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Steiner Quadruple Systems with a Spanning Block Design

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1. Introduction.

A Steiner system $S(t, k, v)$ is a pair (S, s) where S is a v–set and s is a collection of k–subsets (blocks) of S such that every t–subset of S is contained in exactly one member of s. A system $S(3, 4, v)$ is called a Steiner quadruple system of order v (briefly $SQS(v)$). It is well–known that an $SQS(v)$ exists if and only if $v \equiv 2$ or $4 \pmod 6$. [3] A system $S(2, 4, v)$ is also called a block design with block size equals to 4, and it exists if and only if $v \equiv 1$ or $4 \pmod{12}$. [4]

A spanning block design (SBD) of a Steiner quadruple system (Q, q) is a subcollection b of q such that (Q, b) is a block design. It is conjectured by C. C. Lindner that we can construct an $SQS(v)$ which contains an SBD for each $v \equiv 4 \pmod{12}$.

In this paper, we verify this conjecture for $v = 4^k$; k is a positive integer. We also give a recursive construction which shows that if an $SQS(v)$ contains an SBD, then we can construct an $SQS(4v)$ which contains an SBD.

2. The main theorems.

A 1–factor of K_{2n} (complete graph of order $2n$) is a collection of 2–element subsets (edges) of the vertex set $V(K_{2n})$ such that each vertex in $V(K_{2n})$ is contained in precisely one of these edges. A 1–factorization of K_{2n} is a partition of the edges of K_{2n} into 1–factors. We now give the well–known doubling construction for SQS.

Doubling construction. [5]

Let (X,q_1) and (Y,q_2) be any two $SQS(v)$ with $X \cap Y = \phi$. Let $F = \{F_1, F_2, ..., F_{v-1}\}$ and $G = \{G_1, G_2, ..., G_{v-1}\}$ be any two 1–factorizations of X and Y

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respectively, and let α be any permutation on the set $\{1, 2, ..., v-1\}$. Define a collection q of blocks as follows:

(1) Any block belonging to q_1 or q_2 belongs to q; and

(2) If $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $\{x_1,x_2,y_1,y_2\} \in q$ if and only if $\{x_1,x_2\} \in F_i$, $\{y_1,y_2\} \in G_j$, and $i\alpha = j$.

It is a routine matter to see that $(X \cup Y, q)$ is an SQS(2v). We will refer to the blocks obtained from (1), and (2) as blocks of the first and second type respectively.

In what follows we denote a latin cube C of order v by a v–tuple $(L_1, L_2, ..., L_v)$ where $L_1, L_2, ..., L_v$ are pairwise disjoint latin squares of order v.

4v construction.

Let $Q_i = \{(x,i) : x \in S, |S| = v\}$, $i = 1, 2, 3$, and 4, be four disjoint v–sets, and (Q_i, q_i) be an SQS(v). Also, let F_1, F_2, F_3, and F_4 be 1–factorizations of Q_1, Q_2, Q_3, and Q_4 respectively. Define a collection of blocks q on $Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ as follows:

(1) Any block belonging to q_i belongs to q;

(2) The blocks of second type which are obtained from the doubling construction by F_i and F_j, $i \neq j$, belong to q; and

(3) If C is a latin cube of order v, then $\{(x,1), (y,2), (z,3), (w,4)\}$ is a block in q if and only if the (x,y) entry in L_z is w.

It is not difficult to check that (Q,q) is an SQS(4v) with four disjoint subsystems of order v.

A transversal T of a v × v latin square is a collection of v cells no two of which are in the same row or column. We will denote by $|T|$ the number of distinct symbols appearing in the transversal T. A latin cube $C = (L_1, L_2, ..., L_v)$ is called a P–cube if there exist v transversals T_i (from L_i), $i = 1, 2, ..., v$, such that

$|T_i| = v$, $T_i \cap T_j = \phi$, and the $v \times v$ array obtained from $\bigcup_{i=1}^{v} T_i$ forms a latin square of order v. As an example, the latin cube in Figure 2.1 is a P–cube of order 4, where $T_1 = \{(1,1), (2,2), (3,3), (4,4)\}$, $T_2 = \{(1,2), (2,1), (3,4), (4,3)\}$, $T_3 = \{(1,3), (2,4), (3,1), (4,2)\}$ and $T_4 = \{(1,4), (2,3), (3,2), (4,1)\}$.

