38

Some Results on Equalized Total Coloring

HUNG-LIN FU[1]

Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan, R.O.C.

ABSTRACT

A **total k-coloring** of a graph is a map $\pi : V(G) \cup E(G) \to C$ such that $|C| = k$ and no incident or adjacent pair of elements of $V(G) \cup E(G)$ receive the same color. The total chromatic number $\chi_t(G)$ is the least value k for which G has a total k-coloring. A total coloring π is **equalized** if for each pair of distinct colors c_1 and c_2 in C, $||\pi^{-1}(c_1)| - |\pi^{-1}(c_2)|| \leq 1$. In this paper, we study the problem of equalized total colorings and some results are obtained in the direction of verifying the conjecture that G has an equalized total k-coloring for each $k \geq \Delta(G) + 2$.

1. Introduction

Throughout of this paper, all graphs are finite, simple and undirected. Let G be a graph, and its vertex, edge set, and the maximum degree be denoted by $V(G)$, $E(G)$ and $\Delta(G)$ respectively. Other terms and notions not defined in this paper can be found in [5].

A total coloring π of a graph G is a mapping $\pi : V(G) \cup E(G) \to C$ such that no two adjacent vertices receive the same color, no two edges incident with the same vertex receive the same color, and no edge receives the same color as either of the vertices it is incident with. A total k-coloring is a total coloring having image set of size k, and the total chromatic number $\chi_t(G)$ of a graph G is the smallest integer k such that G has a total k-coloring. From the definition of $\chi_t(G)$, it is clear that $\chi_t(G) \geq \Delta(G) + 1$. Behzad [1] made the following conjecture.

Total Coloring Conjecture (TCC). For any graph G, $\chi_t(G) \leq \Delta(G) + 2$.

The TCC has been verified for several classes of graphs [3,6,10,11,12,22,23], especially those graphs with very low or very high degree. Also, similar to the argument of the chromatic index $\chi'(G)$ of G, we can classify those graphs which

[1]Research supported by National Science Council of the Republic of China (NSC80-0208-M009-28) and DIMACS.

satisfy the TCC. A graph G is said to be of **type one** if $\chi_t(G) = \Delta + 1$ and it is of **type two** if $\chi_t(G) = \Delta(G) + 2$. The results obtained so far can be seen in [2,4,6,7,8,9,13,14,16,18,19,20].

In this paper, we study another property of the total coloring which is similar to the **equalized edge coloring** or the **equitable vertex coloring.** An **equalized total coloring** is a total coloring π such that for each pair of distinct colors c_1 and c_2 used in the coloring, $||\pi^{-1}(c_1)| - |\pi^{-1}(c_2)|| \leq 1$. It is well-known that if a graph G has an edge coloring which uses k colors, then G has an equalized edge k-coloring.[21] Also in vertex coloring, A. Hajnal and E. Szemeredi proved that G has an equitable vertex k-coloring for each $k \geq \Delta(G) + 1$.[15] From the above results, we believe that the total coloring should have a similar property. If we expect the result is as good as the edge coloring, then we shall have

Conjecture 1. For every graph G, G has an equalized total k-coloring for each $k \geq \chi_t(G)$.

In this paper, we will show that this conjecture is true for some special graphs, such as the complete graphs, the complete bipartite graphs, $\cdots$, etc. But this conjecture is not true in general. We will give a counterexample in section 2. Then we made another conjecture and some results are obtained in verifying this conjecture.

Conjecture 2. For every graph G, G has an equalized total k-coloring for each $k \geq max\{\chi_t(G), \Delta(G) + 2\}$.

Due to the difficulty of equalizing two color classes of a total coloring, the following problem is worth of mention.

Problem. Is that true that if a graph G has an equalized total k-coloring, then G has an equalized total $(k + 1)$-coloring?

2. The main results

First, we notice that if a graph satisfies the Conjecture 1, then it is also satisfies the Conjecture 2. Hence we shall start with those graphs which satisfy the Conjecture 1. Let T be an independent set in G and G_T^* be obtained by adding a new vertex v^* and the edges v^*u where $u \in V(G)\backslash T$, i.e., $V(G_T^*) = V(G) \cup \{v^*\}$ and $E(G_T^*) = E(G) \cup \{v^*u | u \in V(G)\backslash T\}$. Then the following lemma is clear.

