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Note

A note on ascending subgraph decompositions of complete multipartite graphs ☆

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Abstract

In this note, we prove that the ascending subgraph decomposition conjecture is true for complete multipartite graphs.

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The following conjecture about decomposing a graph G of size $\binom{n+1}{2} \leqslant |E(G)| < \binom{n+2}{2}$ into n ascending subgraphs has been one of the most fascinating problems regularly mentioned by P. Erdös in his talks on 'Unsolved Problems'.

Ascending subgraph decomposition conjecture (ASD conjecture). Let G be a graph of size $\binom{n+1}{2} \leqslant |E(G)| < \binom{n+2}{2}$. Then, $E(G)$ can be partitioned into n sets $E_1, E_2, \ldots, E_n$ which induce subgraphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| < |E(G_{i+1})|$ and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$.

A graph G is said to have an ascending subgraph decomposition $G_1, G_2, \ldots, G_n$ provided that the ASD conjecture holds for G. $G_1, G_2, \ldots, G_n$ are called members of the decomposition.

In order to verify this conjecture, the following revised conjecture attracts more attention than the original one.

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Revised ASD conjecture. Let G be a graph of size $\binom{n+1}{2} \leqslant q < \binom{n+2}{2}$. Then $E(G)$ can be partitioned into n sets $E_1, E_2, \ldots, E_n$ which induce subgraphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$, and $E(G_n) = q - \binom{n}{2}$. So far, quite a few classes have been verified to satisfy this revised ASD conjecture, such as star forests [1,6], forests [4,10], graphs with bounded maximum degrees [5,7], split graphs [9], complete bipartite graphs [8], regular bipartite graphs [3], etc., but it is believed that to prove the conjecture in general is going to be very difficult.

In this note, we shall prove that every complete multipartite graph does have an ascending subgraph decomposition. In order to prove the main result, we need two definitions and several lemmas.

A graph G is said to have an *n-star decomposition* if $|E(G)| \leqslant \binom{n+1}{2}$ and G can be decomposed into n stars $G_1, G_2, \ldots, G_n$ such that (i) all stars have different centers, (ii) $|E(G_i)| \leqslant i$ for all i, and (iii) $|E(G_i)| \leqslant |E(G_j)|$ for $i < j$. And a graph G with size $\binom{n+1}{2} + t$, $0 < t$, is said to have an *(n,t)-star decomposition* if G can be decomposed into $G_1, G_2, \ldots, G_n, T$ such that (i) all G_i's are stars with different centers and (ii) $|E(G_i)| = i$ for $i = 1, 2, \ldots, n$ and $|T| = t$.

Lemma 1. *Let G be a graph with $|E(G)| \leqslant \binom{n+1}{2}$. If $V(G) = X \cup Y$ and the subgraph of G induced by $Y, G[Y]$, has an n'-star decomposition where $n' < n, G[X]$ is an empty graph, $|X| = n - n'$, $|Y| = n$ and $G \setminus G[Y]$ is a complete bipartite graph. Then G has an n-star decomposition.*

Proof. Clearly, $G \setminus G[Y]$ can be decomposed into $n - n'$ stars of size n and all the centers are in X. Let those $n-n'$ stars be $G'_{n'+1}, G'_{n'+2}, \ldots, G'_n$. Since $G[Y]$ has an n'-star decomposition, let it be $G'_1, G'_2, \ldots, G'_{n'}$. Now $|E(G'_j)| - j = n - j$ for $j = n'+1, n'+2, \ldots, n$. Thus, there are at least $n - j$ G'_i's for $i = 1, 2, \ldots, n'$ such that $i - |E(G'_i)| > 0$. Starting from $j = n'+1$, we delete $n - n' - 1$ edges from $G'_{n'+1}$ in which these edges are incident to the centers of G'_i's where $i - |E(G'_i)| > 0$. Then, add these edges to G'_i, respectively. Note that if there are more than $n - n' - 1$ G'_i's with $i - |E(G'_i)| > 0$, we shall add the edge to G'_i which has larger $i - |E(G'_i)|$. By repeating this process, delete $n - n' - 2$ edges from $G'_{n'+2}$, $n - n' - 3$ edges from $G'_{n'+3}$, etc., we conclude the proof. □

