

NEW RESULTS ON GRACEFUL GRAPHS

HUNG-LIN FU and SHUNG-LIANG WU

*Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan, R.O.C.*

A graph $G = (V, E)$ is labelled if each vertex v is assigned a non-negative integer $r(v)$ and each edge $\{v, w\}$ is assigned the absolute value of the difference of the numbers at its endpoints, that is $|r(v) - r(w)|$. The labelling is called *graceful* (or the graph is graceful) if furthermore, we have: (1) The vertices are labelled with distinct integers. (2) The largest value of the vertex labels is equal to the number of edges, i.e., $\max r(v) = e = |E|$. (3) All the edges of G are distinctly labelled with the integers from 1 to e (or equivalently):

$$\{|r(v) - r(w)| : \{v, w\} \in E\} = \{1, 2, \dots, e\}.$$

The well-known conjecture about graceful labelling is that every tree is graceful. There are many graphs for which it is unknown whether they are graceful. In this paper, we utilize the ideas of α -valuation and certain type of construction to obtain more graceful graphs.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with $|E| = e$. A labelling on G is a one-to-one mapping θ from V into the set $N = \{0, 1, 2, \dots, e\}$. For each edge $\{v, w\}$, the weight of $\{v, w\}$ is defined as the value $|\theta(v) - \theta(w)|$ and denoted by $\theta(vw)$. A labelling θ is called a β -labelling of G if the weights of all edges of G are distinct. Recently, a β -labelling has come to be called a graceful labelling. A β -labelling θ is called an α -labelling of G if there is an integer x such that, for any edge $\{v, w\}$, either $\theta(v) \leq x$ and $\theta(w) > x$ or $\theta(v) > x$ and $\theta(w) \leq x$. Figure 1.1 is an example of an α -labelling.

A bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2 . We will call a connected bipartite graph an (m, n) -graph if $|V_1| = m$ and $|V_2| = n$. It is well-known that a tree is an (m, n) -graph for some m and n , and the tree is also called an (m, n) -tree. In what follows, we will say an (m, n) -tree T is an (m, n, θ) -tree provided that T has an α -labelling θ . It is clear that the tree in Fig. 1.1 is a $(3, 3, \theta)$ -tree where θ is the α -labelling. For convenience, if T is an (m, n, θ) -tree, we assume that $\theta(v) < \theta(w)$ for every $v \in V_1$ and $w \in V_2$.

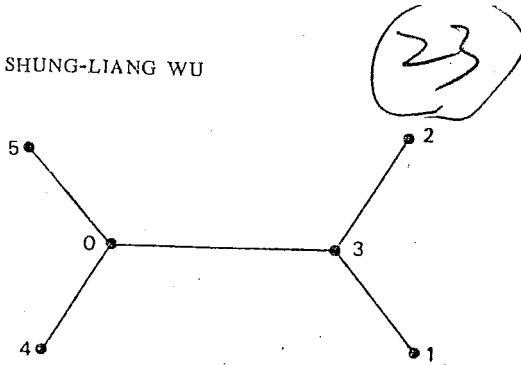


Fig. 1.1

Two graphs G_1 and G_2 are said to be isomorphic if there exists a bijection $\phi: V(G_1) \rightarrow V(G_2)$ such that $\{u, v\} \in E(G_1)$ if and only if $\{\phi(u), \phi(v)\} \in E(G_2)$. It is easy to see that if G_1 and G_2 are isomorphic and G_1 has a graceful labelling (or an α -labelling), then G_2 has a graceful labelling (or an α -labelling).

The problem of graceful labelling has been studied for a while and the famous conjecture on this topic is due to Ringel, who conjectured that all trees can be gracefully labelled. So far this conjecture remains unsolved except for several special classes of trees.

In this paper, we mainly study a few constructions which we can utilize to obtain new classes of graceful graphs.

2. THE MAIN RESULTS

There are a few classes of graphs for which it has been determined whether or not they are graceful. For clearness, we use Table 2.1 to

TABLE 2.1

Graph	Graceful	Condition	Resource
complete graph K_n	No	$n \geq 5$	[7]
complete bipartite graph $K_{m,n}$	Yes	$m, n \geq 1$	[7]
n -cube	Yes	$n \geq 1$	[11]
book B_n	Yes	n .even	[11]
	Yes	$n \equiv 1 \pmod{4}$	[6]
	No	$n \equiv 3 \pmod{4}$	[6]
ladder L_n	Yes	$n \geq 1$	[11]
cycle C_n	Yes	$n \equiv 0$ or $3 \pmod{4}$	[2]
friendship graph F_n	Yes	$n \equiv 0$ or $1 \pmod{4}$	[7]
fan graph F_n	Yes	$n \geq 1$	[1]
wheel graph W_n	Yes	$n \geq 1$	[4]
caterpillars	Yes		[3]
tree	Yes	≤ 4 end-vertices	[10]
net	Yes		[12]

include some of the results. For more details, the reader may refer to the survey paper by J. A. Gallian [6].

