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(m,n)-cycle systems

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Abstract

We describe a method which, in certain circumstances, may be used to prove that the well-known necessary conditions for partitioning the edge set of the complete graph on an odd number of vertices (or the complete graph on an even number of vertices with a 1-factor removed) into cycles of lengths $m_1, m_2, \dots, m_t$ are sufficient in the case $|\{m_1, m_2, \dots, m_t\}| = 2$. The method is used to settle the case where the cycle lengths are 4 and 5.

1. Introduction and notation

The obvious necessary conditions for the existence of a decomposition of the complete graph K_v into cycles $C_1, C_2, \dots, C_t$, of lengths $m_1, m_2, \dots, m_t$, whose edges partition the edge set of K_v are

- $3 \leqslant m_i \leqslant v$ for $i = 1, 2, \dots, t$;
- v is odd; and
- $m_1 + m_2 + \cdots + m_t = v(v-1)/2$.

When v is even, one may, instead, consider partitioning the edge set of the complete graph with a 1-factor removed $K_v \backslash F$ into cycles. In this case, the necessary conditions are

- $3 \leqslant m_i \leqslant v$ for $i = 1, 2, \dots, t$;
- v is even; and
- $m_1 + m_2 + \cdots + m_t = v(v-2)/2$.

The question of whether these necessary conditions are sufficient was asked by Alspach (1981). Although the question remains unsolved in general, the conditions

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have been proven to be sufficient in many cases and there are no known cases where they are not sufficient. Rosa (to appear) has shown that they are sufficient when $v \leqslant 10$.

A decomposition of K_v into cycles all of the same length m which partition the edge set of K_v is usually called an m-cycle system of K_v. The problem of finding all values of v for which there is an m-cycle system of K_v is unsolved for general m, though the necessary conditions have been shown to be sufficient for many values of m. Several results on m-cycle systems of $K_v \backslash F$ also exist. See Lindner and Rodger (1992) for a survey of m-cycle systems.

There exist several results for decompositions of K_v into cycles of more than one length, see Heinrich et al. (1989) for example. One of the results in Heinrich et al. (1989) is that if $m_i \in \{3,4,6\}$ for $i=1,2,\ldots,t$ then the above necessary conditions are sufficient. Recently, Adams et al. (1998) solved the problem for $m_i \in \{3,5\}$, $i=1,2,\ldots,t$.

Here, we present a method of proving (when certain conditions are satisfied) that the above necessary conditions are sufficient when $m_i \in \{m,n\}$ for $i=1,2,\ldots,t$; that is, two different cycle lengths only. We use the method to settle the smallest open case, $m_i \in \{4,5\}$ for $i=1,2,\ldots,t$.

We need the following notation.

- If G_2 is a subgraph of G_1 we denote by $G_1 \backslash G_2$ the graph with vertex set $V(G_1 \backslash G_2) = V(G_1)$ and edge set $E(G_1 \backslash G_2) = E(G_1) \backslash E(G_2)$.
- When a graph G is the union of edge disjoint graphs $G_1, G_2, \ldots, G_t$ we will write $G = G_1 + G_2 + \cdots + G_t$. The use of the $+$ symbol is restricted to the case in which the graphs $G_1, G_2, \ldots, G_t$ are edge disjoint.
- An m-cycle on $\{a_1, a_2, a_3, \ldots, a_m\}$ with edges $a_1a_2, a_2a_3, \ldots, a_{m-1}a_m, a_ma_1$ will be denoted by $(a_1, a_2, a_3, \ldots, a_m)$.
- An (m^r, n^s)-cycle system of a graph G is a set consisting of r m-cycles and s n-cycles whose edges partition $E(G)$.
- For any non-negative integer v, define $S_{m,n}(v) = \{(r,s)\colon\ mr + ns = v \text{ and } r,s \geqslant 0\}$ and for a given graph G, define $\text{Type}_{m,n}(G) = \{(r,s)\colon$ there exists an (m^r, n^s)-cycle system of $G\}$. Where it is clear what m and n are, we will omit the subscripts and just write $S(v)$ and Type(G).
- For $E \subseteq \mathbb{Z} \times \mathbb{Z}$ and $(r,s) \in \mathbb{Z} \times \mathbb{Z}$, define $(r,s) + E = \{(r+x, s+y)\colon (x,y) \in E\}$.
- For non-negative integers u and v with $v \geqslant u$, define $G_u^v = K_v \backslash K_u$ if u and v are odd, and $G_u^v = (K_v \backslash F_1) \backslash (K_u \backslash F_2)$ if u and v are even, where F_1 is a 1-factor of K_v and F_2 is a 1-factor of K_u with $F_2 \subseteq F_1$. If $u=0$ or 1 then we use G^v.

