(43)

A New Construction for a Critical Set in Special Latin Squares

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Abstract

A critical set is a partial latin square which is completable to a latin square and omitting any entry of the partial latin square destroys this property. The size of a critical set is the number of entries in the partial latin square, and a critical set with minimum (maximum) size is called a minimal (maximal) critical set. In this paper, we study the minimal and maximal critical sets, and some results are obtained. Mainly, we prove that every minimal critical set in a latin square of order n has size at least $n+1$. Also, we show that the maximal critical set of a latin square of order 2^n-1 contains at least $4^n-3^n-2^{n+1}+3$ entries.

1 Introduction

A latin square of order n is an $n\times$ array with entries in $N=\{1,2,\dots,n\}$ such that each element of N occurs in each row and each column exactly once. A partial latin square of order n is an $n\times n$ array such that each element of N occurs at most once in each row and at most once in each column. A critical set in a latin square L of order n, is a set $A=\{(i,j;k)\mid i,j,k\in N\}$ such that, L is the only latin square of order n which has element k in position (i,j) for each $(i,j;k)\in A$, and no proper subset of A satisfies the above property. The size of a critical set A is $|A|$ and a critical set with minimum (maximum) size is called a minimal (maximal) critical set.

In [3], Lemma 2.3, Curran and van Rees showed that, if you take the ordered triples $(x,y;z)$ of a critical set, then the i^{th} component of these triples, $i=1,2,3$, must cover at least $n-1$ of the values $1,2,\dots,n$, thereby showing that the size of the minimal critical set is no less than $n-1$. Later, in [4], Lemma 3.1, Donovan et. al. improved this bound to n for each $n\geq 4$. In this paper, we shall prove that the size of a minimal critical set is at least $n+1$ for each $n\geq 5$. Note that this result is also obatined in [2] by a different method.

As to the maximal critical sets, it was shown by Stinson and van Rees [6] that the maximal critical set of a latin square of order 2^n contains at least 4^n-3^n entries. In

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section 3, we shall use a special construction to prove that the maximal critical set of a latin square of order $2^n - 1$ contains at least $4^n - 3^n - 2^{n+1} + 3$ entries.

2 Minimal critical set

Two latin squares are isotopic, if one can be transformed onto the other by permuting rows, permuting columns and renaming the entries. Thus two partial latin squares L and L' are isotopic if there exists an ordered triple (α, β, γ) of permutations such that $L(i, j)$, the (i, j) position in L, is k if and only if $L'(i\alpha, j\beta) = k\gamma$. In particular, two critical sets C and C' are said to be isotopic if they are isotopic as two partial latin squares. Then the following result is easy to see.

Lemma 2.1 ([4]) *Let C be a critical set in a latin square L. Let (α, β, γ)be an isotopism from the critical set C onto C'. Then C' is a critical set in the latin square L' obtained from L by applying the isotopism (α, β, γ).*

In order to obtain the lower bound for the minimal critical sets in a latin square, we shall rely on the following result.

Theorem 2.2 (L. D. Anderson [1]) *A partical latin square of order n with size at most $n+1$ can be completed to a latin square if it doesn't contain a partial latin square as in Figure 1 (Appendix).*

Theorem 2.3 *Let A be a critial set in a latin square of order $n \geq 5$, then $|A| \geq n+1$.*

Proof. Assume that A is a critical set and $|A| \leq n$. We shall claim that there exists a position not in A which we can choose two elements in $N = \{1, 2, \ldots, n\}$ to fill in respectively, and both can be completed to a latin square. This implies that A is not a critical set and thus $|A|$ must be at least $n+1$.

Let L be a latin square of order n. If n cells which jointly contain at least $n-1$ different symbols occupy at least $n-1$ rows and $n-1$ columns of L, then there exists at most one row and/or column which contains two of the cells. In this one row and/or column (if such exists), the entries are distinct so the entries in the cells of the remaining $n-2$ rows/columns use at least $n-3$ distinct symbols.

If they use $n-2$ distinct symbols, then they form a partial transversal of $n-2$ cells. If they use $n-3$ distinct symbols, then in the exceptional row/column which contains two of the cells, there is a cell whose entry is distinct from the $n-3$ symbols, so again we can find a partial transversal of length $n-2$.

