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Abstract

In [1], T. Webb had shown that there exists two 1-factorizations of K_v (the complete graph of order v) which have k edges in common for each $k \in \{v-1, v, \ldots, N-7, N-6, N-4, N\}$ where v is even, $v \geqslant 8$, and $N = v(v-1)/2$. By the way, he conjectured that the theorem also holds for $k \in \{0, 1, \ldots, v-2\}$. In this paper, we first prove the conjecture by constructing two 1-factorizations of K_v with k edges in common for each $k \in \{0, 1, \ldots, v-2\}$ where v is even and $v \geqslant 8$. Secondly, we apply the above result to prove a recursive construction that if $J[v] = I_v$, then $J[2v] = I_{2v}$ [2] for each admissible order v of the Steiner quadruple systems which are larger than 10. Moreover, we can show that some of the results in [2,3] are the direct consequences of the recursive construction.

1. Introduction.

A 1-factor of K_{2m} is a collection of 2-element subsets (edges) of the vertex set $V(K_{2m})$ such that each vertex in $V(K_{2m})$ is contained in precisely one of these edges. A 1-factorization of K_{2m} is a partition of the edges of K_{2m} into 1-factors. We denote the 1-factors of a 1-factorization F of K_{2m} by $F_1, F_2, \ldots, F_{2m-1}$. Let $F = \{F_1, F_2, \ldots, F_{2m-1}\}$ and $G = \{G_1, G_2, \ldots, G_{2m-1}\}$ be two 1-factorizations of K_{2m}. (We note here that the 1-factors has no order.) We say that F and G have k edges in common provided $\sum_{t=1}^{2m-1} | F_t \cap G_t |$ is equal to k. Let $T[v] = \{k \mid$ there exists two 1-factorizations of K_v which have k edges in common$\}$.. It's clear that $v(v-1)/2 \in T[v]$.

Let S be a set of size v. A latin square of order v based on S is a $v \times v$ matrix with the property that each $i \in S$ occurs in each row and each column exactly once. A latin rectangle R based on S is an r×t matrix such that each $i \in S$ occurs in each row and each column at most once. (Note that the definition of a latin rectangle requires that both r and t be $\leqslant v$.) A latin square $L = [l_{i,j}]$ is commutative provided $l_{i,j} = l_{j,i}$ for all $i, j \in S$. If $l_{i,i} = c$ for all $i \in S$ and c is an arbitrary constant in S, then L is called a constant diagonal latin square or a unipotent latin square. For brevity, we shall refer to a commutative unipotent latin square of order v as a CULS(v). A bit of reflection, a CULS(v) exists if and only if v is even. We say that two latin squares of order v, $L = [l_{i,j}]$ and $M = [m_{i,j}]$, have k entries in common provided there are exactly k cells (i, j) for which $l_{i,j} = m_{i,j}$. We avoid redundant counting by saying that two CULS(v) L and M have k entries in common, denoted by $|L \cap M| = k$, if there are exactly k entries in common on the upper triangular parts (exclude the diagonal) of L and M. Unless we mention otherwise, the CULS(v) is based on the set $\{0, 1, \ldots, v-1\}$ and 0 in the diagonal.

2. Constructions of the commutative unipotent latin squares.

It is well known that a CULS(v) $L = [l_{i,j}]$ is equivalent to a 1-factorization F of K_v. ($F = \{F_1, F_2, \ldots, F_{v-1}\}$, $F_k = \{\{i, j\} \mid l_{i,j} = k\}$.) Hence, $k \in T[v]$ if and only if there exists two CULS(v) L and M such that $|L \cap M| = k$. In [1], T. Webb had shown $\{v-1, v, \ldots, N-7, N-6, N-4, N\} \subseteq T[v]$ where v is even, $v \geqslant 8$, and $N = v(v-1)/2$. Now we prove that $\{0, 1, 2, \ldots, v-2\} \subseteq T[v]$.

