

On the Constructions and Applications of Two 1-Factorizations with Prescribed Intersections

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Abstract

In [1], T. Webb had shown that there exists two 1-factorizations of K_v (the complete graph of order v) which have k edges in common for each $k \in \{v-1, v, \dots, N-7, N-6, N-4, N\}$ where v is even, $v \geq 8$, and $N = v(v-1)/2$. By the way, he conjectured that the theorem also holds for $k \in \{0, 1, \dots, v-2\}$. In this paper, we first prove the conjecture by constructing two 1-factorizations of K_v with k edges in common for each $k \in \{0, 1, \dots, v-2\}$ where v is even and $v \geq 8$. Secondly, we apply the above result to prove a recursive construction that if $J[v] = I_v$, then $J[2v] = I_{2v}$ [2] for each admissible order v of the Steiner quadruple systems which are larger than 10. Moreover, we can show that some of the results in [2,3] are the direct consequences of the recursive construction.

1. Introduction.

A 1-factor of K_{2m} is a collection of 2-element subsets (edges) of the vertex set $V(K_{2m})$ such that each vertex in $V(K_{2m})$ is contained in precisely one of these edges. A 1-factorization of K_{2m} is a partition of the edges of K_{2m} into 1-factors. We denote the 1-factors of a 1-factorization F of K_{2m} by $F_1, F_2, \dots, F_{2m-1}$. Let $F = \{F_1, F_2, \dots, F_{2m-1}\}$ and $G = \{G_1, G_2, \dots, G_{2m-1}\}$ be two 1-factorizations of K_{2m} . (We note here that the 1-factors has no order.) We say that

F and G have k edges in common provided $\sum_{t=1}^{2m-1} |F_t \cap G_t|$ is equal to k . Let

$T[v] = \{k \mid \text{there exists two 1-factorizations of } K_v \text{ which have } k \text{ edges in common}\}$. It's clear that $v(v-1)/2 \in T[v]$.

Let S be a set of size v . A latin square of order v based on S is a $v \times v$ matrix with the property that each $i \in S$ occurs in each row and each column exactly

once. A latin rectangle R based on S is an $r \times t$ matrix such that each $i \in S$ occurs in each row and each column at most once. (Note that the definition of a latin rectangle requires that both r and t be $\leq v$.) A latin square $L = [l_{i,j}]$ is commutative provided $l_{i,j} = l_{j,i}$ for all $i, j \in S$. If $l_{i,i} = c$ for all $i \in S$ and c is an arbitrary constant in S , then L is called a constant diagonal latin square or a unipotent latin square. For brevity, we shall refer to a commutative unipotent latin square of order v as a CULS(v). A bit of reflection, a CULS(v) exists if and only if v is even. We say that two latin squares of order v , $L = [l_{i,j}]$ and $M = [m_{i,j}]$, have k entries in common provided there are exactly k cells (i, j) for which $l_{i,j} = m_{i,j}$. We avoid redundant counting by saying that two CULS(v) L and M have k entries in common, denoted by $|L \cap M| = k$, if there are exactly k entries in common on the upper triangular parts (exclude the diagonal) of L and M . Unless we mention otherwise, the CULS(v) is based on the set $\{0, 1, \dots, v-1\}$ and 0 in the diagonal.

2. Constructions of the commutative unipotent latin squares.

It is well known that a CULS(v) $L = [l_{i,j}]$ is equivalent to a 1-factorization F of K_v . ($F = \{F_1, F_2, \dots, F_{v-1}\}$, $F_k = \{(i, j) \mid l_{i,j} = k\}$.) Hence, $k \in T[v]$ if and only if there exists two CULS(v) L and M such that $|L \cap M| = k$. In [1], T. Webb had shown $\{v-1, v, \dots, N-7, N-6, N-4, N\} \subseteq T[v]$ where v is even, $v \geq 8$, and $N = v(v-1)/2$. Now we prove that $\{0, 1, 2, \dots, v-2\} \subseteq T[v]$.