We will start with an example of a graceful graph. Let T_l^k be a tree which is the union of k paths of the same length l (l vertices) and $k - 1$ parallel edges, see Fig. 2.2. Then, it is easy to see that T_l^k has an α -

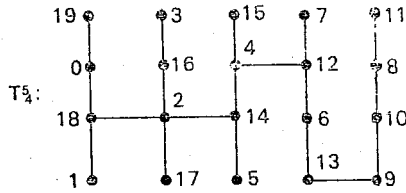


Fig. 2.2

valuation. For clearness, we give a labelling for the case $l = 4$ and $k = 5$. Then the proof should be easy to obtain by looking at the pattern.

The above result is an example of (m, n, θ) -tree. Now we can start with an (m, n, θ) -tree to construct new classes of graceful graphs.

Let T be an (m, n, θ) -tree. A graph $G_l(T)$ is obtained from the union of T and l vertices, w_1, w_2, \dots, w_l and all the edges $\{w_j, v\}$ where $v \in V_1(T)$. (Fig. 2.3). Then we have the following.

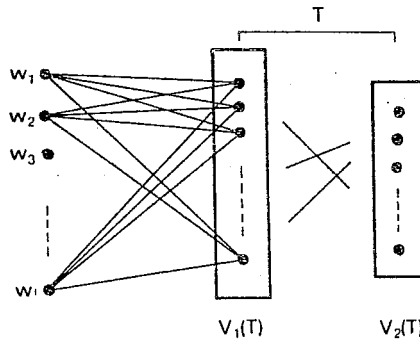


Fig. 2.3

PROPOSITION 2.2. *If T is an (m, n, θ) -tree, then $G_l(T)$ has an α -labelling.*

Proof. Define θ^* on $G_l(T)$ as follows:

- (1) $\theta^*(w_i) = (l - i + 2) \cdot m + n - 1, 1 \leq i \leq l$; and
- (2) $\theta^*(v) = \theta(v), v \in V(T)$.

Then it is a routine matter to check that θ^* is an α -labelling.

Q.E.D.

We note here, as a special case, when the (m, n, θ) -tree has $n = 1$, then we obtain a complete bipartite graph which has an α -labelling. And since every complete bipartite graph can be obtained in this way, we conclude that every complete bipartite graph has an α -labelling. (This result was proved in [6].)

The above proposition can be generalized a little, for we can also add vertices which are adjacent to the vertices of $V_2(T)$. Denote by $G_{l,j}(T)$ the graph which is the union of $G_l(T)$, j vertices u_1, u_2, \dots, u_j , and all edges of the form $\{v, u_i\}$ where $1 \leq i \leq j$ and $v \in V_2$. (Figure 2.4.)

By a similar argument, we have the following.

PROPOSITION 2.3. *If T is an (m, n, θ) -tree, then $G_{l,j}(T)$ has an α -labelling.*

Proof. It is a direct result of the following labelling θ^* defined on $G_{l,j}(T)$.

- (1) $\theta^*(v) = \theta(v)$, for $v \in V_1(T)$;
- (2) $\theta^*(v) = \theta(v) + j \cdot n$, for $v \in V_2(T)$;
- (3) $\theta^*(w_i) = (l - i + 2) \cdot m + (j + 1) \cdot n - 1$, for $1 \leq i \leq l$; and
- (4) $\theta^*(u_i) = (j - i + 1) \cdot n + m - 1$, for $1 \leq i \leq j$.

Q.E.D.

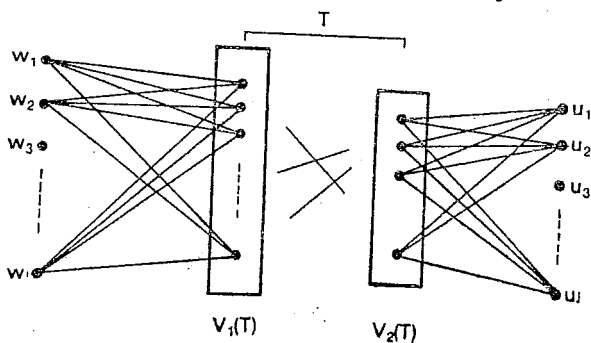


Fig. 2.4

Instead of adding l vertices to an (m, n, θ) -tree T , we can also add a star with k vertices to T (or both) to obtain a new graph $G_l^k(T)$. Fig. 2.5 is an example of this type of graph, $G_2^3(P_4)$.

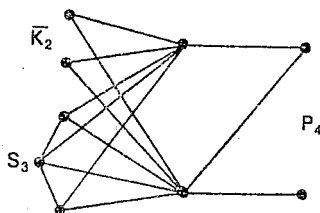


Fig. 2.5

PROPOSITION 2.4 If T is an (m, n, θ) -tree, then the graph $G_1^k(T)$ has a graceful labelling.