Lemma 2.1. Let T be an independent set in G and $\chi'(G_T^*) = g$, then $\chi_t(G) \leq g + 1$.

Proof. Since G_T^* has an edge coloring which used g colors, let φ be such a coloring. Then the proof follows by letting a total coloring of G be π such that (i) $\pi(v) = g + 1$ for each $v \in T$, (ii) $\pi(u) = \varphi(v^*u)$ for each $u \in V(G)\backslash T$, and (iii) $\pi(e) = \varphi(e)$ for each $e \in E(G)$. **Q.E.D.**

From the definition of G_T^*, it is easy to see that $\Delta(G_T^*) \leq \max\{\Delta + 1, n - t\}$. $|V(G)| = n$ and $|T| = t$. Thus by Vizing's Theorem $\chi'(G_T^*) \leq \max\{\Delta+2, n-t+1\}$. This implies that $\chi_t(G) \leq \max\{\Delta(G) + 3, n - t + 2\}$. This upper bound can be improved if either $G\backslash T$ contains a perfect matching M or $G\backslash T$ contains a matching N such that any vertex of degree $\Delta(G)$ in $V(G\backslash T)$ is incident to one of the edges in N.

Lemma 2.2. If $G\backslash T$ contains a perfect matching M or $G\backslash T$ contains a matching N such that any vertex of degree $\Delta(G)$ (major vertex) in $V(G\backslash T)$ is incident to one of the edges in N, then $\chi_t(G) \leq \max\{\Delta(G) + 2, n - t + 2\}$.

Proof. Let G' be a graph obtained by deleting M from G_T^*. Then $\Delta(G') \leq \max\{\Delta(G), n - t\}$ and hence $\chi'(G') \leq \max\{\Delta(G) + 1, n - t + 1\}$. By coloring the edges in M and the vertices in T with the same color, we have the proof of the first case. The second case can be obtained by replaced M with N. **Q.E.D.**

We note here that if $\Delta(G) + 2 \geq n - t + 2$, i.e., $\Delta(G) \geq n - t$, then the result in Lemma 2.2 becomes to be $\chi_t(G) \leq \Delta(G) + 2$. This implies that the graph G satisfies the TCC. Therefore, we have the following

Corollary 2.3. If there exists an independent set T such that (i) $\Delta(G) \geq n - t$, and (ii) either $G\backslash T$ contains a perfect matching or $G\backslash T$ contains a matching N such that any vertex of degree $\Delta(G)$ in $V(G\backslash T)$ is incident to one of the edges in N, then G satisfies TCC.

The above technique is not new at all. For completeness, we give a proof. Now, we return to equalized total coloring.

Lemma 2.4. If there exists an independent set T in G and a matching N (may be empty) in $G\backslash T$ such that $|T| + |N| = \lfloor \frac{|E(G)|+n-t-|N|}{k} \rfloor$, and $\chi'(G_T^*) \leq k$. Then G has an equalized total (k+1)-coloring.

Proof. By [21], $G_T^*\backslash N$ has an equalized edge k-coloring. Since $|T| + |N| = \lfloor \frac{|E(G)|+n-t-|N|}{k} \rfloor$, hence by a similar coloring process as in Lemma 2.1, we have an equalized total $k + 1$-coloring of G. **Q.E.D.**

Now we are ready to verify the conjecture for several classes of graphs.

Proposition 2.5. Let G be a graph of order n with $\Delta(G) = n - 1$. Then G has an equalized total k-coloring for each $k \geq \chi_t(G)$.

Proof. Consider G^* ($T = \emptyset$), i.e., $G^* = G + v^*$. If G^* is of class one, then by deleting v^* from G and color the vertex u in G with the color of the edge v^*u we obtain a total n-coloring of G. This implies that G is of type one. On the other hand, it is easy to see that if G is of type one, then G^* is of class one. Hence $\chi'(G^*) = \chi_t(G)$. Since G^* has an equalized edge k-coloring for each $k \geq \chi'(G^*)$, thus by the coloring process as mentioned above, we obtain an equalized total k-coloring (without a new color) for each $k \geq \chi'(G^*) = \chi_t(G)$. **Q.E.D.**

Corollary 2.6. The complete graph K_n has an equalized total k-coloring for each $k \geq \chi_t(K_n)$.