2. Main results

The proof of Lemma 2.1 is straightforward.

Lemma 2.1. *If* $G = G_1 + G_2 + \cdots + G_t$ *and for* $1 \leqslant i \leqslant t$, $(r_i, s_i) \in \text{Type}(G_i)$, *then* $(\sum_{i=1}^t r_i, \sum_{i=1}^t s_i) \in \text{Type}(G)$.

Theorem 2.2. *Let u, v and w be non-negative integers with $v \geqslant u$ and $v \geqslant w$. If*
(1) *there exists an m-cycle system of G_u^v;*
(2) *there exists an n-cycle system of G_w^v;*
(3) $\text{Type}(G^u) = S(|E(G^u)|)$ *and* $\text{Type}(G^w) = S(|E(G^w)|)$; *and*
(4) $(|E(G^w)| + |E(G^u)|) - |E(G^v)| \geqslant 0$,
then $\text{Type}(G^v) = S(|E(G^v)|)$.

Proof. Since $\text{Type}(G^v) \subseteq S(|E(G^v)|)$ from the definitions of $\text{Type}(G^v)$ and $S(|E(G^v)|)$, it is sufficient to prove that $S(|E(G^v)|) \subseteq \text{Type}(G^v)$. Let (r,s) be any element in $S(|E(G^v)|)$ and let x and y be non-negative integers such that $|E(G_u^v)| = xm$ and $|E(G_w^v)| = yn$. Then $|E(G^v)| = rm + sn$, $|E(G^u)| = |E(G^v)| - |E(G_u^v)| = (r-x)m + sn$ and $|E(G^w)| = |E(G^v)| - |E(G_w^v)| = rm + (s-y)n$.

(A) In the case $r \geqslant x$, it follows from the above equation and Eq. (3) in Theorem 2.2 that $(r-x,s) \in S(|E(G^u)|)$ and $(r-x,s) \in \text{Type}(G^u)$. Since $(x,0) \in \text{Type}(G_u^v)$ and $G^v = G^u + G_u^v$, it follows from Lemma 2.1 that $(r,s) \in \text{Type}(G^v)$.

(B) In the case $s \geqslant y$, it follows from Eq. (3) in Theorem 2.2 that $(r, s-y) \in \text{Type}(G^w)$. Since $(0,y) \in \text{Type}(G_w^v)$ and $G^v = G^w + G_w^v$, we have $(r,s) \in \text{Type}(G^v)$.

(C) In the case $r < x$ and $s < y$, it follows that $|E(G^u)| = (r-x)m + sn < sn$ and $|E(G^w)| = rm + (s-y)n < rm$. Hence, it follows from Eq. (4) that $|E(G^v)| \leqslant |E(G^u)| + |E(G^w)| < rm + sn$. Since $|E(G^v)| = rm + sn$, this is a contradiction. Hence, $r \geqslant x$ or $s \geqslant y$.

It follows from (A)–(C) that $S(|E(G^v)|) \subseteq \text{Type}(G^v)$. This completes the proof. □

We are now ready to prove that for all positive integers v, $\text{Type}_{4,5}(G^v) = S_{4,5}(|E(G^v)|)$. From here on we will omit the subscript 4,5 on Type and S. Note that in the notation (r,s), the first coordinate represents the number of 4-cycles and the second coordinate represents the number of 5-cycles. We make use of the following results.