LEMMA 2.1. *If* $\{0, 1, 2, \ldots, u-2\} \subseteq T[u]$ *and* $u \geqslant 8$, *then* $\{0, 1, \ldots, 2u-2\} \subseteq T[2u]$.

PROOF. Let A be a CULS(u). A can be embedded in a CULS(2u) L, as in figure 2.1, where B is a latin square of order u based on $\{u, u+1, \ldots, 2u-1\}$ and C is a CULS(u), We will denote the latin square (or rectangle) which is obtained

$$L = \begin{array}{|c|c|} \hline A & B \\ \hline B^t & C \\ \hline \end{array}$$

Figure 2.1.

by applying α on the transversals of C (or B) as C_α (B_α). (Note that we only consider the transversals which are occupied by the same elements.) Let M be a CULS(2u) obtained in a similar way by embedding A′ in M, where A′ is a CULS(u), α is the permutation $(u, u+1, u+2, \ldots, 2u-1)$ and β is the permutation

(1 2 3 . . . u−1). (Figure 2.2) It is not difficult to see that $| L \cap M | = | A \cap A' |$.

$$M = \begin{array}{|c|c|} \hline A' & B_{\alpha} \\ \hline B^t_{\alpha} & C_{\beta} \\ \hline \end{array}$$

Figure 2.2.

Since $[0, 1, \ldots, 2u-2, \ldots, u(u-1)/2 - 6, u(u-1)/2 - 4, u(u-1)/2] \subseteq T[u]$ for each $u \geqslant 8$, we conclude the proof by using the fact $u(u-1)/2 - 6 > 2u-2$.

Before going any further we need the following definition. An idempotent commutative latin square of order v based on $\{1, 2, \ldots, v\}$ is a commutative latin square $[l_{i,j}]$ of order v based on $\{1, 2, \ldots, v\}$ such that $l_{i,i} = i$ for each $i \in \{1, 2, \ldots, v\}$.

LEMMA 2.2. *If* $\{0, 1, 2, \ldots, u-2\} \subseteq T[u]$ *and* $u \geqslant 8$, *then* $\{0, 1, \ldots, 2u\} \subseteq T[2u+2]$.

PROOF. From [1,4], an idempotent commutative latin square $P = [p_{i,j}]$ based on $\{1, 2, \ldots, u-1\}$ can be embedded in an idempotent commutative latin square $Q = [q_{i,j}]$ based on $\{1, 2, \ldots, 2u+1\}$. We can construct a CULS(2u+2) $L = [l_{i,j}]$ from Q with a CULS(u) (referred as A) in the upper left hand corner by letting:

(i) $l_{i,i} = 0,\ 1 \leqslant i \leqslant 2u+2$;

(ii) $l_{1,j} = l_{j,1} = j-1,\ 2 \leqslant j \leqslant 2u+2$;

(iii) $l_{i,j} = l_{j,i} = q_{i-1,\, j-1},\ 2 \leqslant i, j \leqslant 2u+2,\ i \neq j$.

Let M be a CULS(2u+2), as in figure 2.3, where A′ is a CULS(u), $\alpha = (1\ 2\ 3\ \ldots\ u-1)(u\ u+1\ u+2\ \ldots\ 2u+1)$. Then we have $| L \cap M | = | A \cap A' |$. The conclusion follows from the same argument as in lemma 2.1.

$$L = \begin{array}{|c|c|} \hline A & B \\ \hline B^t & C \\ \hline \end{array} \qquad M = \begin{array}{|c|c|} \hline A' & B_{\alpha} \\ \hline B^t_{\alpha} & C_{\alpha} \\ \hline \end{array}$$

Figure 2.3.

LEMMA 2.3. $\{0, 1, 2, 3, 4, 5, 6\} \subseteq T[8]$.

