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On the Bipartite Orthogonal Latin Square Graph

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1. Introduction.

Two $n \times n$ latin squares are orthogonal provided that when they are superimposed the ordered pairs obtained are all distinct. An orthogonal latin square graph (OLSG) is one in which the vertices are distinct latin squares of the same order and on the same symbols, and two vertices are adjacent if and only if the latin squares are orthogonal. If G is an arbitrary (finite) graph, we say that G is realizable as an OSLG if there is an OSLG isomorphic to G. The spectrum of G, Spec (G), is defined as the set of all positive integers n such that there is a realization of G by latin squares of order n.

In [1], it has been proved that every graph is realizable and Spec (G) contains all but a finite set of integers, i.e., there exists an integer v_0, such that for every $v \geq v_0$, $v \in$ Spec (G). But no explicit bounds on v_0 are given. In this paper, we give an upper bound for the v_0 when G is bipartite. We also compute the spectrum of several graphs.

2. The main theorems.

Let L and M be a pair of orthogonal latin squares based on the set $S = \{1,2,\ldots,n\}$, and α,β be permutations on S. It is well-known that L_α and M_β are orthogonal, where L_α and M_β are the latin squares obtained from permuting the entries of L and M by α and β respectively. Since, for each $k \neq 1,2,6$, there exists a pair of orthogonal latin squares of order k, we have the following result.

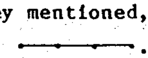
<u>Theorem 2.1.</u> Let $G = (H_1,H_2)$ be a complete bipartite graph with $\max\{|H_1|,|H_2|\} = m$. Then $k \in$ Spec (G), if $k \neq 1,2,6$, and $k! \geq m$.

In what follows, we will use K to denote the set of all positive integers except 1,2, and 6.

<u>Corollary 2.2.</u> If $m \leq 6$, then Spec $(G) = K$.

It is easy to see that we can generalize the theorem 2.1 to complete γ-partite graphs, provided there exist γ mutually orthogonal latin squares of order n.

<u>Lemma 2.3.</u> A pair of orthogonal latin squares of order n can be embedded in a pair of orthogonal latin squares of order $v \geq 3n$ respectively. [3]

In [1], they mentioned, it is interesting to compute the spectrum of the graph P_4 . We will give some results on small bipartite graphs in the following theorems.

Theorem 2.4. Spec $(C_6) \supseteq \{k: k > 12\}$.

Proof. It is well-known that there are three mutually orthogonal latin squares of order 4, C_1, C_2, and C_3. By Lemma 2.3, a pair of latin squares of order 4 can be embedded in a pair of latin squares of order $v \geq 12$, L and M. The proof is clear from Figure 2.1.

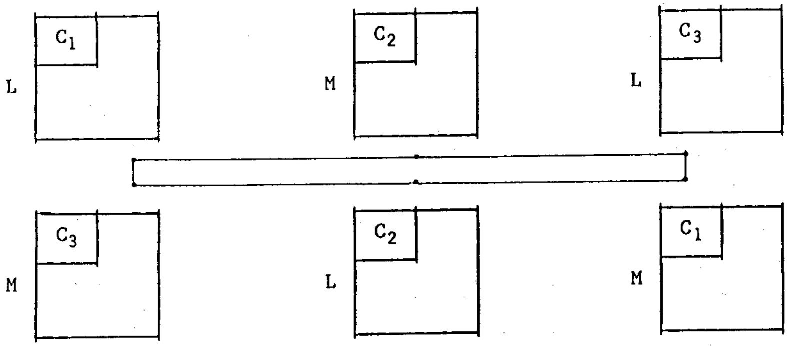
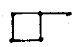


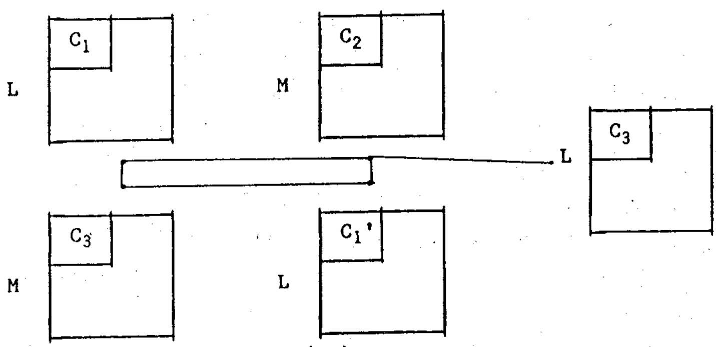
Figure 2.1.

Corollary 2.5. $\text{Spec}(P_4) \supseteq \{k: k \geq 12\}$, and $\text{Spec}(P_5) \supseteq \{k: k \geq 12\}$.

Proof. It is easy to see if H is an induced subgraph of G, then $\text{Spec}(G) \subseteq \text{Spec}(H)$. Since P_4 and P_5 are induced subgraphs of C_6, we have the proof.

Theorem 2.6. If G is a connected bipartite graph of order 5, then $\text{Spec}(G) \supseteq \{k: k \geq 12\}$.

Proof. It suffices to show Spec () $\supseteq \{k: k \geq 12\}$, and it is clear from Figure 2.2.



$C_1' = (C_1)_{(12)}$

Figure 2.2.

