ARS COMBINATORIA

VOLUME THIRTY-FIVE–A

NOVEMBER, 1993

WINNIPEG, CANADA

Two Classes of Graphs Which Have Ascending Subgraph Decompositions

Hung-Lin Fu and Wei-Hsin Hu[1]

Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan, Republic of China

Abstract. Let G be a graph with $\binom{n+1}{2}$ edges. Then we will say that G has an **ascending subgraph decomposition** if the edge set of G can be partitioned into n sets generating graphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \ldots, n$) and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$. In this paper, we show that a split graph with $\binom{n+1}{2}$ edges and $K_{n+3} \setminus S$, where S is a subgraph of size $2n+3$, both have ascending subgraph decomposition.

Dedicated to Roger Entringer on the occasion of his 60th birthday

1. Introduction

In [1], the authors give the following partition conjecture.

Conjecture. *Let G be a graph on $\binom{n+1}{2}$ edges. Then the edge set of G can be partitioned into n sets generating graphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \ldots, n$) and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$.*

A graph G which can be partitioned as described in the conjecture will be said to have an *ascending subgraph decomposition* (abbreviated ASD). The graphs $G_1, G_2, \ldots, G_n$ are members of such a decomposition. The conjecture has been verified for special classes of graphs. We list some of them as theorems.

Theorem 1.1. [2,7] *Any forest with $\binom{n+1}{2}$ edges has an ASD.*

Theorem 1.2. [5] *If a graph G has $\binom{n+1}{2}$ edges and maximum degree $\Delta(G) \leq \lfloor (n-1)/2 \rfloor$, then G has an ASD with each member a matching.*

Theorem 1.3. [3] *If G is a graph with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor (2-\sqrt{2})n \rfloor$, then G has an ASD. Moreover, if G is a forest with $\binom{n+1}{2}$ edges, and $\Delta(G) < \lfloor (3-\sqrt{3})n/2 \rfloor$, then G has an ASD.*

Theorem 1.4. [6] *A complete bipartite graph with $\binom{n+1}{2}$ edges has an ASD.*

A graph $G = (V, E)$ is called a *split graph* if its vertex set can be partitioned into two disjoint subsets U and W such that the graph induced by U is a complete graph and the graph induced by W is an empty graph.

[1] Research supported by the National Science Council of the Republic of China (NSC 79-0208-M009-33).

In this paper, we first prove that a split graph with $\binom{n+1}{2}$ edges has an ASD, and then we show that the graph $K_{n+3} \backslash S$ has an ASD provided that S is a subgraph of K_{n+3} with $2n+3$ edges.

2. The main result

First, we show that a split graph with $\binom{n+1}{2}$ edges has an ASD.

Theorem 2.1. *A split graph G with $\binom{n+1}{2}$ edges has an ASD with each member a star.*

Proof: The proof will be by induction on n. It is trivial for small n. Let the assertion be true for $n = m$ and assume that G has size $\binom{m+2}{2}$. Let $V(G) = \{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_t\}$ where $\{u_1, u_2, \ldots, u_k\}$ induce a complete graph and $\{v_1, v_2, \ldots, v_t\}$ induce an empty graph, and $\deg(u_1) = \Delta(G)$. First, we have $\deg(u_1) \geq m+1$. For otherwise, $\Delta(G) \leq m$ and so $|E(G)| \leq \binom{k}{2} + k(m-k+1) = \frac{k(k-1)}{2} + km - k(k-1) = km - \frac{k(k-1)}{2} = \frac{k(2m-k+1)}{2} \leq \frac{m(m+1)}{2} = \binom{m+1}{2}$. If $\deg(u_1) = m+1$, delete u_1 from G. Then $G' = G \backslash u_1$ has $\binom{m+1}{2}$ edges and has an ASD with each member a star by the induction hypothesis. Hence G has an ASD with each member a star. If $\deg(u_1) > m+1$, delete exactly $m+1$ edges which are incident with u_1 beginning with the edges of the form $\{u_1, v_j\}$. There are two cases to consider. In the case that $\deg(u_1) \geq m+k$, after deleting $m+1$ edges of the form $\{u_1, v_j\}$, we have G' which is also a split graph with $\binom{m+1}{2}$ edges. By induction we conclude that G has an ASD with each member a star. If $\deg(u_1) < m+k$, let G' be the resulting graph after deleting $m+1$ edges. Then u_1 is not adjacent to v_i for $i = 1, 2, \ldots, t$. It is clear that $\{u_2, u_3, \ldots, u_k\}$ induces a complete graph and $\{v_1, v_2, \ldots, v_t, u_1\}$ induces an empty graph. Then we have a split graph with vertex set $\{u_2, u_3, \ldots, u_k, v_0, v_1, \ldots, v_t, u_1\}$. It is a split graph with $\binom{m+1}{2}$ edges. Thus, G has an ASD with each member a star.

