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On the Problem of Total Coloring

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ABSTRACT

A *total coloring* π of a graph G is a mapping π: $V(G) \cup E(G) \rightarrow \{1,2,\ldots\}$ such that (1) no two adjacent vertices or edges have the same image; and (2) the image of each vertex is distinct from the images of its incident edges. The *total chromatic number* $\chi_t(G)$ of a graph G is thd smallest integer k such that G has a total coloring having image set $\{1,,2,\ldots,k\}$. From the definition of $\chi_t(G)$, it is clear that $\chi_t(G) \geqslant \Delta(G) + 1$.

In this paper, we apply some special latin squares to classify the total colorings of two classes of graphs. First, we show that the balanced complete t-partite graph, O_r^t, is of type two, if and only if $t = 2$ or t is even and r is odd. Secondly, we prove that a $(2n-3)$-regular graph G of order $2n$ is of type one, if and only if the complement of G, G^c, is a disjoint union of two 1-factors.

Keywords: total coloring, total chromatic number

I. Introduction and Preliminary Results

Throughout this paper, all the graphs are finite, simple, and undirected. Let G be a graph, and its vertex set, edge set, chromatic index, and the maximum degree be denoted by $V(G)$, $E(G)$, $\chi'(G)$ and $\Delta(G)$, respectivity. We will use O_r^t to denote the balanced complete t-partite graph with r vertices in each partite set. The complete graph of order n is denoted by K_n. A perfect matching of a complete graph K_{2n}, also known as a 1-factor, is a matching with n edges. It is well known that K_{2n} can be decomposed into $2n-1$ 1-factors $F_1, F_2, \ldots, F_{2n-1}$. The collection of 1-factors $F_1, F_2, \ldots, F_{2n-1}$ is called a 1-factorization of K_{2n}. It is easy to see that the union of two disjoint 1-factors is a disjoint union of even cycles. Also, it is clear that a graph $G = K_{2n} \backslash (F_i \cup F_j)$, $1 \leqslant i \neq j \leqslant 2n-1$, is $(2n-3)$-regular. Denote the complement of a graph G by G^c. Then, if G is a $(2n-3)$-regular graph, G^c either is a disjoint union of two 1-factors or contains an odd cycle. Other terms and notations in graph theory not defined in this paper can be found in Bondy and Murty (1976).

A total coloring π of a graph G is a mapping π: $V(G) \cup E(G) \rightarrow \{1,2,\ldots.$ such that no two adjacent vertices receive the same color, no two edges incident with the same vertex receive the same color, and no edge receives the same color as either of the vertices it is incident with. The total chromatic number $\chi_t(G)$ of a graph G is the smallest integer k such that G has a total coloring having image set $\{1,2,\ldots,k\}$. From the definition of $\chi_t(G)$, it is clear that $\chi_t(G) \geqslant \Delta(G) + 1$. Behzad (1965) and Vizing (1964, 1968) made the following conjecture.

Total Coloring Conjecture (TCC). For any graph G, $\chi_t(G) \leqslant \Delta(G) + 2$.

A bit of reflection, if this conjecture is proved to be true, then $\chi_t(G)$ is equal to $\Delta(G)+1$ or $\Delta(G)+2$ which is similar to the argument of chromatic index $\chi'(G)$. Thus, we define a graph to be of **type one** if $\chi_t(G) = \Delta(G)+1$ and of **type two** otherwise.

The following results are known.

Theorem 1.1. (Yap, 1989) Let G be a complete partite graph; then $\chi_t(G) \leqslant \Delta(G) + 2$.

Theorem 1.2. (Yap *et al.*, 1989) For any graph G of order n having $\Delta(G) \geqslant n-4$, $\chi_t(G) \leqslant \Delta(G) + 2$.

Theorem 1.3. (Hilton, 1989/90) Let $n \geqslant 1$, let J be a subgraph of K_{2n}, let $e = |E(J)|$ and let $j(J)$ be the maximum size of a matching in J. Then $\chi_t(K_{2n} \backslash E(J)) = 2n+1$ if and only if $e + j \leqslant n-1$.

Theorem 1.4. (Chen and Fu, 1991) Let G be a graph of order $2n$ with maximum degree $2n-2$. Then G is of type two if and only if G^c is a disjoint union of an edge and a star with $2n-3$ edges.