Corollary 2.7. The complete split graph $K_n + O_r$ has an equalized total k-coloring for each $k \geq \chi_t(K_n + O_r)$ where O_r is a stable set (independent set) of r vertices.

We remark here that in a graph G of order n, G has an equalized total k-coloring for each $k \geq n + 1$.

Now we turn to the complete bipartite graphs.

Proposition 2.8. Every complete bipartite K_{n_1,n_2}, $n_1 \leq n_2$, has an equalized total k-coloring for each $k \geq \chi_t(K_{n_1,n_2})$.

Proof. It is known that K_{n_1,n_2} is of type two if and only if $n_1 = n_2$. Thus we prove this proposition by considering two cases. (i) $n_1 = n_2$. Let $K_{n_1,n_2} = (A, B)$ where A and B are the two partite sets. By Lemma 2.4, it suffices to show that for each $k \geq n_1 + 2$, there exists an independent set T with N an empty set such that

$$t = |T| = \lfloor \tfrac{n_1 n_2 + n_1 + n_2 - t}{k-1} \rfloor \text{ and } G_T^* \text{ has an edge } (k-1)\text{-coloring.} \qquad (1)$$

It is clear that if $k = n_1 + 2$, then T can be chosen as the partite set A. In what follows, we claim that for each $k > n_1 + 2$ the independent set T can be chosen to satisfy (1). By the construction of G_T^*, we observe that $deg_{G^*}(v^*) = n_1 + n_2 - t$. This implies that if $|T|$ is getting smaller (as k increases), then $deg_{G_T^*}(v^*)$ is getting larger. Thus we have to show that if T is chosen from A with size $n_1 - j = \lfloor \frac{n_1^2 + n_1 + j}{k-1} \rfloor$, then $k - 1$ must be at least $n_1 + j + 1$. Suppose not. Let $k - 1 < n_1 + j + 1$, i.e. $k - 1 \leq n_1 + j$. Hence

$$\begin{aligned} n_1 - j = \lfloor \frac{n_1^2 + n_1 + j}{k-1} \rfloor &\geq \lfloor \frac{n_1^2 + n_1 + j}{n_1 + j} \rfloor \\ &= \lfloor \frac{(n_1^2 - j + 1)(n_1 + j) + j^2}{n_1 + j} \rfloor \geq n_1 - j + 1. \end{aligned}$$

This is a contradiction. Hence $k - 1 \geq n_1 + j + 1$. Therefore

$$\chi'(G_T^*) \leq \Delta(G_T^*) + 1 = n_1 + j + 1 \leq k - 1.$$

This concludes the proof of the case 1.

(ii) $n_1 < n_2$. The situation $k = n_2 + 1$ can be obtained by direct coloring (using a latin rectangle), thus we consider $k > n_2 + 1$. Choose T as a subset of B and

$|T| = n_2 - j$. We shall claim $k-1 \geq n_1+j+1$ provided that $n_2-j = \lfloor\frac{n_1n_2+n_1+j}{k-1}\rfloor$. First, if $n_1+j < n_2$, then $k > n_1+j+1$, i.e. $k-1 \geq n_1+j+1$, we are done. Otherwise, if $n_1+j \geq n_2$, and assume that $k-1 < n_1+j+1$, i.e. $k-1 \leq n_1+j$.

$$\begin{aligned} n_2 - j &\geq \lfloor\frac{n_1n_2+n_1+j}{n_1+j}\rfloor \\ &= \lfloor\frac{(n_2-j+1)(n_1+j)+n_1j-n_2j+j^2}{n_1+j}\rfloor \\ &= \lfloor n_2-j+1+\frac{j(n_1+j-n_2)}{n_1+j}\rfloor \\ &\geq n_2-j+1. \end{aligned}$$

This is a contradiction. Hence we have the claim and the proof of case (ii). **Q.E.D.**

So far, for a general bipartite graph G whether there exists an equalized total k-coloring for each $k \geq \chi_t(G)$ is still unknown. But, it can be shown that it is true for a tree. Since the proof is mainly by induction and adjusting the colors of the end vertices and edges, we will not go any further for the details and simply state the result.