Before we prove the next theorem we need a definition. A tree T is said to be a *double star* if T can be obtained by joining the centers of two stars with an edge. For each $k \geq 3$ a double star with its centers of degree $k-1$ and 2 will be denoted by T_k. For convenience we let T_1 be a graph with a single edge and T_2 be a star with two edges. Clearly, $T_1, T_2, \ldots, T_k$ is a set of ascending graphs. Now we are ready to prove the theorem which extends a result in [4]. For completeness, we state the result of [4] first.

Theorem 2.2. [4] *If G is a graph with $\binom{n+1}{2}$ edges, and at most $n+2$ vertices, then G has an ASD.*

Theorem 2.3. *Let G be a graph with $\binom{n+1}{2}$ edges and $|V(G)| \leq n+3$. Then G has an ASD, $G_1, G_2, \ldots, G_n$ with $G_i = T_i$ for $i = 1, 2, \ldots, n$, except when G is the union of two disjoint triangles.*

Proof: The proof will be by induction on n and it is true for $n \leq 4$. Since $|V(G)| \leq n+3$ and $|E(G)| = \binom{n+1}{2}$ it follows that the maximum degree $\Delta(G)$

of G satisfies $n-1 \leq \Delta(G) \leq n+2$. Thus for $n \geq 5$ we have four cases to consider according to $\Delta(G)$.

Case 1. $\Delta(G) = n-1$: Then G has $n+3$ vertices. Let x be a vertex in $V(G)$ whose degree is $n-1$, and $N(x) = \{y | \{x, y\} \in E(G)\} = \{v_1, v_2, \ldots, v_{n-1}\}$. Let $V(G) = N(x) \cup \{x\} \cup \{a, b, c\}$. It is easy to see that there exists a vertex in $N(x)$ which is adjacent to a vertex in $\{a, b, c\}$. Without loss of generality, let the edge be $\{v_1, a\}$. Now by deleting x and $\{v_1, a\}$ we have a graph G', where $|V(G')| \leq n+2$ and $|E(G')| = \binom{n}{2}$. Thus by the induction hypothesis G' has an ASD, $T_1, T_2, \ldots T_{n-1}$, and an ASD of G is easy to obtain.

Case 2. $\Delta(G) = n$: Let x be a vertex of maximum degree in G and consider the graph $G' = G \backslash x$. By the induction hypothesis, G' has an ASD $G'_i \cong T_i$, $i = 1, 2, \ldots, n-1$. Since $|V(G'_{n-1})| = n$ and $|N(x)| = n$, $|V(G'_{n-1}) \cap N(x)| \geq n + n - (n+2) = n-2 \geq 3$. Let G'_{n-1} be as in Figue 2.1.

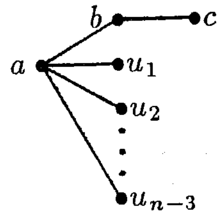

Figure 2.1

If $\{b, c\} \subset N(x)$, the ASD will be obtained by letting $G_n = (S_x \cup \{b, c\}) \backslash \{x, b\}$, $G_{n-1} = (G'_{n-1} \cup \{x, b\}) \backslash \{b, c\}, G_{n-2} = G'_{n-2}, \ldots, G_1 = G'_1$, where S_x is the star induced by the edges incident to x in G. Otherwise we have two situations, to consider. First, only one of b and c is in $N(x)$. Then there exists a u_i, $1 \leq i \leq n-3$, such that $u_i \in N(x)$. Hence the ASD can be obtained by letting $G_n = (S_x \cup \{b, c\}) \backslash \{x, u_i\}$, $G_{n-1} = (G'_{n-1} \cup \{x, u_i\}) \backslash \{b, c\}$, and $G_{n-2} = G'_{n-2}, \ldots, G_1 = G'_1$. Secondly, neither b nor c is in $N(x)$, then $\{a, u_1, u_2, \ldots, u_{n-3}\} \subset N(x)$. The ASD of G will be $G_1 = G'_1$, $G_2 = G'_2, \ldots, G_{n-2} = G'_{n-2}$, $G_{n-1} = (G'_{n-1} \cup \{x, a\}) \backslash \{a, u_1\}$ and $G_n = (S_x \cup \{a, u_1\}) \backslash \{x, a\}$. This concludes the proof of this case.