Lemma 2. *Let G be a graph with $|E(G)| \leqslant \binom{n+1}{2}$. If $V(G) = X \cup Y$, $G[Y]$ has an n'-star decomposition where $n' < n$, $G[X]$ is an empty graph, $|X| = n - n'$ and all the vertices in X have degree not greater than n', then G has an n-star decomposition.*

Proof. Since the bipartite graph obtained from (X, Y) can be decomposed into $n - n'$ stars with centers in X and all of these stars are of size not greater than n', an n-star decomposition of G can be obtained by rearranging the members in the n'-star decomposition of $G[Y]$ and those new stars from (X, Y). □

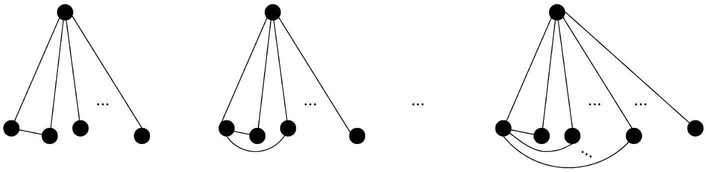

Fig. 1. Pregnant stars.

Lemma 3. *Let G be a graph with $|E(G)| = \binom{n+1}{2} + t$, $0 < t$. If $V(G) = X \cup Y$, $G[Y]$ has an n'-star decomposition where $n' < n$, $G[X]$ is an empty graph, $|X| = n - n'$, $|Y| = n$ and $G \setminus G[Y]$ is a complete bipartite graph. Then G has an (n,t)-star decomposition.*

Proof. Let $G'_1, G'_2, \ldots, G'_{n'}$ be the n' stars obtained by the n'-star decomposition of $G[Y]$ and $G'_{n'+1}, G'_{n'+2}, \ldots, G'_n$ be the stars of size n obtained from the decomposition of the complete bipartite graph (X,Y). Consider $i \leqslant n'$, where $m = i - |E(G'_i)|$ is maximum. Then there are at least m of $G'_{n'+1}, G'_{n'+2}, \ldots, G'_n$ satisfying $|E(G'_j)| > j$ for $j \in \{n'+1, n'+2, \ldots, n\}$. (Choose the ones with larger $|E(G'_j)| - j$.) Therefore, we can delete one edge from each of the above members and add them to G'_i. (Note that the center of G'_i is adjacent to all the centers of G'_j where $j = n'+1, n'+2, \ldots, n$.) By repeating the above process, we have the members $G_1, G_2, \ldots, G_{n'}$ where $|E(G_i)| = i$ for $i = 1, 2, \ldots, n'$. As the larger members, $G_{n'+1}, \ldots, G_n$, we can delete t edges from them suitably and the set of t edges gives the T we need. □

Since the proof of the main result is quite complicated, we believe that an explanation of how we do it will be helpful in going through the details of the proof.

Our goal of decomposition is to obtain $G_1, G_2, \ldots, G_{n-1}, G_n \cup T$ where $|E(G_i)| = i$ and $|T| = t$. For the smaller members, we shall use stars. Although, it is quite possible that we have a decomposition in which every member is a star, but if this is not so, we shall mainly use pregnant stars (small star hiding in a large star) for the larger G_i's, see Fig. 1, and the smaller members remain as stars.

In order to obtain the decomposition, we shall first decompose the complete multipartite graph $K_{m_1, m_2, \ldots, m_k}$, $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k$, into $k-1$ complete bipartite graphs $H_i = K_{m_i, m_{i-1}+m_{i-2}+\cdots+m_1}$, $i = 2, 3, \ldots, k$. Therefore $|E(H_i)| = m_i(m_1 + m_2 + \cdots + m_{i-1})$, $i = 2, 3, \ldots, k$. Then, $K_{m_1, m_2, \ldots, m_k}$ can be depicted as Fig. 2.

