Almost Resolvable Directed m-cycle systems: m ≡ 3 (mod 6)

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ABSTRACT. Let $m \equiv 3 \pmod 6$. We show there exists an almost resolvable directed m-cycle system of D_n if and only if $n \equiv 1 \pmod m$, except possibly if $n \in \{3m+1, 6m+1\}$.

1 Introduction

A directed m-cycle $(v_0, \ldots, v_{m-1})$ is a graph with vertex set $\{v_i \mid i \in \mathbb{Z}_m\}$ and edge set $\{(v_i, v_{i+1}) \mid i \in \mathbb{Z}_m\}$, reducing the subscript modulo m. A directed *m-cycle system* of a directed graph G is an ordered pair (V, C) where C is a set of m-cycles whose edges partition the edge set of G defined on the vertex set V. A directed m-cycle system of *order n* is a directed m-cycle system of D_n, the complete symmetric directed graph. A *parallel class* of a directed m-cycle system (V, C) of G is a set of $|V|/m$ vertex-disjoint directed m-cycles in C, and an *almost parallel class* is a set of $(|V|-1)/m$ vertex-disjoint directed m-cycles in C. The vertex of V that is in no directed m-cycle in the almost parallel class π is said to be the vertex *missing* from π. A directed m-cycle system (V, C) of G is said to be *(almost) resolvable* if C can be partitioned into (almost) parallel classes.

The existence of almost resolvable directed m-cycle systems was settled when $m = 3$ by Bennett and Sotteau [1] in 1981, and when $m = 4$ by Bennett and Xuebin [2]. Just recently substantial progress has been made in other cases: $m = 5$ was solved by Ge and Zhu [6], and the case where m

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is even was completely settled [5]. Unfortunately solving the problem when m is odd has seemed to be very difficult. The solution for $m = 5$ required constructions of skew Room frames with additional properties together with the skew Room frame construction in [7]. It is unlikely that a complete solution to this problem will follow from this technique. However, it turns out that the case where m is divisible by 3 can be nearly solved by making use of frames, as will be shown here.

2 The Result

Let $\vec{K}_{a,a,a}$ be the 3-partite directed graph with vertex set $\mathbb{Z}_a \times \mathbb{Z}_3$ and directed edge set $\{(i,k),(j,k+1) \mid i,j \in \mathbb{Z}_a, k \in \mathbb{Z}_3\}$, reducing the second subscript modulo 3. Let D_u^3 be the u-partite complete directed graph with vertex set $\mathbb{Z}_u \times \mathbb{Z}_3$ and directed edge set $\{((i,k),(j,\ell)) \mid i,j \in \mathbb{Z}_u, k,\ell \in \mathbb{Z}_3, i \neq j\}$. A *holey parallel class* of D_u^3 with *deficiency* $i \in \mathbb{Z}_u$ is a 2-factor of the subgraph of D_u^3 induced by the vertex set $\mathbb{Z}_u \backslash \{i\} \times \mathbb{Z}_3$. A *directed 3-frame* of D_u^3 is a directed 3-cycle system (V, C) of D_u^3 in which C can be partitioned into holey parallel classes.

Lemma 2.1 *For each $i \in \mathbb{Z}_u$, the number of holey parallel classes with deficiency i in any directed 3-frame of D_u^3 is 3.*

Proof: D_u^3 has $18\binom{u}{2}$ directed edges, so any directed 3-frame of D_u^3 has $18\binom{u}{2}/3(u-1) = 3u$ holey parallel classes. Since each vertex has degree $6(u-1)$, so is in $3(u-1)$ holey parallel classes, the result follows. □

The following result is just what we need.

Theorem 2.2 ([3]) *For all $u \geq 4$ except possibly for $u = 6$, there exists a directed 3-frame of D_u^3.*

Several small almost resolvable directed k-cycle systems have been constructed.

Lemma 2.3 ([4], [6]) *For all odd $m \geq 3$ there exists almost resolvable directed m-cycle systems of orders $m+1$ and $2m+1$.*

We also need the following directed m-cycle systems.

Lemma 2.4 *For all $m \equiv 3 \pmod 6$ there exists a directed m-cycle system of $\vec{K}_{m/3,m/3,m/3}$ (having $m/3$ directed cycles).*

Proof: Let $m = 6t+3$. Define $\vec{K}_{m/3,m/3,m/3}$ on the vertex set $\mathbb{Z}_{2t+1} \times \mathbb{Z}_3$. For each $i \in \mathbb{Z}_{2t+1}$, define the directed m cycle $c_i = (c_{i,0}, c_{i,1}, \ldots, c_{i,m-1})$

as follows. Let $c_{i,0} = (0,0)$, and for $1 \leq j \leq m-1$, define $\ell_j \in \mathbb{Z}_3$ by $\ell_j \equiv j \pmod 3$, and let

$$c_{i,j} = \begin{cases} (c_{i,j-1} + i, \ell_j) & \text{if} \quad j \equiv 1 \text{ or } 2 \pmod 3, \\ (c_{i,j-3} + 1, \ell_j) & \text{if} \quad j \equiv 0 \pmod 3. \end{cases}$$

Then $\{\mathbb{Z}_{2t+1} \times \mathbb{Z}_3, \{c_i \mid i \in \mathbb{Z}_{2t+1}\}\}$ is a directed m-cycle system of $\vec{K}_{m/3,m/3,m/3}$ (which clearly has $m/3$ directed cycles). □

We are now ready for our main result.