As can be seen from the above theorems, Theorem 1.1 mainly proves that the TCC holds for complete partite graphs. But, so far, there is no complete answer for whether it is of type one or type two. For the case where it is a balanced complete partite graph, there is a classification by Bermond (1974/75) which is in French. In this paper, we use two special types of latin squares to obtain a constructive method to classify this type of graph which is different what we already have. Theorems 1.3 and 1.4 are the results for the classification of a graph of high maximum degree, $2n$-1 and $2n$−2. In this direction, we obtain a result for the regular graphs of order $2n$ and degree $2n$−3.

Theorem 1.5. A ($2n$-3)-regular graph G of order $2n$ is of type one if and only if the complement of G is a disjoint union of two 1-factors.

II. The Type Two Graphs

Let G be a graph. It is easy to see that the set of all edges and vertices colored with the same color induces a subgraph H which contains a matching M and an independent set such that M and I have no vertex in common. We will call H a **mixed set** and the maximum size of all mixed sets is the mixed number of G denoted by $\tau(G)$.

By observation, we have

Lemma 2.1. A graph G which satisfies TCC is of type two if $\tau(G)\cdot(\Delta(G)+1) < |V(G)| + |E(G)|$.

We can use Lemma 2.1 to obtain the following

Proposition 2.2. A complete bipartite graph is of type two if and only if it is balanced.

Proof. If the graph is balanced, then $|V(G)| + |E(G)| = 2\Delta(G) + [\Delta(G)]^2 > \Delta(G)(\Delta(G)+1)$, where $\tau(G) = \Delta(G)$. Hence, by Lemma 2.1, G is of type two. If G is not balanced, let $G = (A, B)$, $|A| = m < n = |B|$, and $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$. Define a total coloring π as follows: (i) $\pi(a_i) = 0$ for each $a_i \in A$; (ii) $\pi(b_j) = j$, $b_j \in B$; and (iii) $\pi(\{a_i, b_j\}) = i+j-1$ whenever $i+j-1 \leqslant n$, and $\pi(\{a_i, b_j\}) = i+j-1-n$ if $i+j-1 > n$. This is a routine way to check π is a total coloring of G and $\chi_t(G) = n+1 = \Delta(G) + 1$. Thus, G is of type one. Q.E.D.

Let G be a regular graph with valency α and G is of type one. Then, for each vertex v and all the α edges which are incident with v, every color occurs (α+1 colors). Hence, if M_i is the set of edges which are colored with i, and I_i is the set of vertices colored i, we conclude that $2|M_i| + |I_i| = |V(G)|$, $1 \leqslant i \leqslant \alpha+1$. A bit of reflection, $|V(G)|$ and $|I_i|$ are of the same parity. Thus, we have

Proposition 2.3. If t is even and r is odd, then O_r^t is of type two.

Proof. Assume O_r^t is of type one. Since, for each color i, I_i is a subset of one part only and $|I_i|$ is even, there is no way to complete the total coloring due to the fact r is odd. Hence, O_r^t must be type two. Q.E.D.

As a special case, we have

Corollary 2.4. A complete graph is of type two if and only if it is of even order.

In fact, it is not difficult to see that a complete graph of odd order is type one. We omit the details. There is another result which is worthy of mention. If M is a perfect matching of K_{2n}, then $K_{2n}\backslash M$ is of type one. Therefore, we will use it later, we list it as a lemma.

Lemma 2.5. Let M be a perfect matching in K_{2n} and $n \geqslant 3$, then $K_{2n}\backslash M$ is of type one.

Proof. Hanani (1960) proved that K_{2n} can be decomposed into $2n$-1 1-factors $F_1, F_2, \ldots, F_{2n-1}$ ($P_\alpha(n)$ systems) such that there exists a 1-factor F, $|F \cap F_i| = 1$ for each $i = 1,2,\ldots,n$. Thus, we define a total coloring on $K_{2n}\backslash F$ such that the edges in $F_i\backslash F$ are colored with i, $i = 1,2,\ldots,2n-1$, and the color of the vertices u_j and v_j is j, provided that $|F \cap F_j| = \{u_j, v_j\}$ for some j. It is easy to check that this is a total coloring of $K_{2n}\backslash F$ with $2n$-1 colros. Hence, $K_{2n}\backslash F$ is of type one. Since $K_{2n}\backslash M$ is isomorphic to $K_{2n}\backslash F$, we have the proof.