Fig. 2 will give us a rough idea of the decomposition we are looking for. First, we claim that $\sum_{i=1}^k m_i > n \geqslant \sum_{i=2}^k m_i = \delta(G)$. The left-hand inequality is easy to see. Assume that $n < \sum_{i=2}^k m_i$. Then $\delta(G) \geqslant n+1$. This implies that $|V(G)| \geqslant n+2$ and therefore $|E(G)| \geqslant \frac{1}{2}(n+1)(n+2)$. This is a contradiction. Hence, in Fig. 2, there exists an R in $(P_0, P_1]$ such that $RQ = n$. Now, we can draw a dashed line $\overline{RR'}$ such that $\angle R'RQ = 45^\circ$ and this dashed line provides some information for the decomposition. For example, $\triangle B_2$ tells us how many edges can be removed from H_2 in order to have members which are stars, and $\triangle A_1$ shows the deficiency we have in order to construct

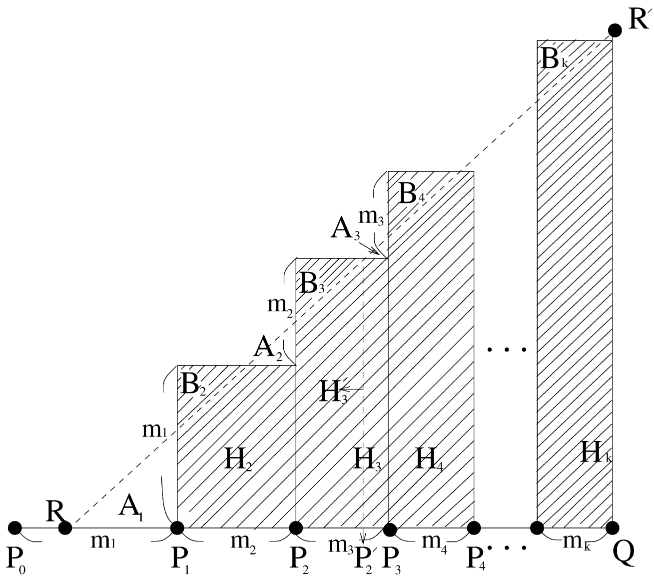

Fig. 2. $K_{m_1,m_2,\ldots,m_k}$.

stars as small members. Also, by Fig. 2 and geometric arguments, since

$$\begin{aligned}&(\text{the height of } H_{i+1}) - (\text{the height of } H_i)\\ &\quad = (m_1 + m_2 + \cdots + m_i) - (m_1 + m_2 + \cdots + m_{i-1})\\ &\quad = m_i\\ &\quad = \text{the width of } H_i, \text{ and } \angle R'RQ = 45^\circ,\end{aligned}$$

the height (vertical) of $\triangle B_2, \triangle B_3, \ldots$ are equal and the height (vertical) of $\triangle A_1, \triangle A_2, \triangle A_3, \ldots$ are not increasing.

Now, the decomposition will be obtained recursively, starting from the small members.

Theorem 4. *Let $G = K_{m_1,m_2,\ldots,m_k}$ with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k \geqslant 1$ and $|E(G)| = \binom{n+1}{2} + t$, $0 \leqslant t \leqslant n$. Then G has an ASD.*

Proof. Let $V(G) = \bigcup_{i=1}^{k} V_i$ where $|V_i| = m_i$. Clearly, G can be decomposed into n stars $S_1', S_2', S_3', \ldots, S_n'$ such that the first $n - (\sum_{i=2}^{k} m_i)$ stars have zero edges, then there are m_2 stars with m_1 edges, m_3 stars with $m_1 + m_2$ edges, etc., and m_k stars with $\sum_{i=1}^{k-1} m_i$ edges.

For $i = 1, 2, \ldots, k$, let A_i denote the sum of $j - |S_j'|$ for all j where the center of S_j' is in V_i and $j - |S_j'| > 0$, and let B_i denote the sum of $|S_j'| - j$ for all j where the center of S_j' is in V_i and $|S_j'| - j > 0$. Now, we consider three cases.