It is possible that we can compute the spectrums of some connected bipartite graphs of small order by the same way as above, we omit the details here. In what follows, we will give a lower bound for a more general bipartite graph.

Let B be a $p\times p$ latin square, and for each ordered pair (i,j), $i,j \in \{1,2,\ldots,p\}$, let A^i_j be a $q\times q$ latin square. Then the generalized direct product of the A^i_j's and B (denoted by $A^i_j \times B$ or $A\times B$ if all of the A^i_j's $= A$) is given by the accompanying diagram (Figure 2.3), where $A^i_j \times B(i,j)$ is the latin square obtained from A^i_j by replacing each entry a in A^i_j by $(a,B(i,j))$, where $B(i,j)$ is the entry in cell (i,j) of B.

$A^1_1\times B(1,1)$	$A^1_2\times B(1,2)$	$\cdots$	$A^1_p\times B(1,p)$
$A^2_1\times B(2,1)$	$A^2_2\times B(2,2)$	$\cdots$	$A^2_p\times B(2,p)$
$\vdots$	$\vdots$		$\vdots$
$A^p_1\times B(p,1)$	$A^p_2\times B(p,2)$	$\cdots$	$A^p_p\times B(p,p)$

Figure 2.3.

It is a routine matter to check if B and $\overline{B}$ are orthogonal latin squares of order p, and for each ordered pair (i,j), A^i_j is orthogonal to $\overline{A}^i_j$, then $A^i_j \times B$ and $\overline{A}^i_j \times \overline{B}$ are orthogonal latin squares of order $p\cdot q$.

Lemma 2.7. Let $G = (H_1,H_2)$ be a bipartite graph with $|H_1| \leq |H_2|$. Let $q \in K\{3,10,14\}$ and $p\in K$. Then $p\cdot q \in \text{Spec}(G)$, if $|H_1| \leq p^2$ and $|H_2| \leq (q!)^{p^2}$.

Proof. Let B, $\overline{B}$ be a pair of orthogonal latin squares of order p. By [2], there are three mutually orthogonal latin squares A_1, A_2, A_3 of order q. To start with, we let $G' = (H'_1,H'_2)$ where $|H'_1| = |H_1|$, $|H'_2|=|H_2|$ and the t-th vertex of H'_1 is the latin square $A^i_j \times \overline{B}$ such that $A^i_j = A_3$ if $t = p(i-1) + j$, and $A^i_j = A_1$ otherwise; and the vertices in H'_2 are $A^i_j \times \overline{B}$ where $A^i_j = A_2\ (\alpha_{i,j})$, $\alpha_{i,j}$ is a permutation on the set $\{1,2,\ldots,q\}$. (Since we can choose A^i_j independently, H'_2 can have as many as $(q!)^{p^2}$ vertices.) Obviously, G' is a complete bipartite graph. In what follows, we assume $H_1= H'_1$, and

v_s in H_2 is corresponding to the vertex v'_s in H'_2. If G is already complete, we are done; otherwise we consider the vertices of H'_2: if v_s is adjacent to each vertex of H_1, let $v_s = v'_s$; if v_s is not adjacent to u_t of H_1, we replace the $A^i{}_j$ in v'_s by $A_3\alpha$ where $t = p(i-1)+j$ and α is a permutation on the set $\{1,2,\ldots,q\}$. After the replacements; if the vertices in H'_2 are all distinct, then we let $H_2 = H'_2$ which concludes the proof. On the other hand, if (W.L.O.G) $v'_1, v'_2,\ldots,v'_k$ are the same latin square, they must have the same adjacency with the vertices of H_1, moreover, if n edges are missing for each vertex v'_i, $i=1,2,\ldots,k$, then $k \leq (q!)^n$. Now we can rearrange the replacements $A_3\alpha$ in $v'_1,v'_2,\ldots v'_k$ to obtain a set of distinct vertices $v_1,v_2,\ldots v_k$. We have the proof.

Theorem 2.8. If $G = (H_1, H_2)$ is a bipartite graph, and $n \in$ Spec (G), then $v \in$ Spec (G) for every $v \geq 3n$.

Proof. Let $v \geq 3n$, and L_1,L_2 be a pair of orthogonal latin squares of order v which contains a pair of orthogonal latin squares of order $n; M_1,M_2$, respectively. By replacing M_1,M_2 with $u \in H_1$, $v \in H_2$ respectively in Lemma 2.3, it's not difficult to see $v \in$ Spec (G).

Corollary 2.9. Let $G = (H_1,H_2)$ be a bipartite graph with $\min\{|H_1|,|H_2|\} \leq 9$ and $\max\{|H_1|,|H_2|\} \leq (24)^9$, then $v \in$ Spec (G) if $v \geq 36$.

By letting $p = 3$, and $q = 4$ in Lemma 2.3, we conclude the proof.

Corollary 2.10. Let $G = (H_1,H_2)$ be a bipartite graph with $|H_1| \leq |H_2|$, then $v \in$ Spec (G) for every $v \geq 3pq^2$, where $p = \min\{x: x\in K, x^2 \geq |H_1|\}$, and $q = \min\{y: y \in K \ \{3,10,14\}, (y!)^p \geq |H_2|\}$.

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