

The Doyen–Wilson Theorem for Minimum Coverings with Triples

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Abstract: In this article necessary and sufficient conditions are found for a minimum covering of K_m with triples to be embedded in a minimum covering of K_n with triples. © 1997 John Wiley & Sons, Inc. *J Combin Designs* 5: 341–352, 1997

1. INTRODUCTION

A Steiner triple system (or simply triple system) of order n is a pair (S, T) , where T is an edge-disjoint collection of triangles (triples) that partition the edge set of K_n (the complete undirected graph on n vertices) with vertex set S . It has been known forever (= since 1847 [5]) that the spectrum for triple systems (= the set of all n such that a triple system of order n exists) is precisely the set of all $n \equiv 1$ or $3 \pmod{6}$. In this case $|T| = n(n-1)/6$.

The triple system (P, T_1) is said to be *embedded* in the triple system (S, T_2) provided that $P \subseteq S$ and $T_1 \subseteq T_2$. We also say that (P, T_1) is a *subsystem* of (S, T_2) . It is trivial to show that if (P, T_1) is a proper subsystem of (S, T_2) then $2|P| + 1 \leq |S|$. Now, a quite natural question to ask is: given integers $m \equiv 1$ or $3 \pmod{6}$ and $n \equiv 1$ or $3 \pmod{6}$ with $2m + 1 \leq n$, does there exist a triple system of order n containing a subsystem of order m ? In 1973, the celebrated work of Jean Doyen and Richard Wilson [2] showed that this is, in fact, the case.

Doyen and Wilson Theorem [2]. *Let $2m + 1 \leq n$, $m \equiv 1$ or $3 \pmod{6}$, and $n \equiv 1$ or $3 \pmod{6}$. Then there exists a triple system of order n containing a subsystem of order m .*

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Over the years, any problem involving trying to prove a similar result for a given combinatorial structure has come to be called a *Doyen–Wilson problem*, and (not too surprisingly) any solution a *Doyen–Wilson theorem*. The Doyen–Wilson Theorem has spawned a cottage industry with respect to just about any combinatorial design that you can think of. The history of work along these lines is much too extensive to go into here. But it doesn't take a lot of imagination to think of two Doyen–Wilson type problems that are natural generalizations of the original result: maximum packings and minimum coverings of K_n with triples. Without going into details, the first of these problems has been partially settled by the combined work in [4] and [6]. The object of this article is the *complete solution* of the Doyen–Wilson problem for the other side of this coin, i.e., *minimum coverings* of K_n with triples.

2. STATEMENT OF THE PROBLEM

Let λK_n denote the multigraph on n vertices in which each pair of vertices is joined by exactly λ edges. Let $E(G)$ denote the collection of edges in the multigraph G . If E and P are collections of edges, then let $E + P$ denote the union of the two collections (so if e occurs x times in E and y times in P , then it occurs $x + y$ times in $E + P$).

A *covering* of K_n with triples is a triple (S, C, P) , where S is the vertex set of K_n , $P \subseteq E(\lambda K_n)$ called the *padding*, and C is a collection of triples that partition $E(K_n) + P$. The *number* n is called the *order* of the covering. So that there is no confusion: an edge $\{a, b\}$ belongs to *exactly* $x + 1$ triples of C , where x is the number of times $\{a, b\}$ belongs to the padding P . If $|P|$ is as small as possible, then (S, C, P) is called a *minimum covering* of K_n with triples (MCT). So, a Steiner triple system is a MCT with padding $P = \emptyset$.

Example 2.1 (MCT (S_1, C_1, P_1) of order 6). $S_1 = \{1, 2, 3, 4, 5, 6\}$, $C_1 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}, \{3, 4, 6\}, \{2, 5, 6\}, \{1, 5, 6\}\}$, and $P_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.

Example 2.2 (MCT (S_2, C_2, P_2) of order 14). $S_2 = \{1, 2, \dots, 14\}$, $C_2 = C_1 \cup \{\{7, 8, 9\}, \{7, 14, 12\}, \{10, 11, 13\}, \{1, 7, 14\}, \{1, 8, 13\}, \{1, 9, 11\}, \{1, 10, 12\}, \{2, 13, 14\}, \{2, 7, 12\}, \{2, 8, 11\}, \{2, 9, 10\}, \{3, 8, 14\}, \{3, 7, 10\}, \{3, 9, 13\}, \{3, 11, 12\}, \{4, 7, 9\}, \{4, 8, 12\}, \{4, 10, 14\}, \{4, 11, 13\}, \{5, 9, 12\}, \{5, 8, 10\}, \{5, 11, 14\}, \{5, 7, 13\}, \{6, 7, 8\}, \{6, 9, 14\}, \{6, 10, 13\}, \{6, 11, 12\}\}$, and $P_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{7, 9\}, \{7, 14\}, \{10, 13\}, \{11, 12\}\}$.

It should come as no surprise that the padding of a MCT is determined by its order (up to a permutation on S). Fort and Hedlund [3] showed that minimum coverings exist for all n , and that if (S, C, P) is a MCT of order n then the padding P is (i) a 1-factor if $n \equiv 0 \pmod{6}$, (ii) a *tripole* = a spanning graph with each vertex having odd degree and containing $(n + 2)/2$ edges if $n \equiv 2$ or $4 \pmod{6}$, (iii) a double edge = $\{\{a, b\}, \{a, b\}\}$ if $n \equiv 5 \pmod{6}$, and, of course, (iv) the empty set if $n \equiv 1$ or $3 \pmod{6}$. Table I shows the minimum paddings.

TABLE I Minimum paddings.

$0 \pmod 6$	$1 \pmod 6$	$2 \pmod 6$	$3 \pmod 6$	$4 \pmod 6$	$5 \pmod 6$
	0		0		
1-factor	Steiner triple system	tripole	Steiner triple system	tripole	double edge

In Example 2.1 the padding P_1 is a 1-factor, and in Example 2.2 the padding P_2 is a tripole.