LEMMA 2.1. *If $\{0, 1, 2, \dots, u-2\} \subseteq T[u]$ and $u \geq 8$, then $\{0, 1, \dots, 2u-2\} \subseteq T[2u]$.*

PROOF. Let A be a CULS(u). A can be embedded in a CULS($2u$) L , as in figure 2.1, where B is a latin square of order u based on $\{u, u+1, \dots, 2u-1\}$ and C is a CULS(u). We will denote the latin square (or rectangle) which is obtained

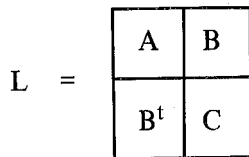


Figure 2.1.

by applying α on the transversals of C (or B) as C_α (B_α). (Note that we only consider the transversals which are occupied by the same elements.) Let M be a CULS($2u$) obtained in a similar way by embedding A' in M , where A' is a CULS(u), α is the permutation $(u, u+1, u+2, \dots, 2u-1)$ and β is the permutation

(1 2 3 ... u-1). (Figure 2.2) It is not difficult to see that $|L \cap M| = |A \cap A'|$.

$$M = \begin{array}{|c|c|} \hline A' & B_\alpha \\ \hline B_\alpha^t & C_\beta \\ \hline \end{array}$$

Figure 2.2.

Since $[0, 1, \dots, 2u-2, \dots, u(u-1)/2 - 6, u(u-1)/2 - 4, u(u-1)/2] \subseteq T[u]$ for each $u \geq 8$, we conclude the proof by using the fact $u(u-1)/2 - 6 > 2u-2$.

Before going any further we need the following definition. An idempotent commutative latin square of order v based on $\{1, 2, \dots, v\}$ is a commutative latin square $[l_{i,j}]$ of order v based on $\{1, 2, \dots, v\}$ such that $l_{i,i} = i$ for each $i \in \{1, 2, \dots, v\}$.

LEMMA 2.2. *If $\{0, 1, 2, \dots, u-2\} \subseteq T[u]$ and $u \geq 8$, then $\{0, 1, \dots, 2u\} \subseteq T[2u+2]$.*

PROOF. From [1,4], an idempotent commutative latin square $P = [p_{i,j}]$ based on $\{1, 2, \dots, u-1\}$ can be embedded in an idempotent commutative latin square $Q = [q_{i,j}]$ based on $\{1, 2, \dots, 2u+1\}$. We can construct a CULS(2u+2) $L = [l_{i,j}]$ from Q with a CULS(u) (referred as A) in the upper left hand corner by letting:

- (i) $l_{i,i} = 0, 1 \leq i \leq 2u+2$;
- (ii) $l_{1,j} = l_{j,1} = j-1, 2 \leq j \leq 2u+2$;
- (iii) $l_{i,j} = l_{j,i} = q_{i-1, j-1}, 2 \leq i, j \leq 2u+2, i \neq j$.

Let M be a CULS(2u+2), as in figure 2.3, where A' is a CULS(u), $\alpha = (1\ 2\ 3\ \dots\ u-1)$ ($u\ u+1\ u+2\ \dots\ 2u+1$). Then we have $|L \cap M| = |A \cap A'|$. The conclusion follows from the same argument as in lemma 2.1.

$$L = \begin{array}{|c|c|} \hline A & B \\ \hline B^t & C \\ \hline \end{array} \qquad M = \begin{array}{|c|c|} \hline A' & B_\alpha \\ \hline B_\alpha^t & C_\alpha \\ \hline \end{array}$$

Figure 2.3.

LEMMA 2.3. $\{0, 1, 2, 3, 4, 5, 6\} \subseteq T[8]$.

PROOF. A CULS(4) A can be embedded in a CULS(8) L as in figure 2.1. C is also a CULS(4). Since $[0, 2, 6] \subseteq T[4]$ and there exists two latin squares with k entry in common for each $k \in \{0, 1\}$. (Figure 2.4) $\{0, 1, 2, 3, 4, 5, 6\} \subseteq \{0, 2, 6\} + \{0, 1\} + \{0, 2, 6\}$. ($A+B = \{a+b \mid a \in A, b \in B\}$). We conclude the proof.

LEMMA 2.4. $\{0, 1, \dots, 8\} \subseteq T[10]$.

PROOF. Let L and M be two CULS(10) obtained as in lemma 2.2. By unplugging B_α with B_i we have the result that B and B_i have exactly $i-1$ entries in common. (Figure 2.5.) Since $\{0, 2, 6\} \subseteq T[4]$, we are done.

