ELSEVIER

Discrete Mathematics 214 (2000) 241–243

DISCRETE MATHEMATICS

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Note

On the α-labeling number of bipartite graphs

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Received 9 November 1998; revised 11 May 1999; accepted 14 June 1999

Abstract

In this note we show that the α-labeling number of a bipartite graph G is bounded, which proves a conjecture of Snevily.

Keywords: α-labeling; Bipartite graph

1. Introduction

A vertex *labeling* (or *valuation*) of a graph G is an assignment γ of labels to the vertices of G that induces for each edge uv a label depending on the vertex labels $\gamma(u)$ and $\gamma(v)$. Rosa [5] called a function γ, a *β-labeling* of a graph G with q edges if γ is an injection from the vertices of G to the set $\{0,1,\dots,q\}$ such that, when each edge uv is assigned the label $|\gamma(u)-\gamma(v)|$, the resulting edge labels are distinct. A β-labeling is now more commonly called a *graceful* labeling. An *α-labeling* is a graceful labeling with the additional property that there exists an integer λ such that for each edge uv either $\gamma(u)\leqslant\lambda<\gamma(v)$ or $\gamma(v)\leqslant\lambda<\gamma(u)$. Note that if G admits an α-labeling then G is necessarily bipartite.

A large number of papers has been devoted to the topic of labelings of graphs (see Gallian [1] for an up-to-date survey). Because of applications in graph decompositions (see [5]), α-labelings are of particular interest.

Rosa showed in [5] that $K_{m,n}$ has an α-labeling for all positive integers m and n and that C_m has an α-labeling if and only if $m\equiv 0 \pmod 4$. On the other hand,

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0012-365X/00/$ - see front matter
PII: S0012-365X(99)00313-1

one can easily show that there exist bipartite graphs which do not have α-labelings. Examples of such graphs include 2-regular graphs with $4m+2$ edges, forests with more than one component and the trees obtained by subdividing each edge in $K_{1,k}$, $k\geqslant 3$ (see [5]).

Let G and H be simple graphs with G a subgraph of H. A *G-decomposition* of H is a partition of $E(H)$, the edge set of H, into subgraphs isomorphic to G. In this case we say that *G divides H* and write $G|H$.

In [6], Snevily introduced the following notion. A bipartite graph G is said to *eventually have an α-labeling* provided that there exists a graph H (a '*host*' graph) which has an α-labeling and $G|H$. The *α-labeling number of G* is defined to be $G_\alpha = \min\{t$: there exists a host graph H of G with $|E(H)| = t\cdot|E(G)|\}$. Snevily showed that if C is a cycle of even length then $C_\alpha \leqslant 2$ and proposed the following conjecture.

Conjecture 1 (*Snevily* [6]). If G is a bipartite graph then $G_\alpha < \infty$.

In this note we shall give a short proof of the above conjecture by showing that every bipartite graph divides a complete bipartite graph.

First, if G is a regular bipartite graph then the following result by Häggkvist shows that, the conjecture is true. Recall that $K_{m,n}$ has an α-labeling for all positive integers m and n [5].

Theorem 1 (Häggkvist [2]). *Let G be a k-regular bipartite graph on* $2n$ *vertices. Then* $G|K_{k^2!n,k^2!n}$.

Thus, it suffices to show that every bipartite graph divides a regular bipartite graph.

Theorem 2. *Every bipartite graph with q edges divides a q-regular bipartite graph.*

Proof. Let G be a bipartite graph with q edges. Let (A,B) be the bipartition of G, where $A=\{a_1,a_2,\ldots,a_s\}$ and $B=\{b_1,b_2,\ldots,b_t\}$. We shall construct a q-regular bipartite graph H with bipartition $(\mathscr{A},\mathscr{B})$ such that $G|H$. Let $\mathscr{A}$ be the disjoint union of $A_1,A_2,\ldots,A_t$ where $A_i=\{a_{i,1},a_{i,2},\ldots,a_{i,s}\}$ for $i=1,2,\ldots,t$ and $\mathscr{B}$ be the disjoint union of $B_1,B_2,\ldots,B_s$ where $B_j=\{b_{j,1},b_{j,2},\ldots,b_{j,t}\}$ for $j=1,2,\ldots,s$. Then, for $1\leqslant k,\ j\leqslant s$ and $1\leqslant i,l\leqslant t$, let $a_{i,k}b_{j,l}$ be an edge in $E(H)$ if and only if $a_{k+j\,(\mathrm{mod}\,s)}$ (which means here that the subscript takes values in the set $\{1,2,\ldots,s\}$) and $b_{l+i\,(\mathrm{mod}\,t)}$ (similarly, the subscript takes values in $\{1,2,\ldots,t\}$) are adjacent in G. It remains to show that H is q-regular and $G|H$.

First, for each $a_{i,k}\in A_i\subseteq\mathscr{A}$, $\deg_H(a_{i,k})=\sum_{j=1}^{s}\deg_G(a_{k+j\,(\mathrm{mod}\,s)})=q$. Similarly, for each $b_{j,l}\in B_j\subseteq\mathscr{B}$, $\deg_H(b_{j,l})=\sum_{i=1}^{t}\deg_G(a_{l+i\,(\mathrm{mod}\,t)})=q$. Hence H is q-regular. By the definition of H it is clear that the bipartite subgraph induced by $A_i\cup B_j$ is isomorphic to G and that $G|H$. $\square$

Combining the results we have proved

Theorem 3. *Let G be a bipartite graph, then* $G_\alpha < \infty$.

The bound obtained in Theorem 3 is clearly quite large. Snevily expects that $T_\alpha \leqslant n$ if T is a tree with n edges. A number of results in the literature on G-decompositions of complete bipartite graphs suggest that n is a reasonable bound for G_α for any bipartite graph G with n edges; in particular, we point out [2–4,7,8]. We note however that we know of no example of a bipartite graph G where $G_\alpha > 2$.

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