Proposition 2.9. Let T be a tree. Then T has an equalized total k-coloring for each $k \geq \chi_t(T)$.

From Proposition 5,6,7,8 and 9, it seems that the Conjecture 1 is quite true. But a counterexample has been found. In Figure 2.1, the graph has no equalized total 7-coloring. Actually, we have a more general result and this class of graphs disprove the Conjecture 1.

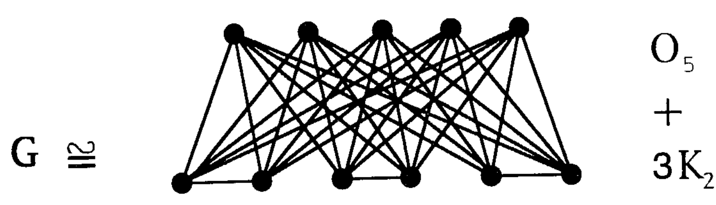

Figure 2.1.

Proposition 2.10. Let $G = nK_2 + O_{2n-1}$ for each $n \geq 3$. Then $\chi_t(G) = 2n+1$, but G does not have an equalized total $(2n+1)$-coloring.

Proof. First, we claim that $\chi_t(G) = 2n+1$. Let G be as in Figure 2.2, and $L = [l_{i,j}]$ be a latin square of order 2n. Define a total coloring π of G by letting (i) $\pi(a_i) = \pi(b_{2k-1}, b_{2k}) = 2n+1$, $i = 1, 2, \cdots, 2n-1$ and $k = 1, 2, \cdots, n$; (ii) $\pi(b_j) = l_{2n,j}$, $j = 1, 2, \cdots, 2n$, and (iii) $\pi(a_ib_j) = l_{i,j}$, $1 \leq i \leq 2n-1$ and $1 \leq j \leq 2n$. Then π is a total $(2n+1)$-coloring which concludes the part one.

Now, since G is a $2n$-regular graph which uses $2n+1$ colors, hence every color c should occur around every vertex, i.e., for each $u \in V(G)$, either u is colored with c or one of the edges incident to u will be colored by c. Therefore it is not difficult to see that if a color c occurs in $V(G)$, then c occurs an odd number of time(s). ($|V(G)|$ is odd.) Also, if c occurs $2t+1$ times in $\{a_1, a_2, \cdots, a_{2n-1}\}$, $t = 0, 1, \cdots, n-1$, then c occurs $t+1$ times in $\{b_1b_2, b_3b_4, \cdots, b_{2n-1}b_{2n}\}$. This implies that if there are more than two colors occur in $\{a_1, a_2, \cdots, a_{2n-1}\}$, then we don't have enough edges of the form $b_{2k-1}b_{2k}$ to use. Hence all the vertices in $\{a_1, a_2, \cdots, a_{2n-1}\}$ must be colored with a common color, say 1. Then $\pi(b_{2k-1}b_{2k}) = 1$, $k = 1, 2, \cdots, n$. Thus 1 occurs $3n-1$ times which is larger than $\lceil \frac{4n-1+2n(2n-1)+n}{2n+1} \rceil = 2n+1$ whenever $n \geq 3$. We conclude the proof. **Q.E.D.**

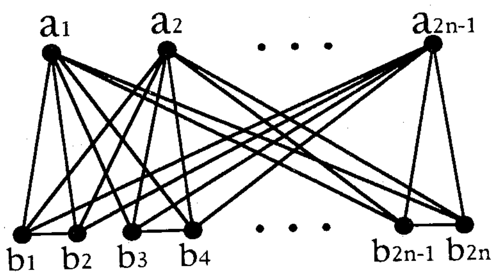

Figure 2.2.

The graph $nK_2 + O_{2n-1}$ does have an equalized total $(2n+2)$-coloring. The coloring can be obtained by letting $T = \{a_1, a_2, \cdots, a_{2n-1}\}$ and $N = \{b_1b_2\}$. Now $\chi'(G_T^* \backslash N) = 2n+1$ and $\lfloor \frac{(4n-1)+2n(2n-1)+n-(2n-1)-1}{2n+1} \rfloor = \lfloor \frac{4n^2+n-1}{2n+1} \rfloor = 2n-1$ but $\lceil \frac{4n^2+n-1}{2n+1} \rceil = 2n$, therefore we have an equalized total $(2n+2)$-coloring. Indeed, $G = nK_2 + O_{2n-1}$ does have an equalized total k-coloring for each $k \geq 2n+2 = \Delta(G)+2$ and we shall omit the proof here.