By the result of Donovan et. al. in [4], we may suppose that $|A| = n$, and also the arguments are up to isotopisms. Since the position in A will occur in at least $n-1$ rows and $n-1$ columns, and the entries occur in A should contain at least $n-1$ elements of N. By above discussion, we may assume that the triples in A by $(i, i; i)$, for $i = 1, 2, \ldots, n-2$, $(n-1, n-1; x)$ and $(a, b; y)$ where $x = n-1$ or $y = n-1$. Now let (c, d) be a position not in A and $c \neq n-1$, $d \neq n-1$. If $n \geq 5$, then there are least two elements t_1 and t_2 in N which do not occur in the c^{th} row and d^{th} column of the partial latin square. Now let (c, d) be filled with t_1 and t_2 respectively, and we obtain A_1 and A_2 respectively. Clearly, A_1 and A_2 are two distinct partial latin squares of size $n+1$, and A_1 and A_2 do not contain any type of partial latin square in Figure 1. This implies that by Theorem 2.2, A_1 and A_2 can be completed respectively. Thus we have the claim and the proof. ♣

3 Maximal critical set

A latin square $L = [L(i,j)]$ of order n is idempotent if $L(i,i) = i$ for each i, unipotent if $L(i,i) = c$ where c is a constant, and commutative if $L(i,j) = L(j,i)$ for all i, j. A quasigroup satisfying the Steiner identities: $x^2 = x$, $x(xy) = y$ and $(yx)x = y$, is called a Steiner quasigroup. From the definition of Steiner quasigroup, we know that a latin squre obtained by a Steiner quasigroup is idempotent and commutative. Let $(P, *)$ be a Steiner quasigroup of order $v \geq 3$ and p, $q \in P$. Then $(P, *)$ contains the following subquasigroup.

$*$	p	q	$p*q$
p	p	$p*q$	q
q	$p*q$	q	p
$p*q$	q	p	$p*q$

The following result is easy to see.

Lemma 3.1 *Let L be a latin square obtained by a Steiner quasigroup. Then any tow distinct elements in L are contained in exactly one latin subsquare of order 3.*

Stinson and von Rees [6] defined the double of a latin square L and a critical set C in L, $2*L$ and $2*C$, as follows:

$$2*L = \begin{array}{|c|c|}\hline L_1 & L_2 \\ \hline L_2 & L_1 \\ \hline \end{array} \qquad 2*C = \begin{array}{|c|c|}\hline L_1 & C_2 \\ \hline C_2 & C_1 \\ \hline \end{array}$$

where L_i and C_i $(i = 1, 2)$ is a copy of L and C respectively, with every symbol x in L replaced by x_i. At the same time they showed that if C is a critical set in L, then $2*C$ is a critical set in $2*L$. Thus a critical set of the latin square representing 2-group of order 2^n can be constructed by above method. Therefore we have the following theorem.

Theorem 3.2 ([6]) *The maximal critical set of latin squares of order 2^n contains at least $4^n - 3^n$ elements.*

Instedad of prolongation [5], we will use a compression to obtain a latin square of order $2^n - 1$ from the latin square of order 2^n.

Let $L = [L(i,j)]$ be the latin square representing 2-group of order 2^n based on $\{1, 2, \ldots, 2^n\}$. Let $M = [M(i,j)]$ be an array constructed by replacing the diagonal entries of L with the 1^{st} row of L, and taking away its 1^{st} column and 1^{st} row. That is for each i, j in $\{1, 2, \ldots, 2^n - 1\}$

$$M(i,i) = L(1, i+1), \text{ and}$$
$$M(i,j) = L(i+1, j+1), \text{ for } i \neq j.$$

Then M has the following properties:

(1) M is an idempotent latin square of order $2^n - 1$ based on $\{2, 3, \ldots, 2^n\}$. (Since L is unipotent and commutative, and its 1^{st} row is $1, 2, 3, \ldots, 2^n$)
(2) M determines a Steiner quasigroup.

Let x and y be two distinct elements in $\{2, 3, \ldots, 2^n\}$. If $x * y = z$ in M, then L contains the following partial latin square.

1	x	y	z
x	1	z	
y	z	1	
z			1

Since L is the latin square representing 2-group, any two entries filled the same element are contained in a latin subsquare of order 2. Thus $x * z = y$ and $y * z = x$. Therefore M is a Steiner quasigroup.

The following example explains the above idea.

L

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

M

2	4	3	6	5	8	7
4	3	2	7	8	5	6
3	2	4	8	7	6	5
6	7	8	5	2	3	4
5	8	7	2	6	4	3
8	5	6	3	4	7	2
7	6	5	4	3	2	8

L is the latin square representing 2-group of order 8 and M is a latin square of order 7. By Theorem 3.2, we can obtain a critical set $C(L)$ of L as follows.

$C(L)$

1	2	3	4	5	6	7	
2	1	4	3	6	5		
3	4	1	2	7		5	
4	3	2	1				
5	6	7		1	2	3	
6	5			2	1		
7		5		3		1	

$C(M)$

2	4	3	6	5		
4	3	2	7		5	
3	2	4		7		
6	7		5	2	3	
5			2	6		
	5		3		7	

Corresponding to $C(L)$ we expect $C(M)$ is a critical set of M. This is a result of the following theorem.