III. The Type One O_r^t

With Lemma 2.5, we are ready to prove

Proposition 3.1. O_{2s}^t is of type one for each $t \geqslant 3$.

Proof. For convenience, let the jth part of O_{2s}^t be the disjoint union of two vertex sets A_{2j-1} and A_{2j}, $A_{2j-1} = \{u_1^{2j-1}, u_2^{2j-1}, \ldots, u_s^{2j-1}\}$ and $A_{2j} = \{u_1^{2j}, u_2^{2j}, \ldots, u_s^{2j}\}$, $j = 1,2,\ldots,t$. By Lemma. 2.5, $K_{2t}\backslash M$ ($M = \{\{v_1, v_2\}, \{v_3, v_4\}, \ldots, \{v_{2t-1}, v_{2t}\}\}$) is of type one, and also $\chi'(K_{2t}\backslash M) = 2t$-2. Let the colors used in the total coloring of $K_{2t}\backslash M$, α, be denoted by $c_1, c_2, \ldots, c_{2t-1}$ and the colors used in the edge coloring of $K_{2t}\backslash M$, β, be denoted by $\delta_1, \delta_2, \ldots, \delta_{2t-2}$. Now, we are ready to find a total coloring π on O_{2s}^t which uses (t-1) $2s$+1 colors. (i) If $\{v_i, v_j\}$ is an edge in $K_{2t}\backslash M$ such that $\beta(\{v_i, v_j\}) = \delta_k$, then the graph H_{ij} obtained by deleting a perfect matching M_{ij} from the complete bipartite graph induced by A_i and A_j in O_{2s}^t is of class one, and we color the edges of $H_{i,j}$ with s-1 colors, $\delta_k^1, \delta_k^2, \ldots, \delta_k^{s-1}$. In total, we use ($2t$-2)($s$-1) colors for π in coloring the edges of $H_{i,j}$, where $\{v_i, v_j\} \in E(K_{2t}\backslash M)$ and $\beta\{v_i, v_j\} = \delta_k$, $k = 1,2,\ldots,2t-2$. (ii) If $\{v_i, v_j\}$ is an edge in $K_{2t}\backslash M$ such that $\alpha(\{v_i, v_j\}) = c_k$, then each edge of $M_{i,j}$ is colored c_k in the total coloring π. (iii) Color all the vertices of A_{2j-1} and A_{2j} with c_h if the vertices $\alpha(v_{2j-1})$

$= \alpha(v_{2j}) = c_h$. From (ii) and (iii), we have used $2t-1$ colors, which is of course due to the fact that $K_{2t} \backslash M$ is type one. In total we use $(t-1)2s+1$ for π. This implies that O^t_{2s} is type one. Q.E.D.

Before we go any further, we need some more definitions. Let S is a v-set. A latin square of order v based on S is a $v \times v$ array with entries from S such that in each row and each column every element of S occurs exactly once. For convenience, let $S = \{1,2,\ldots,v\}$. A latin square $L = [\ell_{ij}]$ is said to be **commutative** provided that $\ell_{ij} = \ell_{ji}$ for every $1 \leqslant i, j \leqslant v$. A latin square $L = [\ell_{ij}]$ is **idempotent** if $\ell_{ii} = i$ for each $i \in S$. It is well-known that an idempotent commutative latin square of order v exists if and only if v is odd. In case $v = 2k$, let $H = \{\{1,2\}, \{3,4\}, \ldots, \{2k-1, 2k\}\}$. The 2-element subsets in H are called holes. A **latin square with holes** H is a latin square such that, for each hole $h \in H$, the subarray formed by $h \times h$ is a subsquare based on h. Since all the holes are of size two, we also refer to the latin square as a **latin square with 2x2 holes H.** Fig. 3.1 is an example of a commutative latin square of order 8 with 2x2 holes H. It has been show by Fu that a communtative latin square of order $2k$ with 2x2 holes H (briefly CLSH($2k$)) exists for each $k \geqslant 3$ (Fu, 1987).