From the fact mentioned above we make the following conjecture.

Conjecture 2. For every graph G, G has an equalized total k-coloring for each $k \geq max\{\chi_t(G), \Delta(G)+2\}$.

In what follows, we shall show two classes of graphs which satisfy the Conjecture 2. Since these two classes of graphs satisfy TCC, hence it suffices to show that G has an equalized total k-coloring for each $k \geq \Delta(G)+2$.

Proposition 2.11. Let G be a graph of order n with $\Delta(G) = n-2$. Then G has an equalized total k-coloring for each $k \geq \Delta(G)+2$, i.e. G satisfies the Conjecture 2.

Proof. Let $G^* = v^* + G$. Now $deg_{G^*}(v^*) = n$ and $deg_{G^*}(v) < n$ for each $v \in V(G)$. Thus G^* is of class 1 and thus G^* has an equalized edge k-coloring for each $k \geq n = \Delta(G) + 2$. By coloring $v \in V(G)$ with the color of v^*v in the equalized edge coloring, we obtain an equalized total k-coloring for G. **Q.E.D.**

Proposition 2.12. Let G be the complete t-partite graph with its partite sets be of sizes $n_1, n_2, \cdots, n_t$ respectively. Then G satisfies the Conjecture 2 if G is of odd order.

Proof. Without loss of generality, let $n_1 \leq n_2 \leq \cdots \leq n_t$ and let $n = \sum_{i=1}^{t} n_i$. It suffices to show that G has an equalized total k-coloring for each k, $n - n_1 + 2 \leq k \leq n$ and $n_1 \geq 3$. Again, we will use Lemma 2.4, i.e., we try to find an independent set T and a matching N in $G \backslash T$, s.t.,

$$|T| + |N| = \lfloor \frac{\sum_{i \neq j} n_i n_j + n - |T| - |N|}{k-1} \rfloor \quad (1)$$

and

$$k - 1 \geq \chi'(G_T^* \backslash N). \quad (2)$$

Since $\lfloor \frac{\sum_{i \neq j} n_i n_j + n - |T| - |N|}{k-1} \rfloor \geq n_1$, hence by letting T be the partite set of size n_1, then $\Delta(G_T^* \backslash N) \leq n - n_1 + 1$ and therefore $\chi'(G_T^* \backslash N) \leq n + n_1 + 1$. ($G_T^*$ is of even order and thus not overfull. By [17], G_T^* is of class 1 so is $G_T^* \backslash N$.) This implies that for each $k \geq n + n_1 + 2$, $k - 1 \geq \chi'(G_T^* \backslash N)$, and (2) is proved. To show (1) is true, let us consider the following inequality:

$$\begin{aligned}
\lfloor \frac{\sum_{i \neq j} n_i n_j + n - |T| - |N|}{k-1} \rfloor &\leq \lfloor \frac{(\frac{n}{t})^2 \binom{t}{2} + 2 - |T| - |N|}{n - n_1 + 1} \rfloor \\
&\leq \lfloor \frac{(\frac{n^2}{t})(\frac{t-1}{2}) + n - n_1}{n - n_1} \rfloor \\
&= \lfloor \frac{(n^2)(\frac{t-1}{t})}{2(n - n_1)} \rfloor + 1 \\
&\leq \lfloor \frac{(n^2)(\frac{t-1}{t})}{2(\frac{t-1}{t})(n)} \rfloor + 1 \\
&= \lfloor \frac{n}{2} \rfloor + 1.
\end{aligned}$$

Hence N can be chosen suitably so that G has an equalized total k-coloring. **Q.E.D.**

We note here that if G is a complete t-partite graph of even order, so far we can only use a similar idea to show that for each $k \geq \Delta(G) + 3$, G has an equalized total k-coloring.

Acknowledgement

The author would like to express his thanks to Dr. Bor-Liang Chen and Dr. A.J. W. Hilton for their helpful comments in working on this problem.

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