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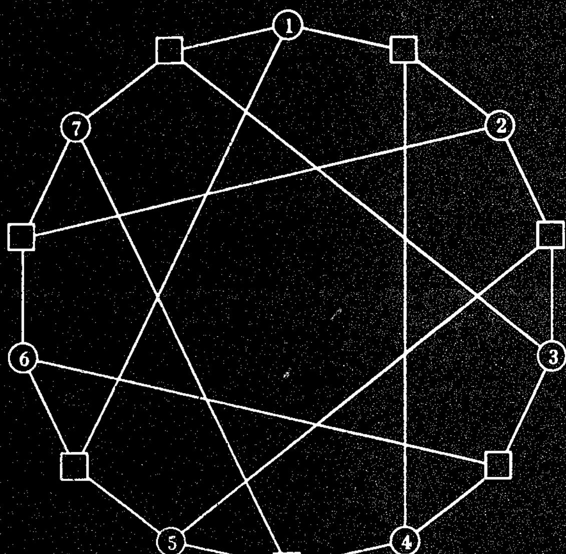

# A Special Partition of the Set $I_n$

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**Abstract.** Let $I_n$ be the set $\{1,2,\ldots,n\}$, and $(s_1,s_2,\ldots,s_k)$ be a $k$-tuple of nonnegative integers such that $\Sigma_{i=1}^{k} s_i = \binom{n+1}{2}$. We will say that $I_n$ can be partitioned into subsets of type $\langle s_1,s_2,\ldots,s_k\rangle$ provided that there exists a collection of $k$ mutually disjoint subsets of $I_n$, $A_1,A_2,\ldots,A_k$ such that $\Sigma_{i=1}^{k} A_i = I_n$ and $\Sigma_{x\in A_i} x = s_i$ for each $i \in \{1,2,\ldots,k\}$. In this paper, we study for which $k$-tuple $(t_1,t_2,\ldots,t_k)$ we can partition $I_n$ into subsets of type $\langle t_1,t_2,\ldots,t_k\rangle$, and we are able to show that $I_n$ can be partitioned into subsets of type $\langle m,m+1,\ldots,m+k-2,\ell\rangle$ where $m \geq 0$, $0 \leq \ell \leq \binom{n+1}{2}$, and $m(k-1)+\binom{k-1}{2}+\ell = \binom{n+1}{2}$. Furthermore, we conjecture that $I_n$ can be partitioned into subsets of type $\langle m_1,m_2,\ldots,m_k\rangle$ provided that $\Sigma_{i=1}^{k} m_i = \binom{n+1}{2}$ and $m_i \geq n$ for each $i \in \{1,2,\ldots,k-1\}$.

## 1. Introduction

Let $I_n$ be the set $\{1,2,\ldots,n\}$, and $(s_1,s_2,\ldots,s_k)$ be a $k$-tuple of nonnegative integers such that $\Sigma_{i=1}^{k} s_i = \binom{n+1}{2}$. We will say that $I_n$ can be partitioned into subsets of type $\langle s_1,s_2,\ldots,s_k\rangle$ provided that there exists a collection of $k$ mutually disjoint subsets of $I_n$, $A_1,A_2,\ldots A_k$, such that $\cup_{i=1}^{k} A_i = I_n$ and $\Sigma_{x\in A_i} x = s_i$ for each $i \in \{1,2,\ldots,k\}$. Some results have been obtained in the study of this problem. We list them as propositions.

**Proposition 1.1.** [4] *$I_n$ can be partitioned into subsets of type $\langle m,m,\ldots,m\rangle$ ($k$-tuple) provided that $km = \binom{n+1}{2}$ and $m \geq n$.*

**Proposition 1.2.** [4,5] *$I_n$ can be partitioned into subsets of type $\langle \underbrace{m+1,\ldots,m+1}_{k_1 \text{ terms}}, \underbrace{m,\ldots,m}_{k_2 \text{ terms}}\rangle$ provided that $k_1(m+1)+k_2 m = \binom{n+1}{2}$, $k_1 > 0$ and $m+1 \geq n$.*

**Proposition 1.3.** [2] *$I_n$ can be partitioned into subsets of type $\langle \underbrace{m,m,\ldots,m}_{k-1 \text{ terms}},\ell\rangle$ provided that $(k-1)m+\ell = \binom{n+1}{2}$ and $m \geq n$.*

We note here that the result in Proposition 1.3 was stated differently. The authors gave a sufficient and necessary condition on $m$ and $k$ such that we can always obtain $k$ mutually disjoint subsets of $I_n$ with constant sum $m$.

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In this paper, we show that $I_n$ can be partitioned into subsets of type $\langle m, m+1, \ldots, m+k-2, \ell\rangle$. With this partition we can easily obtain all the three results mentioned above.