Case 3. $\Delta(G) = n+1$: Let x be a vertex of maximum degree in G and let $N(x) = \{v_1, v_2, \ldots, v_{n+1}\}$. Then there are vertices v_i and v_j, $1 \leq i < j \leq n+1$, such that $\{v_i, v_j\}$ is not an edge of G. Let $G' = (G \backslash x) \cup \{v_i, v_j\}$. By the induction hypothesis, G' has an ASD $G'_i \cong T_i$, $i = 1, 2, \ldots, n-1$. Now if $\{v_i, v_j\} \in G'_1$ then $G'_2 \cup S_x$ must be one of the graphs in Figure 2.2. Each of these graphs can be partitioned into T_1, T_2, T_n. Let $G_i = G'_i$ where $i = 3, 4, \ldots, n-1$, $G_n = T_n$, $G_2 = T_2$ and $G_1 = T_1$. Then we have the ASD for G. If $\{v_i, v_j\} \in G'_2$ then $(G'_2 \cup S_x) \backslash \{v_i, v_j\}$ must be one of the graphs of Figure 2.3 and by a similar argument, G has an ASD with $G_i \cong T_i$ for $i = 1, 2, \ldots, n$.

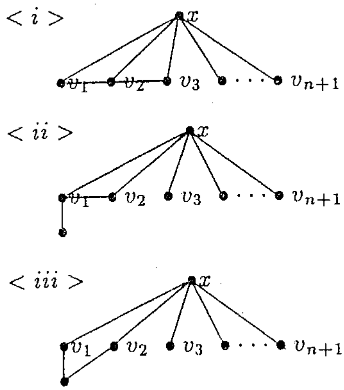

Figure 2.2

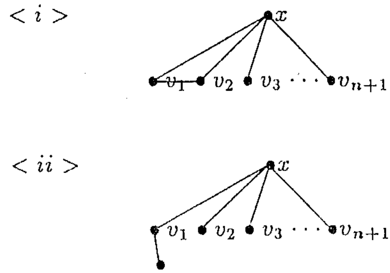

Figure 2.3

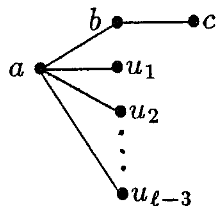
Figure 2.4

Suppose, then, that $\{v_i, v_j\} \in G'_\ell$ where $3 \leq \ell \leq n-1$. We consider three cases.

(i) $\{v_i, v_j\} = \{a, b\}$. Let $G_n = (S_x \cup \{b,c\}) \backslash (\{a,x\} \cup \{x,y\})$ and $G_\ell = (G'_\ell \cup \{a,x\} \cup \{x,y\}) \backslash (\{b,c\} \cup \{a,b\})$ where

$$y \in \begin{cases} \{c\}, & \text{if } \{b,c\} \subset N(x); \text{ and} \\ N(x) \backslash V(G'_\ell) & \text{otherwise.} \end{cases}$$

and $G_i = G'_i$ where $i \in \{1, 2, \ldots n-1\} \backslash \{\ell\}$.

(ii) $\{v_i, v_j\} = \{b, c\}$, let $G'_1 = \{w, z\}$ and W.L.O.G. let $w \in N(x)$. If $a \in N(x)$ then let $G_n = (S_x \cup \{a,b\}) \backslash (\{a,x\} \cup \{x,c\})$, $G_\ell = G'_\ell \cup (\{a,x\} \cup \{x,c\}) \backslash (\{a,b\} \cup \{b,c\})$ and $G_i = G'_i$ for $i \in \{1, 2, \ldots, n-1\} \backslash \{\ell\}$. If $a \notin N(x)$ then let $G_\ell = (G'_\ell \cup \{y,x\}) \backslash \{b,c\}$ where $y \in \{b, u_1\} \backslash \{w\}$. Then $(S_x \cup G'_1) \backslash \{x,y\}$ must be (a) or (b) in Figure 2.5 which clearly can be decomposed into $G_n \cong T_n$ and $G_1 \cong T_1$ and $G_i = G'_i$ where $i \in \{2, 3, \ldots, n-1\} \backslash \{\ell\}$.