The MCT (S_1, C_1, P_1) is said to be *embedded* in the MCT (S_2, C_2, P_2) if and only if $S_1 \subseteq S_2, C_1 \subseteq C_2, P_1 \subseteq P_2$, and if $\{a, b\} \in P_2 \setminus P_1$ then $\{a, b\} \not\subseteq S_1$.

Example 2.3. The MCT (S_1, C_1, P_1) of order 6 in Example 2.1 is embedded in the MCT (S_2, C_2, P_2) of order 14 in Example 2.2.

We list the *possible embeddings* of MCTs in Table II, abbreviating the paddings by: $T =$ tripole, $F =$ 1-factor, and $D =$ double edge. The table should be read as follows: An “ X ” in cell (i, j) means it is *impossible* to embed a MCT of order $i \pmod 6$ in a MCT of order $j \pmod 6$. Otherwise, such an embedding is *possible*. In this case, the cell is filled in with the padding of the embedded MCT in the lower half and the padding of the containing MCT in the upper half.

In the remainder of this article, we will show that if (i, j) is possible, then the necessary conditions for embedding a MCT of order $m \equiv i \pmod 6$ in a MCT of order $n \equiv j \pmod 6$ are also sufficient. Of course, in order to do this it is a good idea to first determine just what the necessary conditions are. This is quite easy to do.

A simple calculation shows that, in every case where (i, j) is possible, we *must* have $n \geq 2m$. It is immediate that $n = 2m$ is possible for the cases $(0, 0), (2, 4),$ and $(4, 2)$ *only*. So, we have the following necessary conditions.

Necessary Conditions. If (i, j) is possible, then $n \geq 2m$. *Except* for the cases $(0, 0), (2, 4),$ and $(4, 2)$ we must have the *strict inequality* $n > 2m$.

TABLE II. Possible embeddings.

$m \setminus n$	0	1	2	3	4	5
0	$F \setminus F$	X	$F \setminus T$	X	$F \setminus T$	X
1	$\emptyset \setminus F$	$\emptyset \setminus \emptyset$	$\emptyset \setminus T$	$\emptyset \setminus \emptyset$	$\emptyset \setminus T$	$\emptyset \setminus D$
2	X	X	$T \setminus T$	X	$T \setminus T$	X
3	$\emptyset \setminus F$	$\emptyset \setminus \emptyset$	$\emptyset \setminus T$	$\emptyset \setminus \emptyset$	$\emptyset \setminus T$	$\emptyset \setminus D$
4	X	X	$T \setminus T$	X	$T \setminus T$	X
5	X	X	X	X	X	$D \setminus D$

Not too surprisingly, since there are 20 cases, we will break the constructions up into several sections. However, the four cases (1, 1), (1, 3), (3, 1), and (3, 3) are covered by the Doyen–Wilson Theorem, so this reduces our work to just 16 cases.

Before describing these 16 cases, we first need to say something about maximum packings of K_n with triples, because we use maximum packings extensively in some of the proofs.

A *packing* of K_n with triangles is a triple (S, T, L) , where S is the vertex set of K_n , T is a collection of edge-disjoint triangles (triples) of K_n , and L is the set of edges not belonging to a triple of T . The *number* n is called the order of the packing (S, T, L) , and the set of unused edges L is called the *leave*. If $|T|$ is as large as possible = if $|L|$ is as small as possible, then (S, T, L) is called a *maximum packing*. Just as with minimum coverings, the leave of a maximum packing is determined by the order. It is a very well known Folk Theorem that maximum packings exist for all n and that the leave is (i) a 1-factor if $n \equiv 0$ or $2 \pmod{6}$, (ii) a tripole if $n \equiv 4 \pmod{6}$, (iii) a 4-cycle if $n \equiv 5 \pmod{6}$, and (iv) the empty set if $n \equiv 1$ or $3 \pmod{6}$ (Steiner triple system).

3. CASES (2, 2), (4, 4), (2, 4), AND (4, 2)

The proofs for (2, 2) and (4, 4) are similar; so are the proofs for (2, 4) and (4, 2). So we will need just two lemmas here.

Lemma 3.1. *Let $n > 2m$, $n \equiv m \equiv 2$ or $4 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order m .*

Proof. In each of the cases (2, 2) and (4, 4) both the containing and embedded MCTs have padding a tripole. We must also have $n > 2m$. We will do the construction for the case (2, 2), the case (4, 4) being similar.

Write $m = 6k + 2$ and $n = 6h + 2$. Since $n > 2m$, we must have $h \geq 2k + 1$ and, therefore, $6h + 1 \geq 2(6k + 1) + 1$. By the Doyen–Wilson Theorem we can embed a triple system of order $6k + 1$ in a triple system of order $6h + 1$. This gives a partition of $K_{6(h-k)}$ based on X into a collection of triples T and $6k + 1$ 1-factors $F = \{f_1, f_2, \dots, f_{6k+1}\}$, which we will denote by (X, F, T) . Let $S_1 = \{1, 2, 3, \dots, 6k + 2\}$ and let (S_1, C_1, P_1) be a MCT of order $m = 6k + 2$ with padding a tripole P_1 . Let α be a mapping from S_1 onto $\{1, 2, 3, \dots, 6k + 1\}$ such that $(6k + 1)\alpha = (6k + 2)\alpha = 6k + 1$, set $S_2 = S_1 \cup X$ ($S_1 \cap X = \emptyset$), and define a collection of triples C_2 as follows:

1. $C_1 \subseteq C_2$,
2. $T \subseteq C_2$, and
3. $\{a, b, i\} \in C_2$ if and only if $\{a, b\} \in f_{i\alpha}$.