1	2	8	5	4	7	6	3
2	1	6	7	8	3	4	5
8	6	3	4	7	2	5	1
5	7	4	3	1	8	2	6
4	8	7	1	5	6	3	2
7	3	2	8	6	5	1	4
6	4	5	2	3	1	7	8
3	5	1	6	2	4	8	7

Fig. 3.1.

The case left is r and t are both odd. Let us start with $t = 3$. The total coloring of O^3_r can be represented by an array as in Fig. 3.2.a, where a_i is the color of vertex v_i, $i = 1,2,\ldots,3r$ and ℓ_{ij} is the color of the edge $\{v_i, v_j\}$. Since two vertices are not adjacent if they are in the same part, ℓ_{ij} is not defined whenever $1 \leqslant i, j \leqslant r$, or $r+1 \leqslant i, j \leqslant 2r$, or $2r+1 \leqslant i, j \leqslant 3r$. As an example, Fig. 3.2.b is a total coloring of O^3_3.

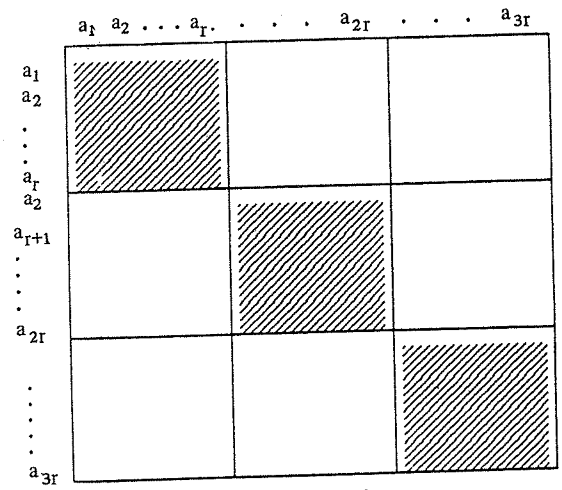

Fig. 3.2.a

	0	0	0	1	2	3	4	5	6
0				4	6	5	1	3	2
0				3	5	4	6	2	1
0				2	1	6	5	4	3
1	4	3	2				0	6	5
2	6	5	1				3	0	4
3	5	4	6				2	1	0
1	1	6	5	0	3	2			
5	3	2	4	6	0	1			
6	2	1	3	5	4	0			

Fig. 3.2.b

Now we will give a general way of finding the total coloring of O^3_r whenever r is odd. Let M_1, M_2 be two idempotent commutative latin squares of order r based on $\{1, 2, \ldots, r\}$ and $\{r+1, r+2, \ldots, 2r\}$, respectively. Moreover, we let the upper triangular part, diagonal, and lower triangular part of a latin square M be denoted by $U(M)$, $D(M)$, and $L(M)$, respectively. Thus, we can construct an array with these three parts whenever they have the same frame. For example, if M_1 and M_2 are two idempotent commutative latin squares of order r mentioned above, then $< U(M_1), D(M_1), L(M_2) >$ (briefly $<1,1,2>$) is an $r x r$ array. Furthermore, let M_0 be a latin square of order r with constant diagonal, and the constant is zero. Then, by Fig. 3.3, it is easy to check that O^3_r is of type one.

Proposition 3.2. O_r^3 is of type one whenever r is odd.

Proof. See Fig. 3.3. We use $2r+1$ colors in total. Q.E.D.

	0 0 · · · 0	1 2 · · · r	r+1 r+2 · · · 2r
0 0 0 ⋮ 0		$\langle 2,2,1 \rangle$	$\langle 1,1,2 \rangle$
1 2 3 ⋮ r	$\langle 1,2,2 \rangle$		$\langle 2,0,1 \rangle$
r+1 r+2 ⋮ 2r	$\langle 2,1,1 \rangle$	$\langle 1,0,2 \rangle$	

Fig. 3.3.

O_r^5 can be obtained similarly. Let M_1, M_2, M_3 and M_4 be four idempotent commutative latin squares of order r based on $\{1,2,\ldots,r\}$, $\{r+1, r+2,\ldots,2r\}$, $\{2r+1, 2r+2,\ldots,3r\}$, and $\{3r+1, 3r+2,\ldots,4r\}$, respectively. Also, M_0 is a latin square with all zeros in diagonal.

Proposition 3.3. O_r^5 is of type one whenever r is odd.

