 = 19$ or 35 respectively, where F_1 is a 2-factor consisting of 5-cycles and at most one 4-cycle. Let $(\mathbb{Z}_n - \mathbb{Z}_{n_1-1}, C_2)$ be a 4-cycle system of $K_{n-n_1+1} - E(F_2)$ where F_2 consists of $(n - n_1)/5$ 5-cycles, none of which includes the vertex $n_1 - 1$; this exists by the first possibility in Case 4. Then $(\mathbb{Z}_n, C_1 \cup C_2 \cup K(\mathbb{Z}_{n_1-1}, \mathbb{Z}_n - \mathbb{Z}_{n_1}))$ is a 4-cycle system of $K_n - E(F)$ as required. □

The necessary condition that n be odd in Theorem 2.1 can be avoided by considering 4-cycle systems in $2K_n - E(F)$, and so this is addressed in the following result.

Theorem 2.2. *Let F be a 2-regular subgraph of $2K_n$. There exists a 4-cycle system of $2K_n - E(F)$ if and only if 4-divides $\epsilon(2K_n) - \epsilon(F)$ and $n \neq 3$.*

Proof. The necessity is clear, so suppose 4 divides $n(n-1) - \epsilon(F)$ and $n \neq 3$. The proof is by induction on n. It is trivial to solve the problem for $n \leq 5$. Again, it suffices to assume that $|V(F)| \geq n - 3$ (for otherwise, vertex disjoint 4-cycles can be added to F).

Suppose that $n \geq 6$, and that for all $4 \leq n' < n$ and for any 2-regular subgraph F of $2K_{n'}$ for which $n(n-1) - \epsilon(F')$ is divisible by 4, there exists a 4-cycle system of $2K_{n'} - E(F')$. We consider three cases in turn. Notice that

$$4 \text{ divides } 4n + 8 = n(n-1) - \epsilon(F) - ((n-2)(n-3) - (\epsilon(F) - 2)). \qquad (*)$$

Case 1. If F contains a 2-cycle $c = (n-2, n-1)$ then let $F' = F - c$. By induction and $*$, there exists a 4-cycle system $(\mathbb{Z}_{n-2}, C_1)$ of $2K_{n-2} - E(F')$, so $(\mathbb{Z}_n, C_1 \cup 2K(\{n-2, n-1\}, \mathbb{Z}_{n-2}))$ is a 4-cycle system of $2K_n - E(F)$.

Case 2. If F contains a cycle of length $x \geq 4$, say $c = (n-1, n-2, \ldots, n-x)$, then let $F' = (F - c) \cup c'$ where $c' = (n-3, \ldots, n-x)$ (so if c is a 4-cycle then c' is a 2-cycle). By induction and $*$, there exists a 4-cycle system $(\mathbb{Z}_{n-2}, C_1)$ of $2K_{n-2} - E(F')$, so $(\mathbb{Z}_n, C_1 \cup 2K(\{n-1, n-2\}, \mathbb{Z}_{n-4}) \cup \{(n-1, n-3, n-2, n-4), (n-1, n-2, n-4, n-3)\})$ is a 4-cycle system of $2K_n - E(F)$.

Case 3. Suppose all cycles in F have length 3. If $n = 6$ then $(\{\mathbb{Z}_3 \times \mathbb{Z}_2, \{((i,0),(i,1), (i+1,1),(i+2,0)), ((i,0),(i,1),(i+2,0), (i+1,1)) \mid i \in \mathbb{Z}_3\})$ is a 4-cycle system of $2K_6 - E(F)$, and if $n = 7$ then $(\{\infty\} \cup (\mathbb{Z}_3 \times \mathbb{Z}_2), \{((i,0),(i,1),(i+1,0),(i+1,1)), ((i,0),(i+1,0),(i,1),\infty), ((i,0),(i+1,1), (i,1),\infty) \mid i \in \mathbb{Z}_3\})$ is a 4-cycle system of $2K_7 - E(F)$, in each case with $F = \cup_{j \in \mathbb{Z}_2}((0,j),(1,j),(2,j))$. Otherwise, since $|V(F)| \geq n - 3$, the necessary condition implies that $n \geq 12$. Let F contain the 3-cycles $c_1 = (n-1, n-2, n-3)$ and $c_2 = (n-4, n-5, n-6)$. Let $(\mathbb{Z}_{n-6}, C_1)$ be a 4-cycle system of $2K_{n-6} - E(F')$ with $F' = F - (c_1 \cup c_2)$, and let $(\mathbb{Z}_n \backslash \mathbb{Z}_{n-7}, C_2)$ be a 4-cycle system of $2K_7 - (c_1 \cup c_2)$. Then $(\mathbb{Z}_n, C_1 \cup C_2 \cup 2K(\mathbb{Z}_{n-7}, \mathbb{Z}_n \backslash \mathbb{Z}_{n-6}))$ is a 4-cycle system of $2K_n - E(F)$. □

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