Then (S_2, C_2, P_2) is a MCT of order n with padding the tripole $P_2 = P_1 \cup f_{6k+1}$ and, of course, containing the MCT (S_1, C_1, P_1) of order m . \square

Lemma 3.2. *Let $n \geq 2m$, and either $m \equiv 2 \pmod{6}$ and $n \equiv 4 \pmod{6}$ or $m \equiv 4 \pmod{6}$ and $n \equiv 2 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order m .*

Proof. We use the same technique for the cases (2, 4) and (4, 2) as in cases (2, 2) and (4, 4). We will do case (2, 4), the case (4, 2) being similar. Now it is possible in the (2, 4) case

to have $n = 2m$. Simply use (2, 2) Construction with (X, F, T) , where $T = \emptyset$. If $n > 2m$, use the Doyen–Wilson Construction to obtain a triple system of order $n - 1$ containing a subsystem of order $m - 1$ and proceed exactly as in the case (2, 2). \square

4. CASES (1, 2), (1, 4), (3, 2), AND (3, 4)

Lemma 4.1. *Let $n > 2m$, $m \equiv 1$ or $3 \pmod{6}$, and $n \equiv 2$ or $4 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order n .*

Proof. In all of these cases, the best possible embedding requires that $n > 2m$. We will give a construction for the (1, 2) case, the other cases being obvious modifications. Let $m = 6k + 1$ and $n = 6h + 2$. By the Doyen–Wilson Theorem we can embed a triple system of order $6k + 1$ in a triple system of order $6h + 1$. If we delete a point from the triple system of order $6h + 1$, which does not belong to the subsystem of order $6k + 1$, we obtain a *maximum packing* of K_{6h} with triples with *leave* a 1-factor. Let (S_1, C_1, L) be this maximum packing, where S_1 is the vertex set of K_{6h} , C_1 is the collection of triples, and L is the leave. Of course (S_1, C_1, L) contains a triple system (X, C_1^*) of order $6k + 1$. Now set $S_2 = \{\infty_1, \infty_2\} \cup S_1$ and define a collection of triples C_2 as follows:

1. $C_1 \subseteq C_2$,
2. write $L = L^* \cup \{x, y\}$ and for each edge $\{a, b\} \in L^*$ place the two triples $\{a, b, \infty_1\}$, $\{a, b, \infty_2\}$ in C_2 and
3. place the 3 triples $\{\infty_1, \infty_2, x\}$, $\{\infty_1, \infty_2, y\}$, and $\{\infty_2, x, y\}$ in C_2 .

Then (S_2, C_2, P) is a MCT of order $n = 6h + 2$ with padding the tripole $P = \{\{\infty_2, \infty_1\}, \{\infty_2, x\}, \{\infty_2, y\}\} \cup L^*$, and, of course, containing the subsystem (X, C_1^*) of order $m = 6k + 1$.

The other 3 cases are handled in a similar fashion. \square

5. CASES (0, 0) AND (5, 5)

Before moving on to the next two cases, we will need the following construction of PBDs, which may or may not be well known.

The $6h + 5$ Construction.. Let (Q, \circ) be an idempotent commutative quasi-group of order $2h + 1$ containing a subquasi-group (P, \circ) of order $2k + 1$. Let α be a cyclic permutation on $Q \setminus P$, set $S = \{\infty_1, \infty_2\} \cup (Q \times \{1, 2, 3\})$, and define a collection of blocks B consisting of one block of size $6k + 5$ and the remaining blocks of size 3 as follows:

1. $\{\infty_1, \infty_2\} \cup (P \times \{1, 2, 3\}) \in B$;
2. denote by c^* the cycle

$$c^* = ((a, 1), (a, 2), (a, 3), (a\alpha, 1), (a\alpha, 2), (a\alpha, 3), (a\alpha^2, 1), (a\alpha^2, 2), (a\alpha^2, 3), \dots, (a\alpha^{x-1}, 1), (a\alpha^{x-1}, 2), (a\alpha^{x-1}, 3)),$$

where $a \in Q \setminus P$ and $x = 2h - 2k$, and use alternate edges in c^* with ∞_1 and ∞_2 , respectively, to form $6(h - k)$ triples, and place these triples in B ; and

3. if $\{a, b\} \not\subseteq P$, place the 3 triples $\{(a, 1), (b, 1), (a \circ b, 2)\}$, $\{(a, 2), (b, 2), (a \circ b, 3)\}$, $\{(a, 3), (b, 3), ((a \circ b)\alpha, 1)\}$ in B .

Then (S, B) is a PBD of order $6h + 5$ containing *exactly* one block of size $6k + 5$ and the remaining blocks all of size 3.

Lemma 5.1. *Let $n > 2m$, $n \equiv m \equiv 5 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order m .*

Proof. Write $m = 6k + 5$ and $n = 6h + 5$. Since $n > 2m$, $2h + 1 \geq 2(2k + 1) + 1$. Hence by Allan Cruse's Theorem [1] there exists an idempotent commutative quasi-group of order $2h + 1$ containing a subquasi-group of order $2k + 1$. Use the $6h + 5$ Construction and replace the block of size $6k + 5$ with a MCT of order $6k + 5$ with padding a *double edge*. \square

Lemma 5.2. *Let $n \geq 2m$, and $n \equiv m \equiv 0 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order m .*

Proof. Again we must have $n \geq 2m$. Write $m = 6k$ and $n = 6h$. By the $6h + 5$ Construction, there exists a PBD (S, B) of order $6h - 1$ containing exactly one block of size $6k - 1$ and the remaining blocks of size 3. This gives a partition of $K_{6(h-k)}$ based on X into a collection of triples T and $6k - 1$ 1-factors F , denoted by (X, T, F) . If we now copy the argument in Case (2, 2) taking (S_1, C_1, P_1) to be a MCT of order $6k$ with padding a 1-factor, the result is a MCT of order n with padding a 1-factor and containing a MCT of order m . \square

6. MAXIMUM PACKINGS

Two of the unsettled cases for the Doyen–Wilson Theorem for *maximum packings* are $(0, 4)$ and $(2, 4)$ [6]. Both cases are crucial for four of the remaining cases. In order to obtain these packings we will need some preliminary results.