Proof. By Fig. 3.4. Q.E.D.

	0 0 · · · 0	1 2 · · · r	r+1 r+2 · · · 2r	2r+1 · · · 3r	3r+1 · · · 4r
0 0 ⋮ 0		M_4	M_3	M_1	M_2
1 2 ⋮ r	M_4		$\langle 2,0,1 \rangle$	$\langle 3,2,2 \rangle$	$\langle 1,3,3 \rangle$
r+1 r+2 ⋮ 2r	M_3	$\langle 1,0,2 \rangle$		$\langle 2,4,4 \rangle$	$\langle 4,1,1 \rangle$
2r+1 2r+2 ⋮ 3r	M_1	$\langle 2,2,3 \rangle$	$\langle 4,4,2 \rangle$		$\langle 3,0,4 \rangle$
3r+1 3r+2 ⋮ 4r	M_2	$\langle 3,3,1 \rangle$	$\langle 1,1,4 \rangle$	$\langle 4,0,3 \rangle$	

Fig. 3.4.

Finally, we consider O_r^t, where r is odd, $t \geqslant 7$ and t is odd. Since t is odd, let $t = 2k+1$, $k \geqslant 3$. Let $L = [\ell_{ij}]$ be a CLSH($2k$) based on $\{1,2,\ldots,2k\}$ with holes in H. Let the vertex set of O_r^t be the disjoint union of t parts A_0, $A_1,\ldots,A_{2k}$ and $A_i = \{(i,1), (i,2),\ldots,(i, r)\}$, $i = 0, 1, \ldots, 2k$. Define a total coloring on O_r^t as follows: (i) the vertices of A_0 are colored with 0, the vertices of A_i are colored with $(i-1)r+1$, $(i-1)r+2, \ldots, ir$ where $i = 1, 2, \ldots, 2k$; (ii) the edge set of compete 3-partite graphs induced by A_0, A_1, A_2; A_0, A_3, A_4; . . . ; A_0, A_{2k-1}, A_{2k} are colored in a way similar to that in Proposition 3.2 by using the colors $0, 1, 2, \ldots, 2r$; $0, 2r+1, 2r+2,\ldots,4r$; . . . ; $0,(2k-2)r+1,(2k-2)+2, \ldots, 2k$, respectively; (iii) the edge set of the complete bipartite graph induced by A_i, A_j, $\{i,j\} \notin H$, is colored by the colors $(x-1)r+1$, $(x-1)r+2, \ldots, xr$ if $\ell_{ij} = x$ in L. By routine checking, we obtain the following:

Proposition 3.4. If $t \geqslant 7$, t and r are odd, then O_r^t is of type one.

Combining Propositions 2.3, 3.1, 3.2, 3.3, and 3.4, we have proved the following theorem.

Theorem 3.5. O_r^t is of type two if $t = 2$ or t is even and r is odd; otherwise, O_r^t is of type one.

IV. Regular Graph of Order 2n and Degree 2n-3

As mentioned in section 2, if G is a regular graph with valency α and is of type one, then the order of G and the number of vertices which are colored commonly are of the same parity; thus we have

Lemma 4.1. If G is a regular graph of even order and G is of type one, then for each color c_i, the number of vertices colored c_i is an even number.

With the above result, we can prove

Proposition 4.2. If G is a $(2n-3)$-regular graph of order $2n$ and G^c contains an odd cycle, then G is of type two.

Proof. Let G be a $(2n-3)$-regular graph of order $2n$ such that G^c contains an odd cycle $C = (v_1, v_2, \ldots, v_{2k+1})$, $v_i \in V(G)$, $1 \leqslant i \leqslant 2k+1$. Let π be a total coloring of G such that G is of type one. A bit of reflection, there are at most two vertices in $V(G)$ which can be colored with the same colors. Thus, if $\pi(v_i) = c_i$, then by Lemma 4.1, exactly one of v_{i-1} and v_{i+1} must be colored c_i. Hence, there exists a vertex v_j in $V(C)$ such that $\pi(v_j)$ occurs exactly once. By Lemma 4.1, we conclude that G can not be of type one. Therefore, G is of type two. Q.E.D.

Now, in order to prove Theorem 1.5, we need to show that if G^c is a disjoint union of two 1-factors, then G is of type one.