Theorem 3.3 *Let L be a latin square of order 2^n representing 2-group and $C(L)$ be the critical set of L constructed as in Theorem 3.2. Let M be the latin square obtained from L by using a compression as explained above. Then $C(M) = \{(i,j;M(i,j)) | i,j \in \{1,2,\ldots,2^n-1\}$, and $(i+1,j+1;L(i+1,j+1)) \in C(L)\}$ is a critical set of M.*

Proof. Since $C(L)$ is a critical set of L, by construction it is easy to see that $C(M)$ can be completed to M and also the completion is unique. Now we claim that removing any entry of $C(M)$ will destroy this property. First, if an entry $(i,j;u)$ of $C(M)$ not in diagonal is removed, correspondingly an entry in $C(L)$ not in the first row is removed. By the fact that $C(L)\backslash\{(i+1,j+1;L(i+1,j+1))\}$ can not be completed uniquely, $C(M)\backslash\{(i,j;u\}$ can not be completed uniquely either. Finally, if we remove any entry $(i,i;x)$ of $C(M)$, then since M determines a Steiner quasigroup $(M,*)$ there exists a 3×3 subsquare of M determined by x and 2^n, which intersects $C(M)$ at exactly one entry $(x,2^n;x*2^n)$. This implies that $C(M)\backslash\{(i,i;x)\}$ is not uniquely completable. A minimal critical set of a latin square of order 3 contains at least 2 entries. Therefore we have the proof. ♣

By direct counting, we obtain the following result for the size of the maximal critical sets.

Corollary 3.4 *The maximal critical set of latin squares of order 2^n-1 contains at least $4^n-3^n-2^{n+1}+3$ elements.*

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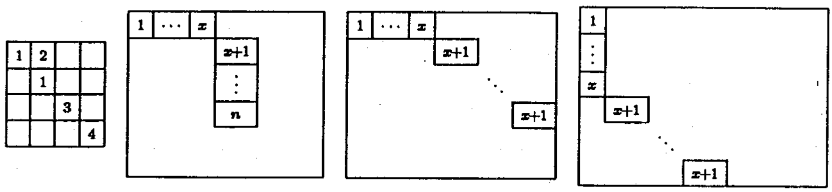
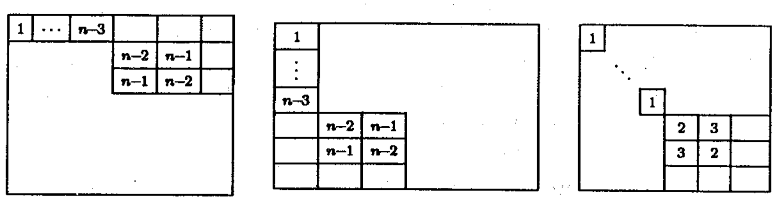
Appendix

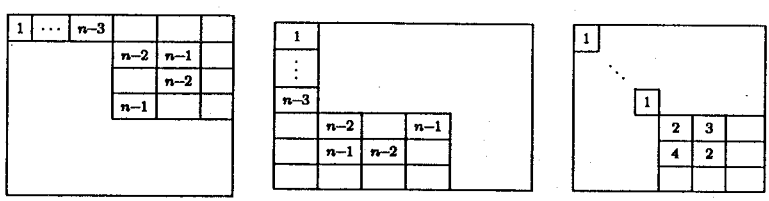


Type 1 ($1 \leq x < n$) Type 2 ($1 \leq x < n$) Type 3 ($1 \leq x < n$)

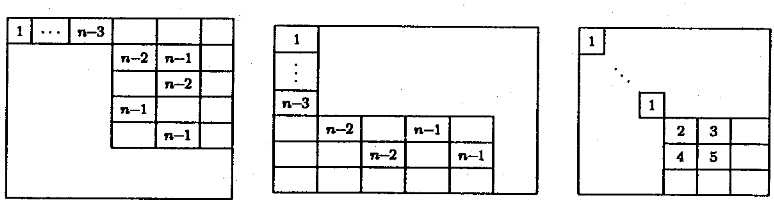
The Noncompletable Partial Latain Squares of Side n with n Nonempty Cells.



Type 4 ($n \geq 3$) Type 5 ($n \geq 3$) Type 6 ($n \geq 3$)



Type 7 ($n \geq 4$) Type 8 ($n \geq 4$) Type 9 ($n \geq 4$)



Type 10 ($n \geq 5$) Type 11 ($n \geq 5$) Type 12 ($n \geq 5$)

The Noncompletable Partial Latin Squares of Side n with $n+1$ Nonempty Cells.