As far as we are concerned, a *difference triple* is a 3-element subset $\{x, y, z\}$ of *distinct positive integers* such that $x + y = z$. The following lemma can be found in [7].

Lemma 6.1 (G. Stern and H. Lenz [7]). *The sets $\{1, 2, 3, \dots, 6k - 1\} \setminus \{4k, 5k\}$ and $\{1, 2, 3, \dots, 6k + 2\} \setminus \{3k + 1, 4k + 2\}$, $k \geq 2$, can be partitioned into difference triples, where 1, 2, and 3 are in different difference triples.*

Let $G(n)$ be a subgraph of K_n with vertex set Z_n . Then $G(n)$ is said to be *cyclic*, provided that the edge $\{x, y\} \in E(G(n))$ if and only if $\{x + 1, y + 1\} \pmod{n} \in E(G(n))$. The edge $\{0, x\} \in E(G(n))$ is said to have *order* m , provided that x generates a subgroup of $(Z_n, +)$ of order m . The edge $\{x, y\}$ in $G(n)$ is said to have *length* $\ell(x, y) = \min\{y - x \pmod{n}; x - y \pmod{n}\}$; so $1 \leq \ell(x, y) \leq n/2$. The following important theorem is also due to Stern and Lenz.

Theorem 6.2 (G. Stern and H. Lenz [7]). *Let $G(n)$ be a cyclic graph. If $G(n)$ contains an edge of even order, then $G(n)$ can be 1-factorized.*

The following two lemmas are extremely important. The first part of each will be used for maximum packings, while the second part of each will be used for minimum coverings. Because of the technicalities in the proof, it is best to “mix” the statements and the proofs.

Lemma 6.3. *Let $n \equiv 4 \pmod{6}$. Then the edges $E(1, 2, 3)$ of K_n of length 1, 2, and 3 can be partitioned into a tripole, 3 1-factors F_1, F_2, F_3 , and triples. Additionally, there exists a tripole P such that $P \cap E(F_1 \cup F_2 \cup F_3) = \emptyset$ and $(E(1, 2, 3) \setminus \{F_1 \cup F_2 \cup F_3\}) \cup P$ can be partitioned into triples.*

Proof. Let $n = 6k + 4$ and let the vertex set of K_{6k+4} be \mathbb{Z}_{6k+4} . Let $T = \{\{3i + 1, 3i + 2, 3i + 3\} | 0 \leq i \leq 2k\}$ be a set of $2k + 1$ triples. Let P' be the tripole consisting of the edges in $\{\{0, 1\}, \{0, 2\}, \{0, 3\}\} \cup \{\{6i - 2, 6i + 1\}, \{6i - 1, 6i + 2\}, \{6i, 6i + 3\} | 1 \leq i \leq k\}$. Finally, we define 3 1-factors as follows:

$$F_1 = \{\{0, 6k + 1\}, \{2, 6k + 3\}\} \\ \cup \{\{6i - 2, 6i - 5\}, \{6i - 1, 6i - 3\}, \{6i, 6i + 2\} | 1 \leq i \leq k\},$$

$$F_2 = \{\{0, 6k + 3\}, \{1, 6k + 2\}\} \\ \cup \{\{6i - 2, 6i - 4\}, \{6i - 1, 6i + 1\}, \{6i, 6i - 3\} | 1 \leq i \leq k\},$$

and

$$F_3 = \{\{0, 6k + 2\}, \{1, 6k + 3\}\} \\ \cup \{\{6i - 2, 6i - 3\}, \{6i - 1, 6i - 4\}, \{6i, 6i + 1\} | 1 \leq i \leq k\}.$$

Then P', F_1, F_2, F_3 , and T give the required decomposition of $E(1, 2, 3)$.

Let P be the tripole consisting of the edges in $\{\{0, 1\}, \{0, 2\}, \{0, 3\}\} \cup \{\{6i - 2, 6i + 2\}, \{6i - 1, 6i + 3\}, \{6i, 6i + 1\} | 1 \leq i \leq k\}$, and let $T' = (T \setminus \{1, 2, 3\}) \cup \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}\} \cup \{\{6i - 2, 6i - 1, 6i + 2\}, \{6i - 1, 6i, 6i + 3\}, \{6i - 2, 6i, 6i + 1\}, \{6i + 1, 6i + 2, 6i + 3\} | 1 \leq i \leq k\}$. Then T' is the required decomposition of $(E(1, 2, 3) \setminus \{F_1 \cup F_2 \cup F_3\}) \cup P$. \square

Lemma 6.4. *Let $n \equiv 2 \pmod{6}$. Then the edges $E(1, 2, 3)$ of K_n of length 1, 2, and 3 can be partitioned into a tripole, one 1-factor, and triples.*