Case 1. n is odd.

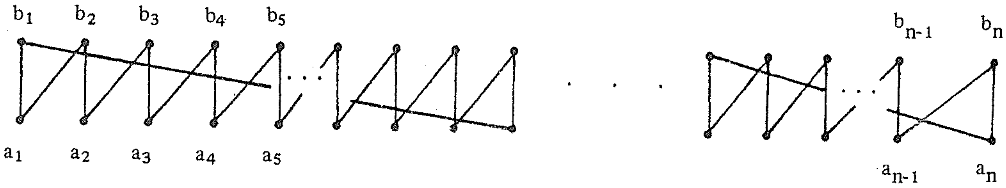

Fig. 4.1.

By Corollary 2.4, we can prove the following proposition.

Proposition 4.3. Let n be an odd number. The graph G obtained by deleting two 1-factors from K_{2n}, i.e., $G = K_{2n} \backslash (F_1 \cup F_2)$, is of type one.

Proof. Since two isomorphic graphs have the same total chromatic number, without loss of generality, we can let $V(G) = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$, $F_1 = \{\{v_i, v_{n+i}\} : 1 \leqslant i \leqslant n\}$ and the edges in F_2 are of the form $\{v_i, v_{n+j}\}$, where $1 \leqslant i \neq j \leqslant n$. In other words, G can be decomposed into three parts G_1, G_2, $G_3 = (A, B)$, where G_1 and G_2 are complete graphs of order n with $V(G_1)$ and $V(G_2)$ equal to $\{v_1, v_2, \dots, v_n\}$ and $\{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$, respectively, and G_3 is a $(n-2)$-regular bipartite graph with $A = V(G_1)$ and $B = V(G_2)$. Define a total coloring π on G such that (1) $\pi|_{G_1}$ is a total coloring of G_1 with n colors $\{c_1, c_2, \dots, c_n\}$ such that $\pi(v_i) = c_i$, (2) $\pi|_{G_2}$ is a total coloring of G_2 with n colors $\{c_1, c_2, \dots, c_n\}$ such that $\pi(v_{n+i}) = c_i$, and (3) $\pi|_{G_3}$ is an edge coloring of G_3 with $n-2$ colors $\{c_{n+1}, c_{n+2}, \dots, c_{2n-2}\}$. (A regular bipartite graph is of class one.) Then, it is clear that G is of type one. Q.E.D.

Case 2. n is even and $n = 2k$.

First, we need a lemma.

Lemma 4.4. If there exist two idempotent commutative latin squares of order $2k+1$, $L = [\ell_{ij}]$ and $M = [m_{ij}]$, such that $(\ell_{1,2k+1}, \ell_{2,2k+1}, \dots, \ell_{2k,2k+1}) = (m_{k,2k+1}, m_{k+1,2k+1}, \dots, m_{k-2,2k+1}, m_{k-1,2k+1})$, or $(\ell_{1,2k+1}, \ell_{2,2k+1}, \dots, \ell_{2k,2k+1}) = (m_{k+3,2k+1}, m_{k+4,2k+1}, \dots, m_{k+1,2k+1}, m_{k+2,2k+1})$, then we have a total coloring of $K_{2n} \backslash (F_1 \cup F_2)$ with $2n-2$ colors, where F_1 and F_2 are two 1-factors of K_{2n}.

Proof. Without loss of generality, we let $F_1 = \{\{a_i, b_i\} \mid i = 1, 2, \dots, n\}$ and F_2 be the set of all edges in Fig. 4.1 which is not in F_1. It is easy to see that $F_1 \cup F_2$ is a union of disjoint even cycles. (The lengths of cycles are in nonincreasing order.) Now let $V(G_1) = \{a_1, a_2, \dots, a_n\}$ and $V(G_2) = \{b_1, b_2, \dots, b_n\}$, as in Fig. 4.1. Let $F_3 = \{\{a_i, b_{i+k-1}\} : 1 \leqslant i \leqslant n$, $i+2$ takes modulo $2n\}$. Clearly, $F_3 \cap (F_1 \cup F_2) = \phi$. Define a total coloring π on $K_{2n} \backslash (F_1 \cup F_2)$ by letting (1) $\pi(\{a_i, a_j\}) = \ell_{ij}$, $1 \leqslant i \neq j \leqslant n$, (2) $\pi(\{b_i, b_j\}) = m_{ij}$, $1 \leqslant i \neq j \leqslant n$, (3) $\pi(a_i) = \pi(b_i) = i$, $1 \leqslant i \leqslant n$, (4) $\pi(\{a_i, b_{i+k-1}\}) = \ell_{i,2k+1}$, $1 \leqslant i \leqslant n$, and $i+2$ takes modulo n, and (5) the restriction of π on the regular bipartite subgraph of $K_{2n} \backslash (F_1 \cup F_2 \cup F_3)$ induced by the two parts $V(G_1)$ and $V(G_2)$ is just the edge-coloring of this regular bipartite graph with valency $n-3$. It is a routine matter to check that π is a total coloring of $K_{2n} \backslash (F_1 \cup F_2)$ with $2n-2$ colors. Similarly, we can prove the second case. Q.E.D.