L_1

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

L_2

2	3	1	4
4	1	3	2
3	2	4	1
1	4	2	3

Figure 2.1

L_3

3	2	4	1
1	4	2	3
2	3	1	4
4	1	3	2

L_4

4	1	3	2
2	3	1	4
1	4	2	3
3	2	4	1

Figure 2.1

Lemma 2.1. There exists an SQS(16) which contains an SBD.

Proof. Let (Q,q) be the SQS(16) obtained from above 4v construction with v = 4 and C be the latin cube in Figure 2.1. It is not difficult to see that (Q,b) is an SBD where b is the collection of 4 blocks from (1) and 16 blocks from (3) by looking at the cells of C which are (x, y, z), $(x, y) \in T_z$. We conclude the proof.

Let $C = (L_1, L_2, \ldots, L_u)$ and $D = (M_1, M_2, \ldots, M_v)$ be two latin cubes of order u and v respectively. The direct product of C and D, denoted by $C \times D$, is the latin cube $(L_1 \times M_1, L_1 \times M_2, \ldots, L_1 \times M_v, L_2 \times M_1, \ldots, L_u \times M_v)$ where $L_i \times M_j$ is the direct product of two latin squares L_i and M_j. Then we have the following lemma.

Lemma 2.2. If C and D are P–cubes, then C × D is a P–cube.

Proof. We let the transversal in $L_i \times M_j$ be the cells which are obtained by the transversals in L_i and M_j. This concludes the proof.

Theorem 2.3. For each $v = 4^k$, k a positive integer, we can construct an SQS(v) which contains an SBD.

Proof. We prove this theorem inductively. If k = 1, it is obvious. Assume it's true for k = 1, and also we have a P–cube of order 4^{k-1}. (Lemma 2.2) By 4v construction, the collection of blocks obtained from the SBD of each SQS(4^{k-1}), and the transversals of the P–cube of order 4^{k-1}, we have an SQS(4^k) which contains an SBD. This concludes the proof.

We note here that this result can also be seen in [1,6], but the constructions are different.

Now we will take a close look at the P–cube. It is not difficult to see if there exists a P–cube of order v, then we can construct two latin squares of order v which are orthogonal. One of them is by definition, and the other is obtained by letting all entries in T_i be i. The problem of constructing a P–cube by using two mutually orthogonal latin squares is still open in general. (For n = 3, two mutually orthogonal latin squares exist, but no P–cube.) But if there exist three mutually orthogonal latin squares of order v, then we are able to construct a P–cube.

Lemma 2.4. If there exist three mutually orthogonal latin squares of order v, then there exists a P–cube of order v.

Proof. Let L = [L(i,j)], M = [M(i,j)],and N = [N(i,j)] be three mutually orthogonal latin squares of order v. Define B(M(i,j), N(i,j)) = L(i,j). Since M and N are orthogonal, B is well–defined. Now let C = [C(i,j,k)] = [B(M(i,j), k)]. We are going to show that C is a P–cube. Because N and L are orthogonal, for each entry k (fixed), $\{L(i,j) : N(i,j) = k\}$ is a set of v distinct elements. Hence, there are v transversals T_k ($T_k = \{(i,j,k) : N(i,j) = k\}$), $k = 1, 2, \ldots, v$, $|T_k| = v$, and if we put these transversals together, we have the latin square L. Since it is not difficult to see that C is a latin cube, we have the proof.

By Lemma 2.4, 4v construction and the fact that there exist three mutually orthogonal latin squares of order $v \geq 16$ [2], we have the following recursive construction.

Theorem 2.5. If there exists an SQS(v) which has an SBD, then we can construct an SQS(4v) which contains an SBD.

Proof. $4 \cdot v(v-1)/12 + v^2 = 4v(4v-1)/12.$

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