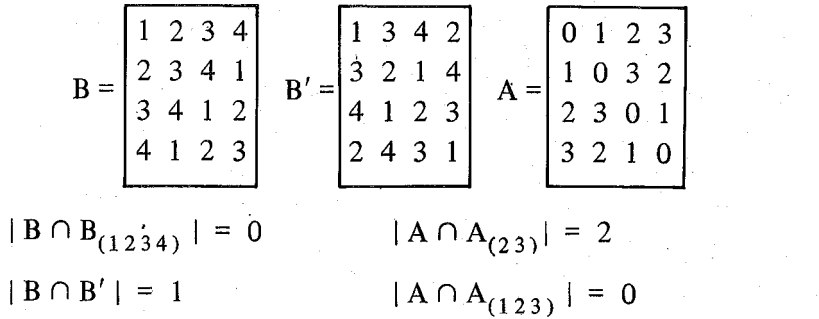
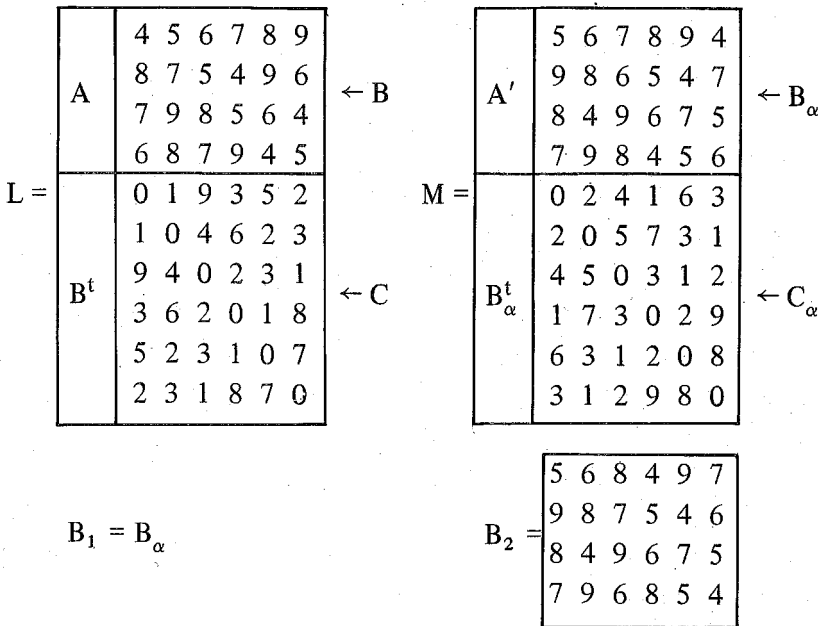


Figure 2.4.



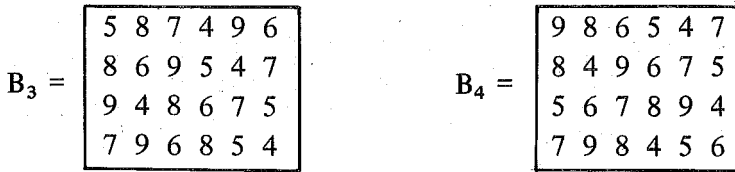
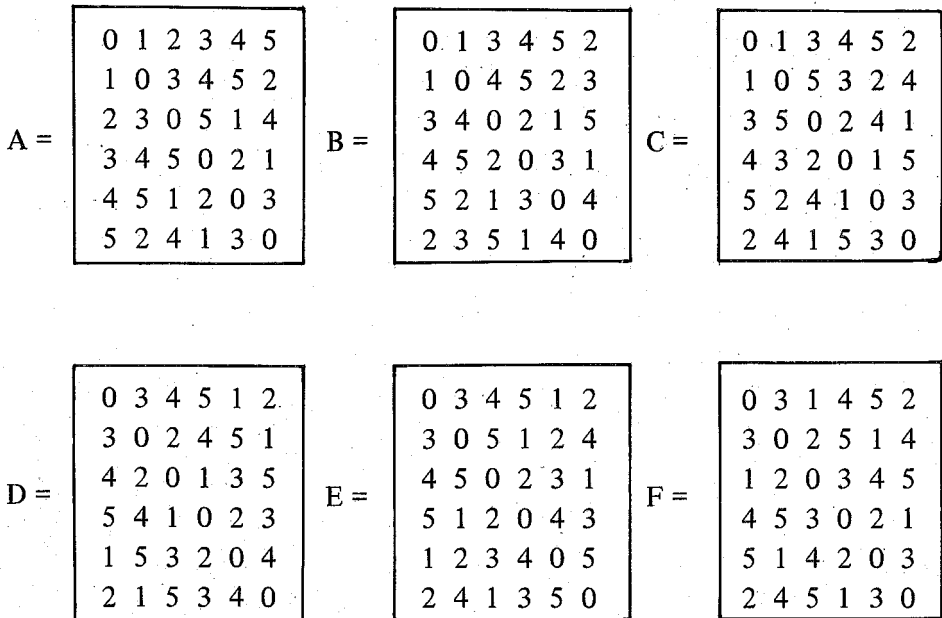


Figure 2.5.