Case 1: $\triangle(G) = m = \sum_{i=1}^{k-1} m_i \leqslant n$. By the argument following Fig. 2, we conclude that $B_1 = 0$, $B_2 = B_3 = \cdots = B_k$ and $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_k$. Since $\sum_{i=1}^{k-1} m_i + m_k = \sum_{i=1}^{k} m_i > n$, hence $m_2 \geqslant m_k > n - m$. Delete $n - m$ stars $X_{n-m}, X_{n-m-1}, \ldots, X_1$ with $n-m, n-m-1, \ldots, 1$ edges, respectively, starting from the left-hand side of $\overline{P_1P_2}$, and then add these $n - m$ stars to $S'_n, S'_{n-1}, \ldots, S'_{m+1}$ to obtain $G_n, G_{n-1}, \ldots, G_{m+1}$. By the reason that S'_j, $j = m+1, \ldots, n$, has center in V_k and S'_j is incident to each vertex of $\bigcup_{i=1}^{k-1} V_i$, $G_{m+1}, G_{m+2}, \ldots, G_n$ are pregnant stars. Now, we construct the small members recursively. First, it is clear that $H_2 \setminus \bigcup_{i=1}^{n-m} X_i$ has an $(n - \sum_{i=3}^{k} m_i)$-star decomposition. By Lemma 1, $H = (H_2 \setminus \bigcup_{i=1}^{n-m} X_i) \cup H'_3$ (see Fig. 2) has an $n' = (n - \sum_{i=4}^{k} m_i - P'_2P_3)$-star decomposition in case that the above graph has at most $\binom{n'+1}{2}$ edges. On the other hand, if the above graph has more than $\binom{n'+1}{2}$ edges, then by Lemma 3, we have an (n', t')-star decomposition for some t'. Here H'_3 is a part of H_3 with height $n_3 = m_1 + m_2$ and base $= |\{j \mid |S_j| > j$ and the center of S_j is in $V_3\}|$. By Lemma 2, the n'-star (or (n', t')-star) decomposition of H can be extended to $H \cup (H_3 \setminus H'_3)$. Continuing the above processes, we have an (n, t)-star decomposition for $G \setminus (\bigcup_{i=m+1}^{n} G_i)$. Then the proof follows by adding T to G_n.

Case 2: $\triangle(G) > n$ and $A_2 = A_3 = \cdots = A_k = 0$. First, if $m_1 > n$ then each star S'_i of positive size has at least n edges. By Theorem in [11], we have the desired ASD with each member a star. On the other hand, let $m_1 \leqslant n$. Let i be the largest integer such that $G[\bigcup_{j=1}^{i} V_j]$ contains edges not greater than $\binom{n'+1}{2}$ edges where $n' = n - \sum_{j=i+1}^{k} m_j$. By Lemma 1, we are able to obtain an n'-star decomposition of $G[\bigcup_{j=1}^{i} V_j]$ following a way similar to what we have in Case 1. Since $A_j = 0$ for each $j \geqslant i$, for each $l > i$, $G[\bigcup_{j=1}^{l} V_j]$ contains more than $\binom{n' + \sum_{j=i+1}^{l} m_j}{2}$ edges. By Lemma 3, $G[\sum_{j=1}^{l} V_j]$ has an (n_l, t_l)-star decomposition where $n_l = n' + \sum_{j=i+1}^{l} m_j$ and some $t_l > 0$. This implies that $G(l = k)$ has an ASD by adding T to the largest member G_n where T is obtained in an (n, t)-star decomposition of G. This concludes the proof of Case 2.