Proof. Let $n = 6k + 2$ and let the vertex set of K_{6k+2} be \mathbb{Z}_{6k+2} . For each $i \in \mathbb{Z}_k$, let $T_i = \{\{6i + 1, 6i + 2, 6i + 3\}, \{6i + 2, 6i + 4, 6i + 5\}, \{6i + 3, 6i + 4, 6i + 6\}, \{6i + 5, 6i + 6, 6i + 7\}\}$, and let $T = \{\{6k + 1, 0, 1\}\} \cup (\cup_{i \in \mathbb{Z}_k} T_i)$. The edges of length 1, 2, and 3 that are not in triples in T form a graph G on the vertex set \mathbb{Z}_{6k+2} in which vertex 0 has degree 4, and all other vertices have degree 2; so G has $6k + 3$ edges altogether. In fact, the edges of G can be partitioned into two cycles: a cycle c_1 of length $2k + 4$, namely $(1, 4, 7, 10, \dots, 6k - 2, 6k + 1, 2, 0, 6k)$; and a cycle c_2 of length $4k - 1$, namely $(v_1, v_2, \dots, v_{4k-1})$, where for $0 \leq i < k$, $v_{4i+1} = 6i$, $v_{4i+2} = 6i + 3$, $v_{4i+3} = 6i + 5$ and $v_{4i+4} = 6i + 8$. So since c_1 has even length, its edges can be partitioned into two sets of $k + 2$ independent edges F_1 and F_2 , and since c_2 has odd length, its edges can be partitioned into two sets of edges H_1 and H_2 , where H_1 is a set of $2k - 1$ independent edges and H_2 is a set of $2k$ edges that are independent, except that vertex $v_1 = 0$ is incident with two edges in H_2 . So $F_1 \cup H_1$ is a 1-factor of K_{6k+2} and $F_2 \cup H_2$ is a tripole of K_{6k+2} , the vertex of degree 3 being vertex 0. So the result is proved. \square

Lemma 6.5. *Let $n = 6h + 4$. If $4 \leq t \leq h$, or if $1 \leq t \leq 3$ and $t \leq h \leq 13$, then there exists a partition of K_n into a tripole, $6t$ 1-factors, and triples.*

Proof. We will handle the cases $4 \leq t \leq h$ first. Write $6h + 4 = 12k + 4$ or $12k + 10$. Then by Lemma 6.1, $\{1, 2, \dots, 3h - 1\} \setminus \{a, b\}$ can be partitioned into a collection of difference triples D_1 , where $\{a, b\} = \{4k, 5k\}$ or $\{3k + 1, 4k + 2\}$ as the case may be. Let $\{1, x_1, y_1\}, \{2, x_2, y_2\}$, and $\{3, x_3, y_3\}$ be the 3 difference triples containing 1, 2, and 3 (1, 2, and 3 belong to distinct difference triples in [7]). Let $D_2 = D_1 \setminus \{\{1, x_1, y_1\}, \{2, x_2, y_2\}, \{3, x_3, y_3\}\}$ and $L = \{x_1, y_1, x_2, y_2, x_3, y_3, a, b, 3h, 3h + 1, 3h + 2\}$. By Lemma 6.3 the edges of K_{6h+4} of lengths 1, 2, and 3 can be partitioned into a tripole H , 3 1-factors $F(3)$, and a collection of triples T_1 . Let $4 \leq t \leq h$ and choose $s = t - 4$ triples $D_3 = \{\{u_1, v_1, w_1\}, \{u_2, v_2, w_2\}, \dots, \{u_s, v_s, w_s\}\}$ from D_2 . Now the edges with lengths belonging to $L \cup \{x|x \text{ is in a triple of } D_3\}$ form a cyclic graph, and since L contains the edge $\{0, 3h + 2\}$ of even order, by Theorem 6.2 this graph can be partitioned into $6t - 3$ 1-factors F . Let $D_4 = \{\{i, x + i, x + y + i\} | \{x, y, x + y\} \in D_2 \setminus D_3, i \in \mathbb{Z}_{6h+4}\}$. Then K_{6h+4} is partitioned into the tripole H , $6t$ 1-factors $F \cup F(3)$, and triples $T_1 \cup D_4$. This takes care of the cases where $4 \leq t \leq h$.

To handle the remaining 8 cases, we will need the following partitions of $\{1, 2, 3, \dots, 3h + 2\}$ for $h = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$.

$3h+2$	Partition of $\{1, 2, 3, \dots, 3h+2\}$
5	$\{1, 2, 3\}, \{4, 5\}$
8	$\{1, 2, 3\}, \{4, 5, 7\}, \{6, 8\}$
11	$\{1, 2, 3\}, \{4, 8, 10\}, \{6, 7, 9\}, \{5, 11\}$
14	$\{1, 2, 3\}, \{4, 7, 11\}, \{5, 8, 13\}, \{6, 10, 12\}, \{9, 14\}$
17	$\{1, 2, 3\}, \{4, 12, 16\}, \{5, 10, 15\}, \{6, 7, 13\}, \{9, 11, 14\}, \{8, 17\}$
20	$\{1, 2, 3\}, \{4, 15, 19\}, \{5, 9, 14\}, \{6, 7, 13\}, \{8, 11, 18\}, \{11, 12, 17\}, \{16, 20\}$
23	$\{1, 2, 3\}, \{4, 17, 21\}, \{5, 10, 15\}, \{6, 7, 13\}, \{8, 14, 22\}, \{9, 11, 20\}, \{12, 16, 18\}, \{19, 23\}$
26	$\{1, 2, 3\}, \{4, 20, 24\}, \{5, 14, 19\}, \{6, 11, 17\}, \{7, 16, 23\}, \{8, 10, 18\}, \{9, 13, 22\}, \{12, 15, 25\}, \{21, 26\}$
29	$\{1, 2, 3\}, \{4, 22, 26\}, \{5, 19, 24\}, \{6, 12, 18\}, \{7, 20, 27\}, \{8, 13, 21\}, \{9, 14, 23\}, \{10, 15, 25\}, \{11, 17, 28\}, \{16, 29\}$
32	$\{1, 2, 3\}, \{4, 25, 29\}, \{5, 18, 23\}, \{6, 21, 27\}, \{7, 13, 20\}, \{8, 14, 22\}, \{9, 15, 24\}, \{10, 16, 26\}, \{11, 17, 28\}, \{12, 19, 31\}, \{30, 32\}$
35	$\{1, 2, 3\}, \{4, 20, 24\}, \{5, 28, 33\}, \{6, 26, 32\}, \{7, 23, 30\}, \{8, 14, 22\}, \{9, 16, 25\}, \{10, 17, 27\}, \{11, 18, 29\}, \{12, 19, 31\}, \{13, 21, 34\}, \{15, 35\}$
38	$\{1, 2, 3\}, \{4, 25, 29\}, \{5, 31, 36\}, \{6, 27, 33\}, \{7, 15, 22\}, \{8, 16, 24\}, \{9, 17, 26\}, \{10, 18, 28\}, \{11, 19, 30\}, \{12, 20, 32\}, \{13, 21, 34\}, \{14, 23, 37\}, \{35, 38\}$
41	$\{1, 2, 3\}, \{4, 36, 40\}, \{5, 32, 37\}, \{6, 22, 28\}, \{7, 23, 30\}, \{8, 26, 34\}, \{9, 16, 25\}, \{10, 17, 27\}, \{11, 18, 29\}, \{12, 19, 31\}, \{13, 20, 33\}, \{14, 21, 35\}, \{15, 24, 39\}, \{38, 41\}$