LEMMA 2.5. $\{0, 1, \dots, 10\} \subseteq T[12]$.

PROOF. A CULS(6) A can be embedded in a CULS(12) L as in lemma 2.1. We have seen that C is also a CULS(6). Hence A and C can be unplugged with any other CULS(6). Since $\{0, 1, 2, 3, 5, 6\} \subseteq T[6]$, we prove the lemma. (Figure 2.6.)



$$|A \cap B| = 3, |A \cap C| = 2, |C \cap D| = 1, |A \cap E| = 0, |C \cap E| = 6, |D \cap F| = 5.$$

Figure 2.6.

LEMMA 2.6. $\{0, 1, 2, \dots, 12\} \subseteq T[14]$.

PROOF. A CULS(6) A can be embedded in a CULS(14) L as in lemma 2.2. Let L and M be the two CULS(14) such that $|L \cap M| = |A \cap A'|$. (Figure 2.7.) With the constructions of B_1, B_2, B_3 such that $|B_i \cap B| = 3+i$ (B_α may be

unplugged with B_i) and $\{0, 1, 2, 3, 5, 6\} \subseteq T[6]$ we conclude the proof. (Figure 2.8.)

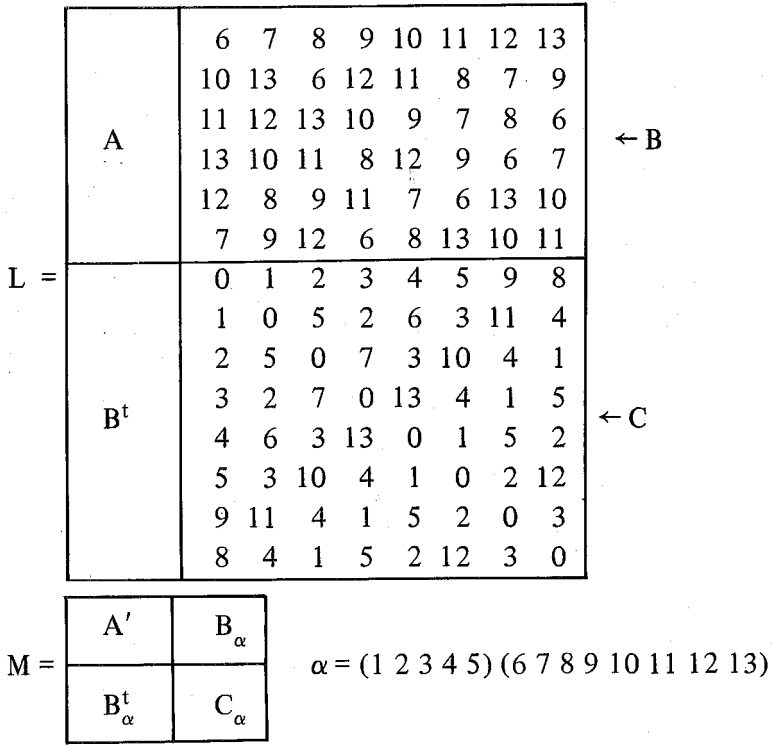
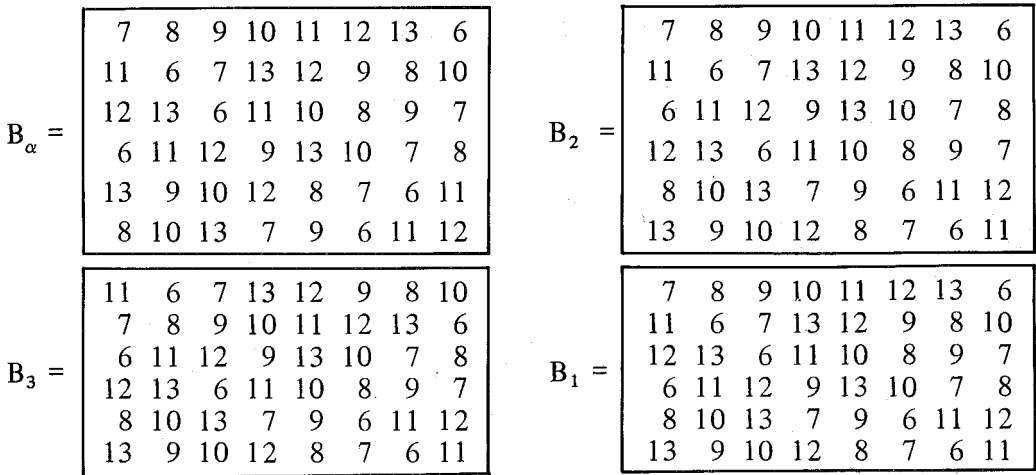


Figure 2.7.



$$|B \cap B_i| = 3 + i, i = 1, 2, 3.$$

Figure 2.8.