Case 3: $\triangle(G) > n$ and $A_2 > 0$. Let s be the integer such that $A_{s-1} > 0$ and $A_s = 0$. ($A_k = 0$ since $\triangle(G) > n$ and $A_2 > 0$.) There are two situations to consider:

(i) $B_{s-1} \leqslant A_{s-1}$. Since $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_{s-1} \geqslant B_{s-1} = B_{s-2} = \cdots = B_2$ and $B_1 = 0$, $\sum_{i=1}^{j-1} A_i \geqslant \sum_{i=2}^{j} B_i$ for $j \leqslant s$. Hence $G[\bigcup_{i=1}^{s} V_i]$ contains at most $\binom{n'+1}{2}$ edges for some $n' = n - \sum_{j=s+1}^{k} m_j$. Then by Lemmas 1 and 2, $G[\bigcup_{i=1}^{s} V_i]$ has an n'-star decomposition. Thus, for $k \geqslant x > s, G[\bigcup_{i=1}^{x} V_i]$ has an n_x-star decomposition or (n_x, t_x)-star decomposition. By the time $x = k$, we have the ASD.

(ii) $B_{s-1} > A_{s-1}$. First, if $s-1 \geqslant 3$, then rearrange $V_1, V_2, \ldots, V_k$ to the order $V_1, V_2, \ldots, V_{s-2}, V_s, V_{s+1}, \ldots, V_k, V_{s-1}$. Now, the proof can be obtained by a similar idea as that of Case 1. Therefore, it is left to consider $s-1=2$. It is easy to see that there are some S'_j's with centers in V_2 and are of size less than their index. Let the number of such stars be u and clearly $|V_2| \geqslant 2u+1$. Let G' be the graph obtained by deleting $2u+1$ vertices from V_2 such that the number of edges deleted is $\binom{2u+1}{2} + (n-2u)(2u+1) = (2u+1)(n-u)$. Let $n' = n - (2u+1)$, then G' has $n'(n'+1)/2 + t$ edges. As in Case 2, G' has an (n', t)-star decomposition $G_1, G_2, \ldots, G_{n'}, T$. Observe that the deleted $2u+1$ vertices and their neighbors form a complete bipartite graph (X, Y, E') where $|X| = 2u + 1$ and $|Y| = n - u$. Therefore, (X, Y, E') can be decomposed into $G_{n'+1}, G_{n'+2}, \ldots, G_n$ such that G_i is a star if $i \leqslant n - u$ and if $i > n - u$ then G_i is a union of two stars with size $n - u$ and $i - (n - u)$ such that it is a double star with common leaves (of small star). Now, combining the two decompositions and adding T to G_n we have the desired ASD. □

For further reading

The following reference is also of interest to the reader: [2].

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References

[1] Y. Alavi, A.J. Boals, G. Chartrand, P. Erdös, O. Oellerman, The ascending subgraph decomposition problem, Congr. Numer. 58 (1987) 7–14.
[2] Fu-Long Chen, Hung-Lin Fu, Partition integral set into subsets with prescribed sums, in preprints.
[3] H. Chen, K. Ma, On the ascending subgraph decompositions of regular graphs, Appl. Math. J. Chinese Univ. Ser. B 13 (2) (1998) 165–170.
[4] R.J. Faudree, R.J. Gould, Ascending subgraph decomposition for forests, Congr. Numer. 70 (1990) 221–229.
[5] R.J. Faudree, R.J. Gould, M.S. Jacobson, L. Lesniak, Graphs with an ascending subgraph decomposition, Congr. Numer. 65 (1988) 33–41.
[6] R.J. Faudree, A. Gyárfás, R.H. Schelp, Graphs which have an ascending subgraph decomposition, Congr. Numer. 59 (1987) 49–54.
[7] H.L. Fu, A note on the ascending subgraph decomposition problem, Discrete Math. 84 (1990) 315–318.
[8] H.L. Fu, Some results on the ascending subgraph decomposition, Bull. Inst. Math. Acad. Sin. 16 (4) (1988) 341–345.
[9] H.L. Fu, W.H. Hu, Two classes of graphs which have ascending subgraph decompositions, Arc Combin. 35-A (1993) 65–70.
[10] Wei-Hsin Hu, On ascending subgraph decomposition and its related problems, Ph.D. Thesis, National Chiao Tung University, Hsin Chu, Taiwan, June 1992.
[11] K. Ma, H. Zhou, J. Zhou, On the ascending star subgraph decomposition of star forest, Combinatorica 14 (3) (1994) 307–320.