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On Latin $(n \times n \times (n-2))$-Parallelepipeds

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1. Introduction

Let $A_1, A_2, \ldots, A_k$ be pairwise disjoint latin squares of (the same) n elements. The ordered k-tuple $A = (A_1, A_2, \ldots, A_k)$ is called a latin (nxnxk)-parallelepiped. In the case k = n, A is called a latin cube of order n.

In [1], P. Horák shows that, for all $n = 2^k$, $k \geqslant 3$, there exists a latin (nxnx(n−2))-parallelepiped that cannot be embedded in any latin cube of order n.

In this paper, generalizing the idea of [1], we prove that for each $n \geqslant 12$ there exists a latin (nxnx(n−2))-parallelepiped that cannot be embedded in any latin cube of order n. Moreover, we show that a latin (nxnx(n−2))-parallelepiped can always be embedded in a latin cube of order m with $m \geqslant 4n$.

2. Main Theorems

We start with the construction of a latin (6x6x4)-parallelepiped which cannot be embedded in a latin cube of order 6. (Figure 2.1).

$$A_1 = \begin{matrix} 1 & 2 & 4 & 3 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \\ 4 & 3 & 2 & 6 & 1 & 5 \\ 3 & 5 & 6 & 1 & 4 & 2 \\ 5 & 6 & 1 & 4 & 2 & 3 \\ 6 & 4 & 5 & 2 & 3 & 1 \end{matrix} \qquad A_2 = \begin{matrix} 2 & 1 & 5 & 4 & 6 & 3 \\ 1 & 3 & 2 & 6 & 4 & 5 \\ 6 & 2 & 3 & 1 & 5 & 4 \\ 5 & 6 & 4 & 2 & 3 & 1 \\ 3 & 4 & 6 & 5 & 1 & 2 \\ 4 & 5 & 1 & 3 & 2 & 6 \end{matrix}$$

$$A_3 = \begin{matrix} 3 & 5 & 1 & 6 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 6 & 4 & 2 & 3 & 1 & 5 \\ 4 & 3 & 5 & 2 & 6 & 1 \\ 1 & 6 & 3 & 4 & 5 & 2 \end{matrix} \qquad A_4 = \begin{matrix} 5 & 6 & 3 & 1 & 4 & 2 \\ 3 & 5 & 6 & 4 & 2 & 1 \\ 1 & 4 & 5 & 2 & 3 & 6 \\ 4 & 2 & 1 & 5 & 6 & 3 \\ 6 & 1 & 2 & 3 & 5 & 4 \\ 2 & 3 & 4 & 6 & 1 & 5 \end{matrix}$$

Figure 2.1

LEMMA 2.1. *(A_1, A_2, A_3, A_4) is a latin (6x6x4)-parallelepiped which cannot be embedded in a latin cube of order 6.*

PROOF. Let $C = [S_{i,j}]$ be the 6x6 array where $S_{i,j} = \{1, 2, \ldots, 6\} \setminus \{A_1(i,j), A_2(i,j), A_3(i,j), A_4(i,j)\}$, ($A_k(i,j)$ is the (i, j)-entry of the latin square A_k). (Figure 2.2.) In order to embed (A_1, A_2, A_3, A_4) into a latin cube of order 6, we have to find A_5 and A_6 such that $A_1, A_2, \ldots, A_6$ are pairwise disjoint latin squares. Hence, if $\{a, b\} = S_{i,j}$, $\{c, d\} = S_{i',j'}$ and $i = i'$ or $j = j'$ (not both), then $A_k(i,j) = a$ and $A_k(i',j') = c$ should imply that $b \neq d$ ($k = 5$ or 6). We start with the entry $A_5(1, 2)$. We will use "$\rightarrow A_5(i,j)$" to denote the next entry to be picked. (1) $A_5(1,2) = 3 \rightarrow A_5(2,2) = 4 \rightarrow A_5(3,2) = 6 \rightarrow A_5(5,2) = 5 \rightarrow A_5(6,2) = 2$, but $A_5(1,2) = 3 \rightarrow A_5(1,1) = 4 \rightarrow A_5(1,3) = 6 \rightarrow A_5(6,3) = 2$, which is not possible for A_5. Similarly, (2) $A_5(1,2) = 4 \rightarrow A_5(4,2) = 3 \rightarrow A_5(6,2) = 1 \rightarrow A_5(5,2) = 2 \rightarrow A_5(3,2) = 5 \rightarrow A_5(3,5) = 6 \rightarrow A_5(3,6) = 2 \rightarrow A_5(3,3) = 1 \rightarrow A_5(2,3) = 5 \rightarrow A_5(4,3) = 3$ which contradicts $A_5(4,2) = 3$. Since it is not possible to find A_5, we have the proof.

4,6	3,4	2,6	2,5	1,3	1,5
4,6	4,6	1,5	2,3	1,5	2,3
3,5	5,6	1,4	3,4	2,6	1,2
1,2	1,3	3,5	4,6	2,5	4,6
1,2	2,5	3,4	1,6	3,4	5,6
3,5	1,2	2,6	1,5	4,6	3,4

Figure 2.2

LEMMA 2.2. *A latin cube of order n can be embedded in a latin cube of order m for every $m \geqslant 2n$.*

PROOF. (We note that this lemma has been proved in [2]. We recall it here, since it is in preprint.) Let $m \geqslant 2n$. It is well known that a latin square A of order n can be embedded in a latin square $L = [\ell_{i,j}]$ of order $m \geqslant 2n$. Set $L_1 = L$ and construct L_t, $t = 2, 3, \ldots, m$, by letting the (i, j)-entry in L_t be $(\ell_{i,j})\alpha_t$ where α_t is the permutation $\begin{pmatrix} \ell_{1,1} & \ell_{1,2} & \cdots & \ell_{1,m} \\ \ell_{t,1} & \ell_{t,2} & \cdots & \ell_{t,m} \end{pmatrix}$. It is easy to see that $(L_1, L_2, \ldots, L_m)$ is a latin cube of order m and contians a latin sub-cube of order n generated from A in the upper-left corners of $(L_1, L_2, \ldots, L_n)$. We can replace this sub-cube with the original given latin cube of order n. This completes the proof.