(iii) $\{v_i, v_j\} = \{a, u_p\}$ where $1 \leq p \leq \ell - 2$. Since $|N(x)| = n+1$, at least one of b or c is in $N(x)$. As in (i) we can decompose $(S_x \cup G'_\ell) \backslash \{a, u_p\}$ into $G_n = T_n$ and $G_\ell = T_\ell$. Let $G_i = G'_i$ for $i \in \{1, 2, \ldots, n-1\} \backslash \{\ell\}$. This gives an ASD for G.

Case 4. $\Delta(G) = n+2$: If there exists a vertex x such that $n-1 \leq \deg(x) \leq n+1$ then go back to Case 1, 2 or 3. Assume, then, that G contains $r(\geq 1)$ vertices

(a) (b)

Figure 2.5

which have degree $n+2$, a vertex z of degree $n-k(k \geq 2)$ and no vertex of degree p such that $n-k < p < n+2$. Then $(n-k)(n+3-r) + r(n+2) \geq n^2+n$. It is easy to check that $r \geq k \geq 2$ and $n-k \geq r$. Since at least one neighbor of z has degree $n+2$, we can delete all edges incident to z and one another edge e to form T_{n-k+1}. We claim that $\tilde{G} = (G\backslash z)\backslash e$ can be decomposed into $T_1, T_2, \ldots, T_{n-k}, T_{n-k+2}, T_{n-k+3}, \ldots, T_n$, i.e., a graph of order $n+2$ with $\binom{n+1}{2} - (n-k+1)$ edges which has at least $k-1$ vertices with degree $n+1$ can be decomposed into $T_1, T_2, \ldots, T_{n-k}, T_{n-k+2}, \ldots, T_n$ whenever $n-k \geq 2$.

Proof of the claim: We prove the claim by induction and it is true for $n=4$ (since $n-k \geq 2$ we have that $n \geq 4$). Assume $\tilde{G}$ has $n+2$ vertices and at least $k-1$ vertices which have degree $n+1$. Let x be a vertex of degree $n+1$ and $N(x) = \{v_1, v_2, \ldots, v_{n+1}\}$. Then there are vertices v_i and v_j, $1 \leq i < j \leq n+1$, such that $\{v_i, v_j\}$ is not an edge of $\tilde{G}$. Let $\tilde{G}' = (\tilde{G}\backslash x) \cup \{v_i, v_j\}$ which by induction can be decomposed into $\tilde{G}'_1, \tilde{G}'_2, \ldots, \tilde{G}'_{n-k}, \tilde{G}'_{n-k+2}, \ldots, \tilde{G}'_{n-1}$ if $n > n-k+2$, or $\tilde{G}'_1, \tilde{G}'_2, \ldots, \tilde{G}'_{n-k}$ if $n = n-k+2$. Let $\{v_i, v_j\}$ be in $\tilde{G}'_\ell$ Then by the same argument as in Case 3 $\tilde{G}' \cup S_x$, can be decomposed into $T_1, T_2, \ldots, T_{n-k}, T_{n-k+2}, \ldots, T_n$.

Thus we have the proof.

Acknowledgement

The authors would like to appreciate the helpful comments of the referee.

References

1. Y. Alavi, A.J. Boals, G. Chartrand, P. Erdös and 0. Oellerman, *The ascending subgraph decomposition problem*, Congr. Number. **58** (1987), 7–14.
2. R.J. Faudree and R.J. Gould, *Ascending subgraph decomposition for forest*, Congressus Numeratium **70** (1990), 221–229.
3. R.J. Faudree, R.J. Gould, M.S. Jacobson and L. Lesniak, *Graphs with an ascending subgraph decomposition.* in preprints.
4. R.J. Faudree, A. Gyárfás and R.H. Schelp, *Graphs which have an ascending subgraph decomposition*, Congr. Number. **59** (1987), 49–54.
5. H.L. Fu, *A note on the ascending subgraph decomposition problem*, Discrete Math. **84** (1990), 315–318.
6. H.L. Fu, *Some results on the ascending subgraph decomposition*, Bulletin of the Institute of Mathematics, Academia Sinica **16, No. 4** (1988), 341–345.
7. H.L. Fu and W.H. Hu, *Ascending subgraph decompositon of the forest*, Report, NSC 78-0208-M00920.