In [3], it was conjectured by Fu that for each $k$-tuple $(m_1, m_2, \ldots, m_k)$, $m_i \geq n$, $i = 1, 2, \ldots, k$, and $\Sigma_{i=1}^k m_i = \binom{n+1}{2}$, $I_n$ can be partitioned into subsets of type $\langle m_1, m_2, \ldots, m_k\rangle$. The conjecture can be reduced to $2n-2 \geq m_i \geq n$, $i = 1, 2, \ldots, k$, and $\Sigma_{i=1}^k m_i = \binom{n+1}{2}$, which was conjectured earlier by Y. Alavi et. al. in [1]. So far, this conjecture is still unsolved. With the results we have obtained, it seems that we can make the following stronger conjecture.

**Conjecture.** *$I_n$ can be partitioned into subsets of type $\langle m_1, m_2, \ldots, m_k\rangle$ provided that $\Sigma_{i=1}^k m_i = \binom{n+1}{2}$ and $m_i \geq n$ for each $i = 1, 2, \ldots, k-1$.*

## 2. The main result

For convenience, we will use an array $A = [a_{ij}]$ with $k$ columns to represent a partition of type $\langle s_1, s_2, \ldots, s_k\rangle$ where the $a_{ij}$ are distinct elements of $I_n$ and $\Sigma_i a_{ij} = s_j$. As an example,

$$\begin{bmatrix} 3 & 1 & 6 & 2 \\ * & 4 & * & 5 \end{bmatrix}$$

is an array which represents a partition of $I_6$ into subsets of type $\langle 3, 5, 6, 7\rangle$.

Now we are ready to prove

**Proposition 2.1.** *Let $m, \ell, k$ be integers such that $k > 0, m > 0, 0 < \ell \leq \binom{n+1}{2}$ and $(k-1)m + \ell + \binom{k-1}{2} = \binom{n+1}{2}$. Then $I_n$ can be partitioned into subsets of type $\langle m, m+1, \ldots, m+k-2, \ell\rangle$.*

Proof: The proof will be by induction. If $m+k-2 \leq n$, then let $A_i = m+i-1$ for $i = 1, 2, \ldots, k-1$ and $L = I_n \setminus \cup_{i=1}^{k-1} A_i$, then we have the partition. If $m+k-2 > n$ but $m \leq n$, then there exists a $j$, $1 \leq j < k-1$, such that $m+j-1 = n$; then let $A_i = m+i-1$ for $i = 1, 2, \ldots, j$. Let $n' = m-1$; then $\Sigma_{i=j+1}^{k-1}(m+i-1) + \ell = \binom{n'+1}{2}$. By the induction hypothesis $I_{n'}$ can be partitioned into subsets of type $\langle m+j, m+j+1, \ldots, m+k-2, \ell\rangle$. So without loss of generality, we can assume that $m > n$ and consider three cases.

**Case 1.** $m \geq 2n - 2k + 3$

Let $B_i = \{n-i+1, n-2k+i+2\}$ for $i = 1, 2, \ldots, k-1$; then by the induction hypotheis, $I_{n-2k+2}$ can be partitioned into subsets of type $\langle m-2n+2k-3, m-2n+2k-2, \ldots, m-2n+3k-5, \ell\rangle$, say $A'_1, A'_2, \ldots, A'_{k-1}, L$. Then $A'_1 \cup B'_1$, $A'_2 \cup B_2, \ldots, A'_{k-1} \cup B_{k-1}, L$ are the subsets of type $\langle m, m+1, \ldots, m+k-2, \ell\rangle$.

**Case 2.** $m < 2n - 2k + 3$ and $m + \lfloor \frac{k-1}{2} \rfloor - 1 \geq 2n - 2k + 3$.

$$
\left[\begin{array}{ll}
n-k+3, n-k+5, \ldots, n-k+2j-1, n-k+2, n-k+4, \ldots, n-k+2j, & n-k+2j+1, \ldots, n, \; * \\
n-k-j+1, n-k-j, \ldots, n-k-2j+3, n-k+1, n-k, \ldots, n-k-j+2, & n-k-2j+2, \ldots, n-2k+3, \; * \\ \cline{2-2}
*, *, \ldots, *, *, *, \ldots, * & \multicolumn{1}{|c}{C_1'} \\
\quad \cdots & \multicolumn{1}{|c}{}
\end{array}\right]
$$

Figure 2.1

In this case we can find $j$ such that $m + j - 1 = 2n - 2k + 3$ and $j \leq \lfloor \frac{k-1}{2} \rfloor$. Consider the array $C_1$ in Figure 2.1.