Theorem 2.5 *Let $m \equiv 3 \pmod 6$. There exists an almost resolvable directed m-cycle system of order n if and only if $n \equiv 1 \pmod m$, except possibly if $n \in \{3m+1, 6m+1\}$.*

Proof: The necessity is clear, so we turn to the sufficiency. Let $n = um+1$. We will define an almost resolvable directed m-cycle system of order n or the vertex set $S = \{\infty\} \cup (\mathbb{Z}_u \times \mathbb{Z}_3 \times \mathbb{Z}_{m/3})$. By Lemma 2.3 we can assume that $u \geq 4$.

For each $w \in \mathbb{Z}_u$, let $(\{\infty\} \cup (\{w\} \times \mathbb{Z}_3 \times \mathbb{Z}_{m/3}), C_w)$ be an almost resolvable directed m-cycle system (see Lemma 2.3), and let $c_w(\infty)$ or $c(w,j,k)$ be the almost parallel class in C_w with deficiency ∞ or (w,j,k) respectively.

For any directed 3-cycle $\theta = (x,y,z)$, let $\{c_0(\theta), \ldots, c_{m/3}(\theta)\}$ be the set of $m/3$ directed cycles classes in a resolvable directed m-cycle system of $\vec{K}_{m/3,m/3,m/3}$ defined on the vertex set $\{x,y,z\} \times \mathbb{Z}_{m/3}$ (see Lemma 2.4). (In our case $\{x,y,z\} \subseteq \mathbb{Z}_u \times \mathbb{Z}_3$.)

For the final ingredient, for each $v \in \mathbb{Z}_u$ let $\pi_{v,1}, \pi_{v,2}$ and $\pi_{v,3}$ be the three holey parallel classes with deficiency v in a directed 3-frame of D_u^3 defined on the vertex set $\mathbb{Z}_u \times \mathbb{Z}_3$ (see Lemma 2.1 and Theorem 2.2).

Define

$$C = \left(\bigcup_{w \in \mathbb{Z}_u} C_w \right) \cup \{c_i(\theta) \mid i \in \mathbb{Z}_{m/3}, \theta \in \pi_{v,j}, v \in \mathbb{Z}_u, j \in \mathbb{Z}_3\}.$$

Then (S, C) is a directed m-cycle system of order n. To see this, we consider several cases. Let $(a,b,c), (d,e,f) \in \mathbb{Z}_u \times \mathbb{Z}_3 \times \mathbb{Z}_{m/3}$.

1. The directed edges $(\infty, (a,b,c))$ and $((a,b,c), \infty)$ are in directed m-cycles in $C_a \subseteq C$.
2. If $a = d$ then $((a,b,c),(d,e,f))$ is in a directed m-cycle in $C_a \subseteq C$.
3. If $a \neq d$, then $((a,b),(d,e))$ is a directed edge in a directed 3-cycle say $\theta \in \pi_{v,j}$ for some $v \in \mathbb{Z}_u$ and some $j \in \mathbb{Z}_3$, so $((a,b,c),(d,e,f))$ is in a directed m-cycle in $c_i(\theta)$ for some i, $1 \leq i \leq m/3$.

Since $|C| = u(m+1) + (m/3)(u-1)(u)3 = n(n-1)/m$ which is the number of directed m-cycles in a directed m-cycle system of order n, the claim follows.

We can easily see that (S, C) is almost resolvable as follows. Let

$$p(\infty) = \{c_w(\infty) \mid w \in \mathbb{Z}_u\},$$

and for each $w \in \mathbb{Z}_u$, $j \in \mathbb{Z}_3$ and $i \in \mathbb{Z}_{m/3}$, let

$$p(w, j, i) = \{c(w, j, i)\} \cup \{c_i(\theta) \mid \theta \in \pi_{w,j}\}.$$

Then clearly $p(\infty)$ and $p(w, j, i)$ are almost parallel classes missing ∞ and (w, j, i) respectively, and $C = p(\infty) \cup (\bigcup_{w \in \mathbb{Z}_u, j \in \mathbb{Z}_3, i \in \mathbb{Z}_{m/3}} p(w, j, i))$. □

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