PROOF. A CULS(4) A can be embedded in a CULS(8) L as in figure 2.1. C is also a CULS(4). Since $[0, 2, 6] \subseteq T[4]$ and there exists two latin squares with k entry in common for each $k \in \{0, 1\}$. (Figure 2.4) $\{0, 1, 2, 3, 4, 5, 6\} \subseteq \{0, 2, 6\} + \{0, 1\} + \{0, 2, 6\}$. ($A+B = \{a+b \mid a \in A, b \in B\}$). We conclude the proof.

LEMMA 2.4. *$\{0, 1, \ldots, 8\} \subseteq T[10]$.*

PROOF. Let L and M be two CULS(10) obtained as in lemma 2.2. By unplugging B_α with B_i we have the result that B and B_i have exactly $i-1$ entries in common. (Figure 2.5.) Since $\{0, 2, 6\} \subseteq T[4]$, we are done.

$$B = \begin{array}{|cccc|} \hline 1&2&3&4\\ 2&3&4&1\\ 3&4&1&2\\ 4&1&2&3\\ \hline \end{array} \quad B' = \begin{array}{|cccc|} \hline 1&3&4&2\\ 3&2&1&4\\ 4&1&2&3\\ 2&4&3&1\\ \hline \end{array} \quad A = \begin{array}{|cccc|} \hline 0&1&2&3\\ 1&0&3&2\\ 2&3&0&1\\ 3&2&1&0\\ \hline \end{array}$$

$$|B \cap B_{(1234)}| = 0 \qquad |A \cap A_{(23)}| = 2$$

$$|B \cap B'| = 1 \qquad |A \cap A_{(123)}| = 0$$

Figure 2.4.

$$L = \begin{array}{|c|cccccc|l} \hline & 4&5&6&7&8&9 & \\ A & 8&7&5&4&9&6 & \leftarrow B\\ & 7&9&8&5&6&4 & \\ & 6&8&7&9&4&5 & \\ \hline & 0&1&9&3&5&2 & \\ & 1&0&4&6&2&3 & \\ B^t & 9&4&0&2&3&1 & \leftarrow C\\ & 3&6&2&0&1&8 & \\ & 5&2&3&1&0&7 & \\ & 2&3&1&8&7&0 & \\ \hline \end{array} \qquad M = \begin{array}{|c|cccccc|l} \hline & 5&6&7&8&9&4 & \\ A' & 9&8&6&5&4&7 & \leftarrow B_\alpha\\ & 8&4&9&6&7&5 & \\ & 7&9&8&4&5&6 & \\ \hline & 0&2&4&1&6&3 & \\ & 2&0&5&7&3&1 & \\ B^t_\alpha & 4&5&0&3&1&2 & \leftarrow C_\alpha\\ & 1&7&3&0&2&9 & \\ & 6&3&1&2&0&8 & \\ & 3&1&2&9&8&0 & \\ \hline \end{array}$$

$$B_1 = B_\alpha \qquad B_2 = \begin{array}{|cccccc|} \hline 5&6&8&4&9&7\\ 9&8&7&5&4&6\\ 8&4&9&6&7&5\\ 7&9&6&8&5&4\\ \hline \end{array}$$

B_3 =

5	8	7	4	9	6
8	6	9	5	4	7
9	4	8	6	7	5
7	9	6	8	5	4

B_4 =

9	8	6	5	4	7
8	4	9	6	7	5
5	6	7	8	9	4
7	9	8	4	5	6

Figure 2.5.

LEMMA 2.5. *{0, 1, . . . , 10}* $\subseteq$ *T[12].*

PROOF. A CULS(6) A can be embedded in a CULS(12) L as in lemma 2.1. We have seen that C is also a CULS(6). Hence A and C can be unplugged with any other CULS(6). Since {0, 1, 2, 3, 5, 6} $\subseteq$ T[6], we prove the lemma. (Figure 2.6.)