Now we are ready for the main theorem.

THEOREM 2.3. *For each $n \geqslant 12$, there exists a latin (nxnx(n−2))-parallelepiped which cannot be embedded in a latin cube of order n.*

PROOF. Let n be any positive integer $\geqslant 12$. By Lemma 2.2, there exists a latin cube $L = (L_1, L_2, \ldots, L_n)$ of order n, which contains a latin cube $B = (B_1, B_2, \ldots, B_6)$ of order 6, in the upper-left corners of $(L_1, L_2, \ldots, L_6)$. By replacing B_i with A_i of Figure 2.1, i = 1, 2, 3, 4, it is easy to see $(L_1, L_2, L_3, L_4, L_7, L_8, \ldots, L_n)$ is a latin (nxnx(n−2))-parallelepiped which cannot be embedded in a latin cube of order n.

LEMMA 2.4. *Any latin (nxnx(n−2))-parallelepiped can be embedded in a latin cube of order 2n.*

PROOF. Let $A = (A_1, A_2, \ldots, A_{n-2})$ be a latin (nxnx(n−2))-parallelepiped and C be the nxn array $[S_{i,j}]$ such that $S_{i,j} = \{1, 2, \ldots, n\} \setminus \{A_k(i, j) | k = 1, 2, \ldots, n-2\}$. It is not difficult to see $\{S_{i,1}, S_{i,2}, \ldots, S_{i,n}\}$ satisfies the Hall's condition. Hence we can construct an nxn array $B_{n-1} = [B_{n-1}(i, j)]$ such that the ith row is an SDR (system of distinct representatives) of $\{S_{i,1}, S_{i,2}, \ldots, S_{i,n}\}$. It is a direct result that each row of the nxn array $B_n = [B_n(i, j)]$ has distinct elements where $B_n(i, j) = S_{i,j} \setminus \{B_{n-1}(i, j)\}$. Now let $B'_{n-1} = [B'_{n-1}(i, j)]$ and $B'_{n-1}(i, j) = B_{n-1}(i, j) + n$ if there exists $i' > i$ such that $B_{n-1}(i, j) = B_{n-1}(i', j)$, otherwise $B'_{n-1}(i, j) = B_{n-1}(i, j)$. Similarly we construct B'_n. Moreover, we let $A_{n-1} = B'_{n-1}$, and $A_n(i, j) = B'_n(i, j) + n$ if $B'_n(i, j)$ occurs in the jth column of $A_{n-1}(= B'_{n-1})$, otherwise $A_n(i, j) = B'_n(i, j)$. We are ready to construct a latin cube of order 2n which contains the parallelepiped $(A_1, A_2, \ldots, A_{n-2})$. (Figure 2.3.) In the figure, we define $C_k = [C_k(i, j)]$ as follows: $C_k(i, j) = A_k(i, j) + n$, if $A_k(i, j) \leqslant n$; $C_k(i, j) = A_k(i, j) - n$, if $A_k(i, j) > n$.

L_k $\quad 1 \leqslant k \leqslant n-2$

A_k	C_k
C_k	A_k

L_{n-1}

A_{n-1}	C_{n-1}
A_n	C_n

L_n

A_n	C_n
A_{n-1}	C_{n-1}

L_{n+1}

C_{n-1}	A_{n-1}
C_n	A_n

L_{n+2}

C_n	A_n
C_{n-1}	A_{n-1}

L_{k+n+2} $\quad 1 \leqslant k \leqslant n-2$

C_k	A_k
A_k	C_k

Figure 2.3

It is not difficult to see that L_k is a latin square for each $1 \leqslant k \leqslant n-2$, and $n+3 \leqslant k \leqslant 2n$. If L_{n-1} is a latin square, so are L_n, L_{n+1}, and L_{n+2}. It suffices to check that L_{n-1} is a latin square of order 2n. Since, by construction, $L_{n-1}(i, j) \neq L_{n-1}(i', j)$, and $L_{n-1}(i, j) \neq L_{n-1}(i, j')$ for each $i' \neq i$ and $j' \neq j$, we conclude that L_{n-1} is a latin square. $L_k(i, j) \neq L_{k'}(i, j)$ for each $k' \neq k$ is a direct result of the way we defined L_k. Hence $L = (L_1, L_2, \ldots, L_{2n})$ is a latin cube of order 2n which contains the parallelepiped $(A_1, A_2, \ldots, A_{n-2})$. This completes the proof.

THEOREM 2.5. *Any latin (nxnx(n−2))-parallelepiped can be embedded in a latin cube of order m for every $m \geqslant 4n$.*

PROOF. By Lemmas 2.2, and 2.4.

References

[1] PETER HORAK, *Latin parallelepipeds and cubes*, J. Comb. Theory, Series A 33, 213-214 (1982).

[2] HUNG-LIN FU, *Steiner quadruple systems of order 4v with prescribed intersections*, Ars Combinatoria 21 (1986), pp. 89-103.

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