By Lemma 6.3, the edges of K_{6h+4} of length 1, 2, and 3 can be partitioned into a tripole, 3 1-factors, and triples. The edges of length belonging to the 2-element subset $\{x, 3h + 2\}$ can be partitioned into 3 1-factors as well, since the edge $\{0, 3h + 2\}$ has even order. Combining these 1-factors gives 6 1-factors. If $\{a, b, c\} \neq \{1, 2, 3\}$, we can either cyclically develop the triple $\{0, x, x + y\}$ (giving $6h + 4$ triples) or use this triple along with the edges of lengths in $\{x, 3h + 2\}$ to obtain an *additional* 6 1-factors. Since there are $t - 1$ triples other than $\{1, 2, 3\}$, for each h we can obtain up to $6t$ 1-factors.

Combining all of the above results completes the proof. □

Lemma 6.6. *Let $m \equiv 0 \pmod{6}$ and $n \equiv 4 \pmod{6}$, and $n > 2m$. There exists a maximum packing of order n containing a maximum packing of order m .*

Proof. Let $n - m = 6h + 4$ and $m = 6t$. We will first handle the cases where $4 \leq t \leq h$ or $1 \leq t \leq 3$ and $6h + 4 \leq 82$. By Lemma 6.5, K_{n-m} can be partitioned into a tripole,

$m = 6t$ 1-factors, and triples. Let (S_1, T_1, L_1) be a maximum packing of K_m with vertex set S_1 and (X, H, F, T) a decomposition of K_{n-m} into a tripole H , $6t$ 1-factors F , and triples T ; where K_{n-m} has vertex set X , and $X \cap S_1 = \emptyset$. Let α be a 1 – 1 mapping of S_1 onto the collection of 1-factors $F = \{f_1, f_2, \dots, f_{6t}\}$. Define a maximum packing (S_2, T_2, L_2) of K_n with vertex set $S_2 = S_1 \cup X$ as follows:

1. $T_1 \subseteq T_2$,
2. $\{a, b, x\} \in T_2$ if $\{a, b\} \in f_{x\alpha}$,
3. $T \subseteq T_2$, and
4. $L_2 = L_1 \cup H$.

Then the maximum packing (S_2, T_2, L_2) of order n contains the maximum packing (S_1, T_1, L_1) of order m .

We now handle the cases where $m = 6t = 6, 12$, or 18 and $n = 6h + 4 \geq 88$. So, let (S^*, T^*, L^*) be a maximum packing of K_m , where $m = 6t = 6, 12$, or 18 and embed this maximum packing into a maximum packing (S_1, T_1, L_1) of order 42 (this is just an application of the Doyen–Wilson Theorem). Since $n = 6h + 4 \geq 88$, $n - m \geq 46$. By Lemma 6.5, K_{n-m} can be partitioned into a tripole, 42 1-factors, and triples. So the maximum packing (S_1, T_1, L_1) and, therefore, (S^*, T^*, L^*) can be embedded in a maximum packing of order $n = 6h + 4$. \square

Lemma 6.7. *Let $n = 6h + 2$. If $4 \leq t < h$, or if $1 \leq t \leq 3$ and $t < h \leq 13$, then there exists a partition of K_n into a tripole, $6t + 2$ 1-factors, and triples.*

Proof. Not too surprisingly, the proof is almost identical to the proof of Lemma 6.5. If $4 \leq t < h$, the proof is the same as the proof of Lemma 6.5 with the following modifications. L is replaced with $\{x_1, y_1, x_2, y_2, x_3, y_3, a, b, 3h, 3h + 1\}$ and Lemma 6.3 is replaced with Lemma 6.4. (Note that the edge $\{0, 3h + 1\}$ now has even order.)

As with the proof of Lemma 6.5, we will need the following partitions of $\{1, 2, 3, \dots, 3h + 1\}$ for $1 \leq h \leq 13$.