THEOREM 2.7. $\{0, 1, 2, \dots, v-2\} \subseteq T[v]$, for every even $v \geq 8$.

PROOF. The theorem follows directly from lemma 2.1 to lemma 2.6.

3. Applications.

From theorem 2.7 we have $\{0, 1, 2, \dots, v-2\} \subseteq T[v]$. In [1], T. Webb had shown that $\{v-1, v, \dots, N-6, N-4, N\} \subseteq T[4]$ where $N = v(v-1)/2$. Hence we can construct a pair of 1-factorizations of K_v with k edges in common for every even $v \geq 8$, and $k \in \{0, 1, 2, \dots, N-6, N-4, N\}$.

A Steiner quadruple system (SQS) is a pair (Q, q) where Q is a finite set and q is a collection of 4-subsets of Q (called blocks) such that every 3-subset of Q is contained in exactly one block of q . By counting, $|q| = q_v = v(v-1)(v-2)/24$. In [2,3] we already have some results in the constructions of Steiner quadruple systems having a prescribed number of blocks in common. Let $J[v] = \{k \mid \text{there exists two SQS of order } v \text{ which have } k \text{ quadruples in common}\}$, and $I_v = \{0, 1, \dots, q_v-14\} \cup \{q_v-12, q_v-8, q_v\}$, for $v \geq 8$. Here we can use the well-known doubling construction for quadruple systems and theorem 2.7 to prove a more general recursive construction: If $J[v] = I_v$, then $J[2v] = I_{2v}$ for every admissible order $v > 10$.

LEMMA 3.1. *Let (X, A) and (Y, B) be any two SQS(v) with $X \cap Y = \phi$. Let $F = \{F_1, F_2, \dots, F_{v-1}\}$ and $G = \{G_1, G_2, \dots, G_{v-1}\}$ be any two 1-factorizations of K_v on X and Y , respectively, and let α be any permutation on the set $\{1, 2, \dots, v-1\}$. Then (Q, q) is an SQS($2v$) where $Q = X \cup Y$ and the blocks of q are as follows:*

- (1) Any block belonging to A or B belongs to q , and
- (2) If $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $\{x_1, x_2, y_1, y_2\} \in q$ if and only if $\{x_1, x_2\} \in F_j, \{y_1, y_2\} \in G_j$, and $i\alpha = j$. [5]

THEOREM 3.2. *If $J[v] = I_v$, then $J[2v] = I_{2v}$ for every admissible order of the Steiner quadruple systems which is greater than 10.*

PROOF. Let (Q, q) and (Q', q') be two SQS($2v$) obtained by the doubling construction of (X, A) , (Y, B) and (X', A') , (Y', B') respectively. Let $X = X'$ and $Y = Y'$, then $Q = Q'$. Moreover, we let $G = G'$ and α is the identity permutation. A bit of reflection, we can see if F and F' have k edges in common then q and q' have $kv/2$ Type (2) quadruples in common. Hence, if A and A' have exactly d quadruples in common, B and B' have exactly d' quadruples in common and F and F' have exactly k edges in common then q and q' shall have $d + d' +$

$kv/2$ quadruples in common exactly. A direct computation shows that $I_{2v} \subseteq J[2v]$. In [2], it was shown that $J[2v] \subseteq I_{2v}$. We conclude the proof.

THEOREM 3.3. ([2]). $I_v \setminus \{q_v - h \mid h = 17, 18, 19, 21, 25\} \subseteq J[v]$ for all $v = 2^n$, $n \geq 5$.

PROOF. The theorem follows directly from $I_{16} \setminus \{103, 111, 115, 119, 121, 122, 123\} \subseteq J[16]$ and the proof of the theorem 3.2.

THEOREM 3.4. ([3]). $I_v \setminus \{q_v - h \mid h = 17, 19, 21, 25\} \subseteq J[v]$ for all $v = 5 \cdot 2^n$, $n \geq 4$.

PROOF. The theorem follows from $I_{40} \setminus \{q_v - h \mid h = 17, 19, 21, 25\} \subseteq J[40]$ and the proof of the theorem 3.2.

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