A =

0	1	2	3	4	5
1	0	3	4	5	2
2	3	0	5	1	4
3	4	5	0	2	1
4	5	1	2	0	3
5	2	4	1	3	0

B =

0	1	3	4	5	2
1	0	4	5	2	3
3	4	0	2	1	5
4	5	2	0	3	1
5	2	1	3	0	4
2	3	5	1	4	0

C =

0	1	3	4	5	2
1	0	5	3	2	4
3	5	0	2	4	1
4	3	2	0	1	5
5	2	4	1	0	3
2	4	1	5	3	0

D =

0	3	4	5	1	2
3	0	2	4	5	1
4	2	0	1	3	5
5	4	1	0	2	3
1	5	3	2	0	4
2	1	5	3	4	0

E =

0	3	4	5	1	2
3	0	5	1	2	4
4	5	0	2	3	1
5	1	2	0	4	3
1	2	3	4	0	5
2	4	1	3	5	0

F =

0	3	1	4	5	2
3	0	2	5	1	4
1	2	0	3	4	5
4	5	3	0	2	1
5	1	4	2	0	3
2	4	5	1	3	0

$|A \cap B| = 3$, $|A \cap C| = 2$, $|C \cap D| = 1$, $|A \cap E| = 0$, $|C \cap E| = 6$, $|D \cap F| = 5$.

Figure 2.6.

LEMMA 2.6. *{0, 1, 2, . . . , 12}* $\subseteq$ *T[14].*

PROOF. A CULS(6) A can be embedded in a CULS(14) L as in lemma 2.2. Let L and M be the two CULS(14) such that $|L \cap M| = |A \cap A'|$. (Figure 2.7.) With the constructions of B_1, B_2, B_3 such that $|B_i \cap B| = 3{+}i$ (B_α may be unplugged with B_i) and $\{0, 1, 2, 3, 5, 6\} \subseteq T[6]$ we conclude the proof. (Figure 2.8.)

L =

	6	7	8	9	10	11	12	13	
	10	13	6	12	11	8	7	9	
A	11	12	13	10	9	7	8	6	← B
	13	10	11	8	12	9	6	7	
	12	8	9	11	7	6	13	10	
	7	9	12	6	8	13	10	11	
	0	1	2	3	4	5	9	8	
	1	0	5	2	6	3	11	4	
	2	5	0	7	3	10	4	1	
B^t	3	2	7	0	13	4	1	5	← C
	4	6	3	13	0	1	5	2	
	5	3	10	4	1	0	2	12	
	9	11	4	1	5	2	0	3	
	8	4	1	5	2	12	3	0	

M =

A'	B_α
B^t_α	C_α

$\alpha = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10\ 11\ 12\ 13)$

Figure 2.7.

B_α =

7	8	9	10	11	12	13	6
11	6	7	13	12	9	8	10
12	13	6	11	10	8	9	7
6	11	12	9	13	10	7	8
13	9	10	12	8	7	6	11
8	10	13	7	9	6	11	12

B_2 =

7	8	9	10	11	12	13	6
11	6	7	13	12	9	8	10
6	11	12	9	13	10	7	8
12	13	6	11	10	8	9	7
8	10	13	7	9	6	11	12
13	9	10	12	8	7	6	11

B_3 =

11	6	7	13	12	9	8	10
7	8	9	10	11	12	13	6
6	11	12	9	13	10	7	8
12	13	6	11	10	8	9	7
8	10	13	7	9	6	11	12
13	9	10	12	8	7	6	11

B_1 =

7	8	9	10	11	12	13	6
11	6	7	13	12	9	8	10
12	13	6	11	10	8	9	7
6	11	12	9	13	10	7	8
8	10	13	7	9	6	11	12
13	9	10	12	8	7	6	11

$|B \cap B_i| = 3 + i, i = 1, 2, 3.$

Figure 2.8.

THEOREM 2.7. *$\{0, 1, 2, \ldots, \nu-2\} \subseteq T[\nu]$, for every even $\nu \geqslant 8$.*

PROOF. The theorem follows directly from lemma 2.1 to lemma 2.6.

