Proc. Natl. Sci. Counc. ROC(A)
Vol. 17, No. 2, 1993. pp. 121-123

(Scientific Note)

A Disproof of the Upper Embeddable Conjecture

HUNG-LIN FU AND MING-CHUN TSAI

Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan, R.O.C.

(Received: October 20, 1992; Accepted: December 8, 1992)

ABSTRACT

Let G be a connected graph. G is called *upper embeddable* if the maximum genus of G, $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$ where $\beta(G) = |E(G)|-|V(G)| + 1$ is the *Betti number* of G. In 1972, R. D. Ringeisen conjected that every 3-edge connected graph is upper embeddable and the conjecture was disproved by Bouchet in 1976 in which the rotational embedding scheme was utilized. In this paper, we will use a simple counting argument to disprove the conjecture. As a consequence, we also construct a 3-connected graph G_n for each nonnegative integer n such that the *Betti deficiency* of G_n is n.

Keywords: upper embeddable, Betti deficiency

I. Introduction and the Main Results

A compact orientable 2-manifold is a surface that may be thought of as a sphere on which has been placed a number of "handles" or, equivalently, a sphere in which has been inserted a number of "holes". The number of handles (or holes) is referred to as the **genus of the surface**. By the **genus** $\gamma(G)$ **of a graph** G is meant the smallest genus of all surfaces (compact orientable 2-manifolds) on which G can be embedded.

If G is embedded on a surface S, then the components of $S - G$ are the **regions** of the embedding. A region is called 2-**cell** if any simple closed curve in that region can be continuously deformed or contracted in that region to a single point. Equivalently, a region is a 2-**cell** if it is topologically homeomorphic to 2-dimensional Euclidean space. An embedding of a graph G on the surface S is called a **2-cell embedding** of G on S if all the regions so determined are 2-cells. Let G be a connected graph. The **maximum genus** $\gamma_M(G)$ of G is the maximum among the genera of all surfaces on which G can be 2-cell embedded. And G is called **upper embeddable** if $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$ where $\beta(G) = |E(G)| - |V(G)| + 1$ is the **Betti number** of G.

Bouchet (1976) used the *rotational embedding scheme* (Behzad et al., 1979) to disprove the conjecture of Ringeisen (1972): G is upper embeddable if G is 3-edge connected. In this paper, we will use another method to disprove this conjecture. Before we prove the following lemma, we need a definition.

In Nordhaus *et al.*, (1971) it was shown that $\gamma_M(G) \le \left\lfloor \frac{\beta(G)}{2} \right\rfloor$. This upper bound motivates the study of the difference $\xi(G) = \beta(G) - 2\gamma_M(G)$ called the *Betti deficiency* of G. If for a spanning tree T of a graph G, we let $\xi(G, T)$ denote the number of components of the cotree $G - E(T)$ which have odd size (= number of edges) then it is well known that $\xi(G) = min\{\xi(G, T)$: T is a spanning tree of $G\}$(Xuong, 1979).

Lemma 1. If G is a connected (p, q)-graph (order p and size q) which contains k vertex disjoint triangles, then $\xi(G) \ge 2k + p-q-1$.

proof. Let $A_1, A_2,..., A_k$ be k vertex disjoint triangles in G. For any spanning tree T of G,

$$0 \le |E(A_i) \cap E(T)| \le 2,\ i = 1, 2, \ldots, k.$$

Thus, we can let

$$|E(A_i) \cap E(T)| = \begin{cases} 2, & i = 1,2,\ldots,k_1; \\ 1, & i = k_1+1,\ldots,k_2;\ and \\ 0, & i = k_2+1,\ldots,k. \end{cases}$$

where $0 \le k_1 \le k_2 \le k$.
This implies that

$$|E(A_i) \cap (E(G) - E(T))| = \begin{cases} 1, & i = 1,2,\ldots,k_1; \\ 2, & i = k_1+1,\ldots,k_2;\ and \\ 3, & i = k_2+1,\ldots,k. \end{cases}$$

where $0 \le k_1 \le k_2 \le k$.
Now, we define

$$E_1 = (E(G) - E(T)) \cap (E(G) - \cup_{i=1}^{k} E(A_i)).$$

Thus

$$|E_1| = |(E(G)-E(T)) \cap (E(G)-\bigcup_{i=1}^{k} E(A_i))|$$
$$= |(E(G)-E(T)) \setminus ((E(G)-E(T)) \cap \bigcup_{i=1}^{k} E(A_i))|$$
$$= |(E(G)-E(T))| - |\bigcup_{i=1}^{k} E(A_i) \cap (E(G) - E(T))|$$
$$= (q-p+1)-[k_1+2(k_2-k_1)+3(k-k_2)].$$