Theorem 2.3 (See Bryant et al., 1997). *Let u and v be odd with $u < v$. Then there exists a 4-cycle system of G_u^v if and only if $v \equiv u \pmod 8$.*

Theorem 2.4 (See Sotteau, 1981). *The complete bipartite graph $K_{x,y}$ can be decomposed into edge disjoint 4-cycles if and only if x and y are even.*

Theorem 2.5 (See Bryant et al., 1996). *Let u and v be odd. Then there exists a 5-cycle system of G_u^v if and only if:*
(a) $v \geqslant 3u/2 + 1$, *and*
(b) $u \equiv v \equiv 3 \pmod{10}$, *or* $u, v \equiv 1$ *or* $5 \pmod{10}$, *or* $u, v \equiv 7$ *or* $9 \pmod{10}$.

Theorem 2.6 (See Bryant and Khodkar, to appear). *Let u and v be even. Then there exists a 5-cycle system of G_u^v if and only if:*
(a) $v \geqslant 3u/2 + 2$, *and*
(b) $u, v \equiv 0$ *or* $2 \pmod{10}$, *or* $u, v \equiv 4$ *or* $8 \pmod{10}$, *or* $u \equiv v \equiv 6 \pmod{10}$.

Corollary 2.7. *For* $t \geqslant 4$ *we have* $(0,2)+(2t-4,0)+\mathrm{Type}(G^{2t-3}) \subseteq \mathrm{Type}(G^{2t+1})$.

Proof. Since $G^{2t+1} = G^5 + K_{2t-4,4} + G^{2t-3}$ the result follows by Theorem 2.4 and Lemma 2.1. □

Theorem 2.8. *Let* v *be odd. Then* $\mathrm{Type}(G^v) = S(|E(G^v)|)$ *for* $v \geqslant 5$.

Proof. In the case $v \leqslant 25$, $v=31$ and $v=33$, it follows from Rosa [10] and the appendix that Theorem 2.8 holds. Hence, it is sufficient to prove that Theorem 2.8 holds in the case $v \geqslant 27$, $v \neq 31$ and $v \neq 33$.

In the case $v \geqslant 27$, let (r,s) be any element in $S(|E(G^v)|)$ and let $u(v)$ denote the largest odd integer u such that there exists a 5-cycle system of G_u^v for a given odd integer v. Since $|E(G^v)| = v(v-1)/2$, it follows from Theorem 2.5 that $4r+5s = v(v-1)/2, u(23)=13$, $u(25)=15$, $u(27)=17$, $u(29)=17$, $u(31)=15$, $u(33)=13$, $u(35)=21$, $u(37)=19$, $u(39)=19$, $u(41)=25$, $u(43)=23$, $u(45)=25$, $u(47)=29$, $u(49)=29$ and $u(51)=31$. By Theorem 2.5, there exists a positive integer y such that $|E(G^v_{u(v)})| = 5y$, where $y=(v(v-1)-u(v)(u(v)-1))/10$. Since $|E(G^v)| = 4r+5s$ and $|E(G^v_{v-8})| = |E(G^v)| - |E(G^{v-8})| = v(v-1)/2-(v-8)(v-9)/2 = 4(2v-9)$, it follows that $|E(G^{v-8})| = 4(r-(2v-9))+5s$.

(A) In the case $r \leqslant u(v)(u(v)-1)/8$, it follows that $|E(G^{u(v)})| = |E(G^v)| - |E(G^v_{u(v)})| = 4r + 5(s-y)$. Since $4r+5s = v(v-1)/2$, it follows that $s \geqslant y$ if and only if $r \leqslant u(v)(u(v)-1)/8$. Hence $(r,s-y) \in S(|E(G^{u(v)})|)$ and $(r,s-y) \in \mathrm{Type}(G^{u(v)})$ by induction on v. Since $G^v = G^{u(v)} + G^v_{u(v)}$, it follows from Lemma 2.1 that $(r,s) \in \mathrm{Type}(G^v)$.

(B) In the case $r \geqslant 2v-9$, it follows that $(r-2v+9,s) \in S(|E(G^{v-8})|)$. Since $(r-2v+9,s) \in \mathrm{Type}(G^{v-8})$ by induction on v, we have $(r,s) \in \mathrm{Type}(G^v)$.