By Lemma 4.4, we conclude that if we can construct a pair of idempotent commutative latin squares of order $n+1 = 2k+1$ that satisfy the condition mentioned in Lemma 4.4, then we have proved our main theorem. In what follows, we will work on the construction of two such squares.

First, we need some definitions. A *partial latin square* of side order ν on the symbols $1, 2, \dots, \nu$ is a $\nu \times \nu$ array in which each cell either is empty or contains one of the symbols $1, 2, \dots, \nu$, and no symbol occurs twice in each row or twice in each column. A partial latin square is said to be *completed*, if we can fill the empty entries with the symbols $1, 2, \dots, \nu$ to make a latin square of order ν. A bit of reflection, if we can always complete a partial latin square of order $2n+1$ as in Fig. 4.2 into an idempotent commutative latin square of order $2n+1$ with prescribed $a_1, a_2, \dots, a_{2n}$, where $a_i \neq i$, $i = 1, 2, \dots, 2n$, then we can construct the two squares we need.

$$\begin{array}{|ccccc|}
\hline
1 & & & & a_1 \\
 & 2 & & & a_2 \\
 & & \cdot & & \cdot \\
 & & & \cdot & \cdot \\
 & & & \cdot & \cdot \\
 & & & & a_{2n} \\
 & & & & 2n+1 \\
\hline
\end{array}$$

Fig. 4.2.

In what follows, we will say that an idempotent commutative latin square (ICLS) $L = [\ell_{ij}]$ has structure

$<a_1, a_2, \ldots, a_{2n}>$ if $\ell_{i,\,2n+1} = a_i$ for each $i = 1,2,\ldots, 2n$. The structure can also be represented by a permutation π on the set $\{1,2,\ldots,2n\}$, where $\pi(i) = a_i$, $i = 1, 2, \ldots, 2n$. Since π can be written as a product of disjoint cycles $\pi_1, \pi_2, \ldots, \pi_h$, $\pi=\pi_1\pi_2\cdots\pi_h$, such that the length of the cycle π_i is equal to ℓ_i and $\ell_1 \leqslant \ell_2 \leqslant \ldots \leqslant \ell_k$, we will say π has the *form* $(\ell_1, \ell_2, \ldots, \ell_h)$. Let L and M be two ICLS's which have the same form. Then, it is not difficult to see that L can be obtained from M by permuting rows, columns and entries. The following lemma shows that we can always construct an ICLS of order $2k+1$ which has the form $(2k)$ for each odd k. Moreover, in this ICLS, the structures $< a_1, a_2, \ldots a_{2k} >$ and $<a_{k+2}, a_{k+3}, \ldots, a_{2k}, a_1, \ldots, a_{k+1}>$ are of the same form.

Lemma 4.5. There exists an ICLS $(2k+1)$, where k is odd and $k \geqslant 3$, such that its structure is $<k+1, k+2, \ldots, 2k, 2, 3, \ldots, k, 1>$. Moreover, this structure and $<3,4,\ldots, k, 1, k+1, \ldots, 2k, 2>$ have the same form $(2k)$.