$3h + 1$	Partition of $\{1, 2, 3, \dots, 3h + 1\}$
7	$\{1, 2, 3\}, \{4, 5, 6, 7\}$
10	$\{1, 2, 3\}, \{4, 5, 9\}, \{6, 7, 8, 10\}$
13	$\{1, 2, 3\}, \{4, 8, 12\}, \{5, 6, 11\}, \{7, 9, 10, 13\}$
16	$\{1, 2, 3\}, \{4, 7, 11\}, \{5, 8, 13\}, \{6, 9, 15\}, \{10, 12, 14, 16\}$
19	$\{1, 2, 3\}, \{4, 8, 12\}, \{5, 9, 14\}, \{6, 10, 16\}, \{7, 11, 18\}, \{13, 15, 17, 19\}$
22	$\{1, 2, 3\}, \{4, 9, 13\}, \{5, 10, 15\}, \{6, 11, 17\}, \{7, 14, 21\}, \{8, 12, 20\}, \{16, 18, 19, 22\}$
25	$\{1, 2, 3\}, \{4, 10, 14\}, \{5, 11, 16\}, \{6, 12, 18\}, \{7, 17, 24\}, \{8, 15, 23\}, \{9, 13, 22\}, \{19, 20, 21, 25\}$
28	$\{1, 2, 3\}, \{4, 11, 15\}, \{5, 12, 17\}, \{6, 13, 19\}, \{7, 14, 21\}, \{8, 19, 27\}, \{9, 16, 25\}, \{10, 22, 24\}, \{20, 23, 27, 28\}$
31	$\{1, 2, 3\}, \{4, 12, 16\}, \{5, 13, 18\}, \{6, 14, 20\}, \{7, 15, 22\}, \{8, 21, 29\}, \{9, 19, 28\}, \{10, 17, 27\}, \{11, 25, 26\}, \{23, 24, 30, 31\}$
34	$\{1, 2, 3\}, \{4, 13, 17\}, \{5, 14, 19\}, \{6, 15, 21\}, \{7, 16, 23\}, \{8, 18, 26\}, \{9, 20, 29\}, \{10, 22, 32\}, \{11, 27, 30\}, \{12, 25, 31\}, \{24, 28, 33, 34\}$
37	$\{1, 2, 3\}, \{4, 14, 18\}, \{5, 15, 20\}, \{6, 16, 22\}, \{7, 17, 24\}, \{8, 28, 36\}, \{9, 26, 35\}, \{10, 23, 33\}, \{11, 21, 32\}, \{12, 19, 31\}, \{13, 27, 34\}, \{25, 29, 30, 37\}$
40	$\{1, 2, 3\}, \{4, 15, 19\}, \{5, 16, 21\}, \{6, 17, 23\}, \{7, 18, 25\}, \{8, 31, 39\}, \{9, 29, 38\}, \{10, 27, 37\}, \{11, 24, 35\}, \{12, 22, 34\}, \{13, 20, 33\}, \{14, 30, 36\}, \{26, 28, 32, 40\}$

By Lemma 6.4, the edges of K_{6h+2} of length 1, 2, and 3 can be partitioned into a tripole, one 1-factor, and triples. The edges of length belonging to the 4-element subset

$\{x, y, z, 3h + 1\}$ can be partitioned into 7 1-factors as well, since the edge $\{0, 3h + 1\}$ has even order. Combining these 1-factors gives 8 1-factors. The remainder of the proof is the same as the corresponding part in Lemma 6.5 with obvious modifications.

This completes the proof. \square

Lemma 6.8. *Let $m \equiv 2 \pmod{6}$ and $n \equiv 4 \pmod{6}$, and $n > 2m$. There exists a maximum packing of order n containing a maximum packing of order m .*

Proof. Let $n - m = 6h + 2$ and $m = 6t + 2$. We will begin with the cases where $4 \leq t < h$ or where $1 \leq t \leq 3$ and $6h + 2 \leq 80$. By Lemma 6.7, K_{n-m} can be partitioned into a tripole, $6t + 2$ 1-factors, and triples. The remainder of the proof is identical to the corresponding proof in Lemma 6.6.

If $m = 8, 14,$ or 20 , we can embed a maximum packing of order m in a maximum packing of order 42. Since $n - m \geq 80$, $n \geq 88$ and so by Lemma 6.6 this maximum packing of order 42 can be embedded in a maximum packing of order n . \square

7. THE REMAINING CASES: (1, 5), (3, 5), (1, 0), (3, 0), (0, 4), AND (2, 4)

The cases (1, 5) and (3, 5) follow easily from Lemma 6.7.

Lemma 7.1. *Let $n \geq 2m$, $m \equiv 1$ or $3 \pmod{6}$, and $n \equiv 5 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order m .*

Proof. Let $m \equiv 0$ or $2 \pmod{6}$, $n \equiv 4 \pmod{6}$, and $n \geq 2m$. By Lemma 6.7, there exists a maximum packing (S_1, T_1, L_1) of order n containing a maximum packing (S_2, T_2, L_2) of order m . In this case, L_2 is a 1-factor and L_1 is a tripole with $L_2 \subseteq L_1$. Let $S_3 = \{\infty\} \cup S_2$ and define a collection of triples T_3 as follows:

1. Let $(\{\infty\} \cup S_2, T^*)$ be a Steiner triple system of order $m + 1 \equiv 1$ or $3 \pmod{6}$ and place the triples of T^* in T_3 .
2. Write $L_2 \setminus L_1 = \{t_1, \ell_1, \ell_2, \dots, \ell_x\}$, where $t_1 = \{\{a, b\}, \{a, c\}, \{a, d\}\}$. Place the 3 triples $\{\infty, a, c\}, \{\infty, a, b\}, \{\infty, a, d\}$ in T_3 ; and for each $\ell_i = \{u, v\} \in L_2 \setminus L_1$ place the triple $\{\infty, u, v\}$ in T_3 .
3. Place the triples in $T_1 \setminus T_2$ in T_3 .

Then (S_3, T_3, P) is a MCT of order $n + 1 \equiv 5 \pmod{6}$ with padding the *double* edge $P = \{\{\infty, a\}, \{\infty, a\}\}$ and containing a Steiner triple system of order $m + 1 \equiv 1$ or $3 \pmod{6}$. \square

We can now use Lemma 7.1 to easily obtain the cases (1, 0) and (3, 0).