(C) In the case $v \leqslant r < 2v-9$, it follows from Corollary 2.7 that $(r,s) \in \mathrm{Type}(G^v)$. Since $v \leqslant u(v)(u(v)-1)/8$ in the case $v \geqslant 27, v \neq 31$ and $v \neq 33$, it follows from (A)–(C), Rosa (to appear) and the appendix that $S(|E(G^v)|) \subseteq \mathrm{Type}(G^v)$ for any odd integer $v \geqslant 5$. Since $\mathrm{Type}(G^v) \subseteq S(|E(G^v)|)$ for any odd integer $v \geqslant 5$, this completes the proof. □

It is worth noting that when $v \geqslant 47, 2(|E(G^{v-8})| + |E(G^{u(v)})| - |E(G^v)|) = u(v)(u(v)-1)-8(2v-9) \geqslant 0$. Hence, once Theorem 2.8 is proved for the case $v<47$, the case $v \geqslant 47$ follows immediately by induction from Theorems 2.2, 2.3 and 2.5.

Theorem 2.9. *Let* $v \geqslant 4$ *be even. Then* $\mathrm{Type}(G^v) = S(|E(G^v)|)$.

Proof. For $v \leqslant 10$ see Rosa (to appear). For $v \geqslant 12$ apply Theorem 2.2 with $w = v-2$ and an integer u which satisfies Theorem 2.2 parts (1) and (4). Since $v \geqslant 12$ one can see that such an integer u always exists. □

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Appendix

In this appendix, we prove that $\text{Type}_{4,5}(G^v) = S_{4,5}(|E(G^v)|)$ for $v = 11, 13, 15, 17, 19, 21, 23, 25, 31$ and 33. We will omit the subscript 4,5 on Type and S. Note that in the notation (r,s), the first coordinate represents the number of 4-cycles and the second coordinate represents the number of 5-cycles.

$\text{Type}(K_{11}) = S(|E(K_{11})|)$: $S(|E(K_{11})|) = \{(0,11),(5,7),(10,3)\}$. Corollary 2.7 takes care of $(10,3)$. By Theorem 2.5 we have $(0,11) \in \text{Type}(K_{11})$. To see $(5,7) \in \text{Type}(K_{11})$ let the vertex set of K_{11} be $\{1,\ldots,11\}$. Let the 5-cycles be $(2,3,4,5,6)$, $(2,4,6,3,5)$, $(7,8,9,10,11)$, $(7,9,11,8,10)$, $(1,2,9,4,7)$, $(1,4,11,6,9)$ and $(1,6,8,3,11)$, and let the 4-cycles be $(1,3,9,5)$, $(1,8,4,10)$, $(2,7,3,10)$, $(2,8,5,11)$ and $(5,7,6,10)$.

$\text{Type}(K_{13}) = S(|E(K_{13})|)$: $S(|E(K_{13})|) = \{(2,14),(7,10),(12,6),(17,2)\}$. Sine $K_{13} = K_{4,2} + K_9 \backslash K_7 + K_{11}$, by Theorems 2.4 and 2.5 and Lemma 2.1, $(2,14),(7,10),(12,6)$ are in $\text{Type}(K_{13})$. Since $K_{13} = K_{13} \backslash K_5 + K_5$, by Theorem 2.3 and Lemma 2.1, $(17,2) \in \text{Type}(K_{13})$.

$\text{Type}(K_{15}) = S(|E(K_{15})|)$: $S(|E(K_{15})|) = \{(0,21),(5,17),\ldots,(25,1)\}$. Since $K_{15} = K_{6,2} + K_9 \backslash K_7 + K_{13}$, by Theorems 2.4 and 2.5 and Lemma 2.1, $(5,17)$, $(10,13)$, $(15,9)$, $(20,5)$ are in $\text{Type}(K_{15})$. By Theorem 2.5 we have $(0,21) \in \text{Type}(K_{15})$. Since $K_{15} = K_{15} \backslash K_7 + K_7$, by Theorem 2.3 and Lemma 2.1, $(25,1) \in \text{Type}(K_{15})$.

$\text{Type}(K_{17}) = S(|E(K_{17})|)$: $S(|E(K_{17})|) = \{(4,24),(9,20),\ldots,(34,0)\}$. Since $K_{17} = K_{17} \backslash K_9 + K_9$, by Theorem 2.5 and Lemma 2.1, $(4,24),(9,20) \in \text{Type}(K_{17})$. Finally, Corollary 2.7 and Theorem 2.3 take care of other types.