Proof. It is well known that an ICLS(k) can be embedded in an ICLS$(2k+1)$. (Fig. 4.3.) Thus, the entries $1,2,\ldots,k$ occur in A and C only, and the last column of the ICLS can be $<k+1, k+2, \ldots, 2k, 2, 3, \ldots, k, 1, 2k+1>^T$ by permuting the rows of B (and the columns of B^T) and the entries, $1,2,\ldots,k$ in C, respectively. Hence, we obtain an ICLS $(2k+1)$ which has the structure $<k+1, k+2,\ldots,2k,2,3,,\ldots,k,1>$. Since, the permutations $\begin{pmatrix} 1 & 2 & 3 \ldots k & k+1 & \ldots & 2k \\ k+1 & k+2 & \ldots 2k & 2 & 3 \ldots & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \ldots & k & \ldots & 2k \\ 3 & 4 & \ldots k & k+1 & \ldots 2k & 2 \end{pmatrix}$ are cycles of length $2k$, we have the proof.

Q.E.D.

As for the case when k is even, we can apply the fact that an ICLS$(k-1)$ can be embedded in an ICLS$(2k+1)$, and the structure of this ICLS$(2k+1)$ can be $<k, k+1, \ldots, 2k-2, 2k-1, 2k, 2, 3, \ldots, k-1, 1>$ (Fig. 4.4). Furthermore, it is not difficult to check that this structure and $<3, 4, \ldots, k-1, 1, k, \ldots, 2k, 2>$ are of the same form $(2k)$. Thus, we have

ICLS(2k+1):

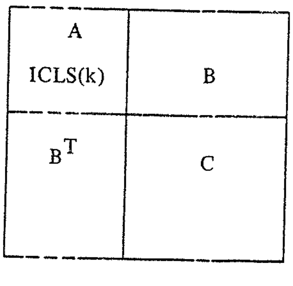

Fig. 4.3.

ICLS(2k+1):

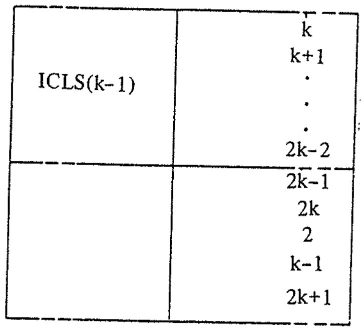

Fig. 4.4.

Lemma 4.6. There exists an ICLS$(2k+1)$, where k is even, such that its structure is $<k, k+1, \ldots, 2k-2, 2k-1, 2k, 2, 3, \ldots, k-1, 1>$. Moreover this structure and $<3, \ldots, k-1, 1, k, k+1, \ldots, 2k, 2>$ both have the form $(2k)$.

By Lemma 4.5 and 4.6, since we have constructed the two squares so as to satisfy the condition in Lemma 4.4, we have concluded the case when n is even. (We note here again that two ICLS$(2k+1)$ of the same form can be obtained mutually).

Now we have established Theorem 1.5.

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References

Behzad, M. (1965) *Graphs and Their Chromatic Numbers.* Doctoral Thesis, Michigan State Univ., U.S.A.

Bermond, J.C. (1974/75) Nombre chromatique total du graphe r-parti complet. *J. London Math. Soc.*, **2**(9), 279-285.

Bondy, J. A. and U.S.R. Murty (1976) *Graph Theory with Applications.* Elsevieer, North Holland.

Chen, B.L. and H.L. Fu (1991) The total colorings of the graphs of order 2n and maximum degree 2n-2 (in press).

Fu, C.M. (1987) *The Intersection Problem of Pentagon Systems.* Ph.D. Dissertation, Auburn University, U.S.A.

Hanani, H. (1960) On quadruple systems. *Canada J. Math.*, **12**, 145-157.

Hilton, A. J. W. (1989/90) A total-chromatic number analogue of Plantholt's theorem. *Discrete Math.*, **79**, 169-175.

Vizing, V. G. (1964) On an estimate of the chromatic class of a p-graph (Russian). *Diskret. Analiz*, **3**, 25-30.

Vizing, V. G. (1968) Some unsolved problems in graph theory. *Uspehi Mat. Nauk*, **23**, 117-134 (*Russian Math. Surveys*, **23** (1968), 125-142).

Yap, H. P. (1989) Total colorings of graphs. *Bull. London Math. Soc.*, **21**, 159-163.

Yap, H. P., F. J. Wang and Z. Zhang (1989) Total chromatic number of graphs of high degree. *J. Austral. Math. Soc.* (Series A), **47**, 445-452.