As shown in $C_1$, if we can partition the set $I_{n-2k+2}$ into subsets of type $\langle m - 2n + 2k + 2j - 4, m - 2n + 2k + 2j - 3, \ldots, m - 2n + 3k - 5, \ell \rangle$ which is represented by $C_1'$, then we are done. Since this can be obtained by the induction hypothesis, we have the proof of this case.

**Case 3.** $m + \lfloor \frac{k-1}{2} \rfloor - 1 < 2n - 2k + 3$.

In this case we consider two situations. If $k - 1$ is odd, then consider the following array. Let $C_0$ be the array in Figure 2.2.

$$
\begin{bmatrix}
n-k+3, & n-k+5, \ldots, & n-1, & n-k+2, & n-k+4, \ldots, & n \\
m-n+k-3, & m-n+k-4, \ldots, & m-n+\frac{1}{2}k-1, & m-n+\frac{3}{2}k-3, & m-n+\frac{3}{2}k-4, \ldots, & m-n+k-2
\end{bmatrix}
$$

Figure 2.2

Let $A_i$ = {the entries in the $i$-th column of $C_0$} and $L = I_n \setminus \cup_{i=1}^{k-1} A_i$; then $A_1, A_2, \ldots, A_{k-1}, L$ are the subsets of type $\langle m, m + 1, \ldots, m + k - 2, \ell \rangle$.

If $k - 1$ is even, then consider the array $C_e$ below.

$$
\begin{bmatrix}
n-k+3, & n-k+5, \ldots, & n, & n-k+2, & n-k+4, \ldots, & n-1 \\
m-n+k-3, & m-n+k-4, \ldots, & m-n+\frac{1}{2}(k-1)-1, & m-n+\frac{3}{2}(k-1)-1, & m-n+\frac{3}{2}(k-1)-2, \ldots, & m-n+k-1
\end{bmatrix}
$$

Figure 2.3

As for the case when $k - 1$ is odd, we have the desired partition.

**Corollary 2.2.** [2,5] *Let $k, m, \ell$ be positive integers such that $m \geq n$, $0 < \ell \leq \binom{n+1}{2}$ and $(k-1)m + \ell = \binom{n+1}{2}$. Then $I_n$ can be partitioned into subsets of type $\langle \underbrace{m, m, \ldots, m}_{k-1 \text{ terms}}, \ell \rangle$.*

Proof: Since $\Sigma_{i=1}^{k-1} [m - (n - i + 1)] + \ell = \binom{n-k+2}{2}$, by Proposition 2.1 $I_{n-k+1}$ can be partitioned into subsets of type $\langle m - n, m - n + 1, \ldots, m - n + k - 2, \ell \rangle$, say, $A_1', A_2', \ldots, A_{k-1}', L$. Let $A_i = A_i' \cup \{n - i + 1\}$ for $i = 1, 2, \ldots, k - 1$; then $A_1, A_2, \ldots, A_{k-1}, L$ are the subsets of type $\langle m, m, \ldots, m, \ell \rangle$. ∎

**Corollary 2.3.** [4,5] *Let $k_1, k_2, m$ be integers such that $m \geq n$ and $k_1(m+1) + k_2 m = \binom{n+1}{2}$, then $I_n$ can be partitioned into subsets of type* $\langle \underbrace{m+1, m+1, \ldots, m+1}_{k_1 \text{ terms}}, \underbrace{m, m, \ldots, m}_{k_2 \text{ terms}} \rangle$.

Proof: Since $\Sigma_{i=1}^{k}[(m+1)-(n-i+1)] + \Sigma_{i=1}^{k_2}[m-(n-k_1-i+1)] = \binom{n-k_1-k_2+1}{2}$. then by Proposition 2.1 $I_{n-k_1-k_2}$ can be partitioned into subsets $A'_1, A'_2, \ldots, A'_{k_1+k_2-1}, L$ of type $\langle m-n+1, m-n+2, \ldots, m-n+k_1+k_2-1, m-n+k_1 \rangle$. Let $A_i = A'_i \cup \{n-i+1\}$ for $i \in \{1,2,\ldots,k_1+k_2\} \backslash \{k_1+1\}$ and $A_{k_1+1} = L \cup \{n-k_1\}$. Then $A_1, A_2, \ldots, A_{k_1+k_2}$ are the subsets of type $\langle \underbrace{m+1, m+1, \ldots, m+1}_{k_1 \text{ terms}}, \underbrace{m, m, \ldots, m}_{k_2 \text{ terms}} \rangle$. ∎

Thus, we have given a shorter proof for all the results mentioned above in Proposition 1.1, 1.2, 1.3. Moreover, we have further evidence for believing that the stronger conjecture mentioned in this paper is a true conjecture.

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