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A NOTE ON THE EMBEDDING OF A LATIN PARALLELEPIPED INTO A LATIN CUBE

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ABSTRACT

In this note, we propose a counter example to a Kochol's conjecture about the embedding of a latin parallelepiped into a latin cube.

I. INTRODUCTION

Let $L_1 = [l_{i,j}^{(1)}]$, $L_2 = [l_{i,j}^{(2)}]$, ..., $L_k = [l_{i,j}^{(k)}]$ be latin squares of elements 1, 2, ..., n. The ordered k-tuple $C_k = (L_1, L_2, ..., L_k)$ is called a **latin** $(n \times n \times k)$**-parallelepiped** if the elements $l_{i,j}^{(1)}$, ..., $l_{i,j}^{(k)}$ are all different for every pair (i,j) such that $1 \leq i,j \leq n$. (Sometimes we say that these latin squares are **disjoint** in this case.) When $k = n$, C_k is called a **latin cube** of order n.

Eger (1981), in the Sixth Hungarian Colloquium on Combinatorics, proposed a problem asking if there exists an $(n \times n \times (n-k))$-parallelepiped to be added to a given $(n \times n \times k)$-parallelepiped to form a latin cube of order n[3]. Fu[1], Horak[2] and Kochol[3,4,5] studied on this problem and got some excellent results. In particular, Kochol proved the following therem in [3] and proposed a conjecture in [5].

Theorem. For any $d \geq 2$ and $n \geq 2d+1$ there exists a latin $(n \times n \times (n-d))$-parallelepiped that cannot be embeded into a latin cube of order n.

Conjecture. Every latin $(n \times n \times (n-d))$-parallelepiped can be embeded into a latin cube of order n if and only if $n \leq 2d$.

II. AN EXAMPLE

Here we propose an example to show that the Conjecture is false.

Example. Consider $n = 5$, $d = 3$, and let

$$L_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 1 & 4 & 5 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 4 & 2 & 3 & 1 \end{bmatrix}, \text{ and}$$

$$L_2 = \begin{bmatrix} 2 & 3 & 1 & 5 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 & 3 \\ 5 & 1 & 3 & 4 & 2 \\ 3 & 2 & 4 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} \{3,4,5\} & \{1,4,5\} & \{2,4,5\} & \{1,2,3\} & \{1,2,3\} \\ \{1,3,5\} & \{1,2,4\} & \{1,3,4\} & \{2,4,5\} & \{2,3,5\} \\ \{2,4,5\} & \{2,3,5\} & \{1,2,3\} & \{1,3,4\} & \{1,4,5\} \\ \{1,2,3\} & \{2,3,4\} & \{2,4,5\} & \{1,3,5\} & \{1,4,5\} \\ \{1,2,4\} & \{1,3,5\} & \{1,3,5\} & \{2,4,5\} & \{2,3,4\} \end{bmatrix}$$

Figure 1.

Let $H = [S_{i,j}]$ be the 5×5 array where $S_{i,j} = \{1, 2, 3, 4, 5\} - \{l_{i,j}^{(1)}, l_{i,j}^{(2)}\}$, ($l_{i,j}^{(k)}$ is the (i,j) entry of the latin square L_k, $k = 1, 2$.) (Figure 1). We claim that there does not exist any latin square $L = [l_{i,j}]$ disjoint from L_1 and L_2 with $l_{5,5} = 4$.

To the contrary, suppose that there is such a latin square L with $l_{5,5} = 4$. Since $S_{3,5} - \{4\} = S_{4,5} - \{4\} = \{1,5\}$, we have $l_{1,5}, l_{2,5} \in \{2,3\}$. In each of the following cases, we will use "$l_{i,j} \to k$" to denote that $l_{i,j}$ is forced to be k.

Case (i). $l_{1,5} = 2$ and $l_{2,5} = 3$.

(a) If $l_{2,1} = 1$, then $l_{2,3} \to 4$, $l_{2,2} \to 2$, $l_{2,4} \to 5$, $l_{5,4} \to 2$, and then $l_{5,1}$ has no element to be taken.

(b) If $l_{2,1} = 5$ and $l_{3,1} = 2$, then $l_{5,1} \to 1$, $l_{4,1} \to 3$, $l_{1,1} \to 4$, $l_{1,3} \to 5$, $l_{1,2} \to 1$, $l_{1,4} \to 3$, $l_{5,3} \to 3$, $l_{5,2} \to 5$, $l_{5,4} \to 2$, $l_{3,2} \to 3$, $l_{3,3} \to 1$, $l_{3,4} \to 4$, and then $l_{2,4}$ has no element to be taken.

(c) If $l_{2,1} = 5$ and $l_{3,1} = 4$, then $l_{1,1} \to 3$, $l_{1,4} \to 1$, $l_{3,4} \to 3$, $l_{4,4} \to 5$, $l_{5,4} \to 2$, $l_{2,4} \to 4$, $l_{2,3} \to 1$, $l_{2,2} \to 2$, $l_{5,1} \to 1$, $l_{4,1} \to 2$, $l_{4,3} \to 4$, $l_{4,5} \to 1$, $l_{3,5} \to 5$, and then $l_{3,2}$ has no element to be taken.

Case (ii). $l_{1,5} = 3$ and $l_{2,5} = 2$.

(a) If $l_{2,4} = 4$, then $l_{2,2} \to 1$, $l_{2,3} \to 3$, $l_{2,1} \to 5$, $l_{1,1} \to 4$, $l_{3,1} \to 2$, $l_{3,3} \to 1$, $l_{3,4} \to 3$, $l_{3,5} \to 5$, and then $l_{3,2}$ has no element to be taken.

(b) If $l_{2,4} = 5$ and $l_{1,4} = 1$, then $l_{5,4} \to 2$, $l_{4,4} \to 3$, $l_{3,4} \to 4$, $l_{5,1} \to 1$, $l_{4,1} \to 2$, $l_{3,1} \to 5$, $l_{2,1} \to 3$, $l_{1,1} \to 4$, $l_{1,2} \to 5$, $l_{5,2} \to 3$, $l_{4,2} \to 4$, $l_{4,3} \to 5$, and then $l_{5,3}$ has no element to be taken.

(c) If $l_{2,4} = 5$ and $l_{1,4} = 2$, then $l_{5,4}$ has no element to be taken.

Combining the above cases, we conclude that we can not extend L_1 and L_2 into a latin cube. Thus we disprove the Conjecture.

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REFERENCES

[1] Fu, H. L., On latin ($n \times n \times (n-2)$)-parallelepipeds, Tamkang J. of Mathematics 17 (1986), pp. 107~111.

[2] Horak, P., Latin parallelepipeds and cubes, J. of Combinatorial Theory (A) 33 (1982), pp. 213~214.

[3] Kochol, M., Relatively narrow latin parallelepipeds that cannot be extended to a latin cube, in preprints.

[4] Kochol, M., Latin ($n \times n \times (n-2)$)-parallelepipeds not completing to a latin cube, Math. Slovaca 39 (1989), pp. 121~125.

[5] Kochol, M., Latin parallelepipeds not completing to a cube, Math. Slovaca 41 (1991), pp. 3~9.

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拉丁長方體鑲成拉丁立方體之註記

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摘　要

本文中，我們提出一個關於拉丁長方體鑲成拉丁立方體之Kochol臆斷的反例。