3. Applications.

From theorem 2.7 we have $\{0, 1, 2, \ldots, v-2\} \subseteq T[v]$. In [1], T. Webb had shown that $\{v-1, v, \ldots, N-6, N-4, N\} \subseteq T[4]$ where $N = v(v-1)/2$. Hence we can construct a pair of 1-factorizations of K_v with k edges in common for every even $v \geqslant 8$, and $k \in \{0, 1, 2, \ldots, N-6, N-4, N\}$.

A Steiner quadruple system (SQS) is a pair (Q, q) where Q is a finite set and q is a collection of 4-subsets of Q (called blocks) such that every 3-subset of Q is contained in exactly one block of q. By counting, $|q| = q_v = v(v-1)(v-2)/24$. In [2,3] we already have some results in the constructions of Steiner quadruple systems having a prescribed number of blocks in common. Let $J[v] = \{k \mid$ there exists two SQS of order v which have k quadruples in common$\}$, and $I_v = \{0, 1, \ldots, q_v-14\} \cup \{q_v-12, q_v-8, q_v\}$, for $v \geqslant 8$. Here we can use the well-known doubling construction for quadruple systems and theorem 2.7 to prove a more general recursive construction: If $J[v] = I_v$, then $J[2v] = I_{2v}$ for every admissible order $v > 10$.

LEMMA 3.1. *Let (X, A) and (Y, B) be any two SQS(ν) with $X \cap Y = \phi$. Let $F = \{F_1, F_2, \ldots, F_{\nu-1}\}$ and $G = \{G_1, G_2, \ldots, G_{\nu-1}\}$ be any two 1-factorizations of K_ν on X and Y, respectively, and let α be any permutation on the set $\{1, 2, \ldots, \nu-1\}$. Then (Q, q) is an SQS(2ν) where $Q = X \cup Y$ and the blocks of q are as follows:*

(1) Any block belonging to A or B belongs to q, and

(2) If $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $\{x_1, x_2, y_1, y_2\} \in q$ if and only if $\{x_1, x_2\} \in F_i$, $\{y_1, y_2\} \in G_j$, and $i\alpha = j$. [5]

THEOREM 3.2. *If $J[\nu] = I_\nu$, then $J[2\nu] = I_{2\nu}$ for every admissible order of the Steiner quadruple systems which is greater than 10.*

PROOF. Let (Q, q) and (Q′, q′) be two SQS(2v) obtained by the doubling construction of (X, A), (Y, B) and (X′, A′), (Y′, B′) respectively. Let X = X′ and Y = Y′, then Q = Q′. Moreover, we let G = G′ and α is the identity permutation. A bit of reflection, we can see if F and F′ have k edges in common then q and q′ have kv/2 Type (2) quadruples in common. Hence, if A and A′ have exactly d quadruples in common, B and B′ have exactly d′ quadruples in common and F and F′ have exactly k edges in common then q and q′ shall have d + d′ +

kv/2 quadruples in common exactly. A direct computation shows that $I_{2v} \subseteq J[2v]$. In [2], it was shown that $J[2v] \subseteq I_{2v}$. We conclude the proof.

THEOREM 3.3. ([2]). *$I_v \setminus \{q_v - h \mid h = 17, 18, 19, 21, 25\} \subseteq J[v]$ for all $v = 2^n$, $n \geqslant 5$.*

PROOF. The theorem follows directly from $I_{16} \setminus \{103, 111, 115, 119, 121, 122, 123\} \subseteq J[16]$ and the proof of the theorem 3.2.

THEOREM 3.4. ([3]). *$I_v \setminus \{q_v - h \mid h = 17, 19, 21, 25\} \subseteq J[v]$ for all $v = 5 \cdot 2^n$, $n \geqslant 4$.*

PROOF. The theorem follows from $I_{40} \setminus \{q_v - h \mid h = 17, 19, 21, 25\} \subseteq J[40]$ and the proof of the theorem 3.2.

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