Lemma 7.2. *Let $n > 2m$, $m \equiv 1$ or $3 \pmod{6}$, and $n \equiv 0 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order m .*

Proof. By Lemma 7.1 there exists a MCT (S_1, T_1, P_1) of order $n - 1$ containing a triple system of order m . Now P_1 is a double edge, say $\{\infty, a\}$ and $\{\infty, a\}$. Let $S_2 = S_1 \setminus \{\infty\}$ and let (S_2, T_2, L_2) be the maximum packing of order $n - 2$ obtained by deleting ∞ from (S_1, T_1, P_1) . Then L_2 is a tripole and (S_2, T_2, L_2) contains the same triple system of order m as the MCT (S_1, T_1, P_1) . As a consequence, none of the edges in L_2 have both vertices in the triple system of order m . Now set $S_3 = \{\infty_1, \infty_2\} \cup S_2$ and $L_2 = \{\ell_1, \ell_2, \dots, \ell_x, t\}$, where $t = \{\{a, b\}, \{a, c\}, \{a, d\}\}$. Define a collection of triples T_3 as follows:

1. $T_2 \subseteq T_3$,
2. for each $\ell_i = \{u, v\} \in L_2$ place the 2 triples $\{\infty_1, u, v\}, \{\infty_2, u, v\}$ in T_3 , and
3. place the 5 triples $\{\infty_1, c, d\}, \{\infty_1, a, b\}, \{\infty_2, a, d\}, \{\infty_2, a, c\}$, and $\{\infty_1, \infty_2, b\}$ in T_3 .

Then (S_3, T_3, P) is a MCT of order n , contains a Steiner triple system of order m , and has padding the 1-factor $P = (L \setminus t) \cup \{\{c, d\}, \{a, \infty_2\}, \{b, \infty_1\}\}$. \square

The following lemmas are modifications of Lemmas 6.5 and 6.6.

Lemma 7.3. *Let $n = 6h + 4$. Then for any $1 \leq t \leq h$, there exists a tripole P such that $K_n \cup P$ can be partitioned into $6t$ 1-factors and triples. Furthermore, none of the edges in the 1-factors belong to the padding P .*

Proof. In the proof of Lemma 6.5 substitute the second half of Lemma 6.3. \square

Lemma 7.4. *Let $n = 6h + 2$. Then for any $1 \leq t \leq h$, there exists a partition of K_n into $6t + 1$ 1-factors and triples.*

Proof. By Lemma 6.1, $\{1, 2, 3, \dots, 3h - 1\} \setminus \{a, b\}$ can be partitioned into a collection of difference triples D , where $\{a, b\} \subseteq \{1, 2, 3, \dots, 3h - 1\}$. Choose any $t - 1$ triples D^* from D . Now the edges with lengths belonging to the set $\{a, b, 3h, 3h + 1\} \cup \{x \mid x \text{ is in a triple of } D^*\}$ form a cyclic graph, and since $\{0, 3h + 1\}$ has even order, by Theorem 6.2 this graph can be partitioned into $7 + 6(t - 1) = 6t + 1$ 1-factors. This gives a partition of K_{6h+2} into $6t + 1$ 1-factors and triples $\{\{i, x + i, y + i\} \mid \{x, y, x + y\} \in D \setminus D^*, i \in \mathbb{Z}_{6h+2}\}$. \square

We can now take care of the two remaining cases $(0, 4)$ and $(2, 4)$.

Lemma 7.5. *Let $n \geq 2m$, $m \equiv 0$ or $2 \pmod{6}$, and $n \equiv 4 \pmod{6}$. Then there exists a MCT of order n containing a MCT of order m .*

Proof. We will handle the case $m \equiv 0 \pmod{6}$ first. So let $m = 6t$ and $n - m = 6h + 4$. Then, of course, $1 \leq t \leq h$. By Lemma 7.3 there exists a tripole P such that $K_{n-m} \cup P$ can be partitioned into $6t$ 1-factors F and triples T and none of the edges in the 1-factors in F belong to the padding P . Let $F = \{f_1, f_2, \dots, f_{6t}\}$ and (S_1, T_1, P_1) a MCT of order m (P_1 is a 1-factor). Let X be the vertex set of K_{n-m} , where $X \cap S_1 = \emptyset$, and define a collection of triples T_2 on $S_2 = X \cup S_1$ as follows:

1. $T_1 \subseteq T_2$,
2. let α be a 1 - 1 mapping from S_1 onto $\{1, 2, \dots, 6t\}$ and place $\{a, b, x\alpha\}$ in T_2 if $\{a, b\} \in f_{x\alpha}$, and
3. $T \subseteq T_2$.

Then (S_2, T_2, P_2) is a MCT of order n containing the MCT (S_1, T_1, P_1) of order m . The padding $P_2 = P \cup P_1$ is a tripole.

If $m \equiv 2 \pmod{6}$, let $m = 6t + 2$ and $n - m = 6h + 2$. Then $1 \leq t \leq h$. By Lemma 7.4, there exists a partition of K_{n-m} into $6t + 1$ 1-factors $F = \{f_1, f_2, \dots, f_{6t+1}\}$ and triples T . Let (S_1, T_1, P_1) be a MCT of order m (P_1 is a tripole). Let X be the vertex set of K_{n-m} , where $X \cap S_1 = \emptyset$, and define a collection of triples T_2 on $S_2 = X \cup S_1$ as follows:

1. $T_1 \subseteq T_2$,
2. let α be a mapping from S_1 onto $\{1, 2, 3, \dots, 6t + 1\}$, where $x_\alpha = y_\alpha = 6t + 1$, $x \neq y$, and place $\{a, b, u\}$ in T_2 if and only if $\{a, b\} \in f_{u\alpha}$, and
3. $T \subseteq T_2$.

Then (S_2, T_2, P_2) is a MCT of order n containing the MCT (S_1, T_1, P_1) of order m . The padding is $P_2 = P_1 \cup f_{6k+1}$.

Combining both cases completes the proof. \square

8. MAIN THEOREM

If we combine the Doyen–Wilson Theorem [2] along with Lemmas 3.1, 3.2, 4.1, 5.1, 5.2, 7.1, 7.2, and 7.5, we have the following *complete solution* of the Doyen–Wilson Problem for minimum coverings of K_n with triples.

Theorem 8.1. *Let $n \geq 2m$. Then the necessary conditions in Section 2 to embed a MCT of order m in a MCT of order n are sufficient.*

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