$\text{Type}(K_{19}) = S(|E(K_{19})|)$: $S(|E(K_{19})|) = \{(4,31),(9,27),\ldots,(39,3)\}$. Since $K_{19} = K_{19} \backslash K_9 + K_9$, by Theorem 2.5 and Lemma 2.1, $(4,31),(9,27) \in \text{Type}(K_{19})$. Finally, Corollary 2.7 takes care of the remaining types.

$\text{Type}(K_{21}) = S(|E(K_{21})|)$: $S(|E(K_{21})|) = \{(0,42),(5,38),\ldots,(50,2)\}$. Since $K_{21} = K_{21} \backslash K_{11} + K_{11}$, by Theorem 2.5 and Lemma 2.1, $(0,42),(5,38),(10,34) \in \text{Type}(K_{21})$. Since $K_{21} = K_{8,4} + K_{17} \backslash K_9 + K_{13}$, by Theorems 2.4 and 2.5 and Lemma 2.1, $(15,30) \in \text{Type}(K_{21})$. Finally, Corollary 2.7 takes care of the remaining types.

$\text{Type}(K_{23}) = S(|E(K_{23})|)$: $S(|E(K_{23})|) = \{(2,49),(7,45),\ldots,(62,1)\}$. Since $K_{23} = K_{23} \backslash K_{13} + K_{13}$, by Theorem 2.5 and Lemma 2.1, $(2,49),(7,45),(12,41),(17,37) \in \text{Type}(K_{23})$. From Corollary 2.7 it follows that $(22,33),(27,29),\ldots,(57,5) \in \text{Type}(K_{23})$ and since $K_{23} = K_{23} \backslash K_{15} + K_{15}$, by Lemma 2.1 and Theorem 2.3 we have $(62,1) \in \text{Type}(K_{23})$.

$\text{Type}(K_{25}) = S(|E(K_{25})|)$: $S(|E(K_{25})|) = \{(0,60),(5,56),\ldots,(75,0)\}$. Since $K_{25} = K_{25} \backslash K_{15} + K_{15}$, by Theorem 2.5 and Lemma 2.1, $(0,60),(5,56),\ldots,(25,40) \in \text{Type}(K_{25})$. From Corollary 2.7 it follows that $(20,44),(25,40),\ldots,(70,4) \in \text{Type}(K_{25})$ and by Theorem 2.3 (with $u = 1$ and $v = 25$), we have $(75,0) \in \text{Type}(K_{25})$.

$\text{Type}(K_{31}) = S(|E(K_{31})|)$: $S(|E(K_{31})|) = \{(0,93),(5,89),\ldots,(115,1)\}$. Since $K_{31} = K_{31} \backslash K_{15} + K_{15}$, by Theorem 2.5 and Lemma 2.1, $(0,93),(5,89),\ldots,(25,73) \in \text{Type}(K_{31})$. From Corollary 2.7 it follows that $(30,69),(35,65),\ldots,(110,5) \in \text{Type}(K_{31})$ and since $K_{31} = K_{31} \backslash K_{23} + K_{23}$, by Lemma 2.1 and Theorem 2.3 we have $(115,1) \in \text{Type}(K_{31})$.

$\text{Type}(K_{33}) = S(|E(K_{33})|)$: $S(|E(K_{33})|) = \{(2,104),(7,100),\ldots,(132,0)\}$. Since $K_{33} = K_{33}\backslash K_{13} + K_{13}$, by Theorem 2.5 and Lemma 2.1, $(0,90) + \text{Type}(K_{13}) \subseteq \text{Type}(K_{33})$. Since $K_{33} = K_{33}\backslash K_{25} + K_{25}$, by Theorem 2.3 and Lemma 2.1, $(57,0) + \text{Type}(K_{25}) \subseteq \text{Type}(K_{33})$. Finally, for the remaining types, we apply Theorems 2.4 and 2.5 and Lemma 2.1 with $K_{33} = K_{12,4} + K_{29}\backslash K_{17} + K_{21}$.

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