Let $H = (V, E)$ such that $V(H) = V(G)$ and $E(H) = (E(G) - E(T)) \cap \bigcup_{i=1}^{k} E(A_i)$. Since $|E(A_i) \cap (E(G)-E(T))|$ is even if and only if $k_1 + 1 \le i \le k_2$, we have,

$$o(H) = k_1 + (k - k_2)$$

where $o(H)$ denotes the number of odd size component of H. For any edge e of E_1, $o(H) - 1 \le o(H + e)$ and

$$E_1 \cup E(H) = (E(G)-E(T)) \cap (E(G)-\bigcup_{i=1}^{k} E(A_i)) \cup ((E(G)-E(T)) \cap \bigcup_{i=1}^{k} E(A_i))$$
$$= E(G) - E(T).$$

Hence we have

$$\begin{aligned}\xi(G, T) &\ge o(H) - |E_1| \\ &= (k_1 + k - k_2) - \{(q - p + 1) - [k_1 + 2(k_2 - k_1) + 3(k - k_2)]\} \\ &= 2k + p - q + 1 + 2(k - k_2) \\ &\ge 2k + p - q - 1.\end{aligned}$$

Q.E.D.

Since it is well known that G is upper embeddable if and only if $\xi(G) \le 1$, the existence of a 3-connected graph with $2k + p - q - 1 \ge 2$ will disprove the conjecture.

Corollary 2. If G is a connected (p, q)-graph which contains k vertex disjoint triangles and $k > \frac{q-p}{2} + 1$, then G is not upper embeddable.

By applying Lemma 1, we are able to obtain the following

Proposition 3. For each nonnegative integer n, there exists a 3-connected graph G_n such that $\xi(G_n) = n$.

Proof. Let G_0 be the Petersen graph and $G_1 = K_4$. We can see that for $n = 0, 1$, G_n is 3-connected and $\xi(G_n) = n$. For $n \ge 2$, we consider G_n as in Fig. 1. Since for each $n \ge 2$ and any vertex $v \in V(G)$, $G_n - v$ contains no cut vertex, G_n is 3-connected. By Lemma 1,

$$\begin{aligned}\xi(G_n) &\ge 2(2(n + 1)) + p - q - 1 \\ &= 4(n + 1) + 6(n + 1) - 9(n + 1) - 1 \\ &= n.\end{aligned}$$

Furthermore, we find a spanning tree T_0 of G_n with $\xi(G_n, T_0) = n$ as the bold lines in Fig. 1, so $\xi(G_n) \le n$. Thus we have $\xi(G_n) = n$. Q.E.D.

Remark

Recently, we were informed by Dr. M. Skoviera that a similar idea has been obtained in his Ph. D. thesis which is written in Slovak. Since the proof is short and independent, we submit it for publication.

References

Behzad, M., G. Chartrad, and L. Lesniak-Foster (1979) *Graphs and Digraphs*. Prindle, Weber and Schmidt, Boston.

Bouchet, A. (1976) Genre maximum d'un Δ-graphe. Colloque international CNRS "Problemes Combinatoires et Theorie des Graphes". Orsay, 57-60.

Nordhaus, E.A., B.M. Stewart, and A.T. White (1971) On the maximum genus of a graph. *J. Combinatorial* Theory, **B11**, 258-267.

Ringeisen, R.D. (1972) Upper and lower imbeddable graphs. In: *Graph Theory and Applications*, pp.261-268. Springer-Verlag, Berlin.

Xuong, N.H. (1979) How to determine the maximum genus of a graph. *J. Combinatorial Theory*, **B26**, 217-225.

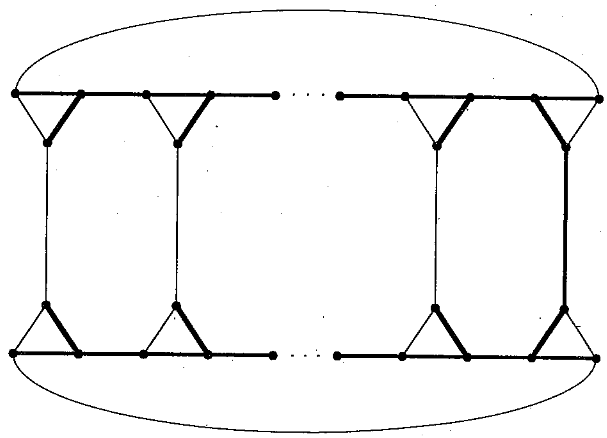

Fig. 1. G_n: $(n+1)$ pairs of triangles.

A Disproof of the Upper Embeddable Conjecture

最大可嵌入猜測的一個反證

傅 恆 霖 • 蔡 明 春

國立交通大學應用數學研究系所

摘　　要

如果一個連通圖的最大虧格等於此圖的貝蒂數除以二的高斯值，則稱此圖爲最大可嵌入圖，其中貝蒂數等於圖的邊數減去圖的點數加一。在一九七二年，R. D. Ringeisen推測：任意三度連通圖皆是最大可嵌入圖，而且Bouchet在一九七六年利用旋轉的嵌入技巧來反證這個推測。在這篇論文中，我們將利用簡單的計算來反證這個推測。最後，我們同樣地做到了對任意正整數N，都可以建構一個貝蒂差值等於N的三度連通圖。