Proof. Define a labelling θ^* on $G_1^k(T)$ as follows:

- (1) $\theta^*(v) = \theta(v), v \in v_1(T)$;
 - (2) $\theta^*(v) = \theta(v) + m, v \in V_2(T)$;
 - (3) $\theta^*(v_0) = m, v_0$ is the center of the star;
 - (4) $\theta^*(v_i) = (i + 2)m + n + i - 1, 1 \leq i \leq k - 1, v_i$ in the star; and
 - (5) $\theta^*(w_i) = e - (i - 1)m, e = (l + k + 1)m + n + k - 2, 1 \leq i \leq l$.
- By direct checking, θ^* is a β -labelling for $G_1^k(T)$. Q.E.D.

Let T be a tree with a graceful labelling and G be any graph. Let the graph denoted by $T \otimes G$, be obtained from the union of T, G and all the edges joining each vertex in G to each vertex in T . In [9], it was verified that $T \otimes \bar{K}_n$ is graceful, where \bar{K}_n is the complement of K_n . Here we can replace \bar{K}_n by S_n (a star with n vertices) and obtain the following result.

PROPOSITION 2.5 If T is a graceful tree (with n vertices), then $T \otimes S_k$ is graceful.

Proof. Let θ be a graceful labelling of T . Define a labelling θ^* on $T \otimes S_k$ as follows:

- (1) $\theta^*(v) = \theta(v), \text{ for } v \in V(T)$;
- (2) $\theta^*(v_0) = (k + 1)n + (k - 2), v_0$ is the center of S_k ; and
- (3) $\theta^*(v_i) = (i + 1)n + i - 1, \text{ for } 1 \leq i \leq k - 1, \text{ and } v_i \in S_k$.

Then, it is not difficult to check that θ^* is a graceful labelling. Q.E.D.

We remark here that, if T has k (or $k + 1$) vertices, then $T \otimes S_k$ is a subgraph of K_{2k} (or K_{2k+1}) which has $\lfloor (2k)^2/4 \rfloor + 2k - 2$ edges (or $\lfloor (2k + 1)^2/4 \rfloor + (2k + 1) - 2$ edges). This is the size of maximum subgraph of K_{2k} (or $K_{2k + 1}$) which is graceful whenever $2k \leq 6$ (or $2k + 1 \leq 7$) [8]. Thus, we obtain a more general way to construct a maximum graceful subgraph of a complete graph K_n whenever $n \leq 7$. In Figure 2.6, we have

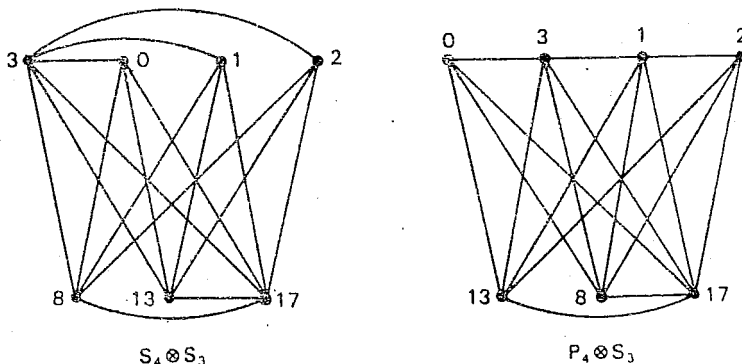


Fig. 2.6

two non-isomorphic graceful subgraphs of K_7 which are of maximum size.

Finally, we consider the cartesian product of two graphs. The cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2 = (V, E)$, where $V = V_1 \times V_2$ and $E = \{(x_1, x_2), (y_1, y_2) : x_1 = y_1 \text{ and } \{x_2, y_2\} \in E_2 \text{ or } x_2 = y_2 \text{ and } \{x_1, y_1\} \in E_1\}$.

PROPOSITION 2.6. *If T is an (n_1, n_2, θ) -tree with $|n_1 - n_2| \leq 1$, then for $m \geq 2$, $T \times P_m$ has an α -labelling.*

Proof. Let the vertices in $V_1(T)$ be u_1, u_2, \dots, u_{n_1} , and the vertices in $V_2(T)$ be v_1, v_2, \dots, v_{n_2} , moreover $n_1 \leq n_2$. Then the vertices of $T \times P_m$ are (u_i, j) , $i = 1, 2, \dots, n_1$, and (v_l, k) $l = 1, 2, \dots, n_2$; $1 \leq j, k \leq m$. (Figure 2.7). By direct computation, $|E(T \times P_m)| = m(n_1 + n_2 - 1) + (m - 1)(n_1 + n_2) = e$. Let θ^* be defined as follows:

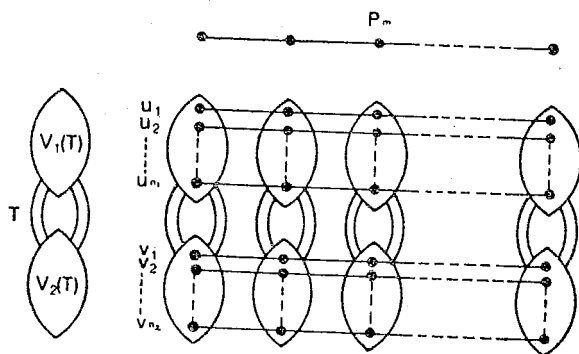


Fig. 2.7

- (1) $\theta^*(u_i, 1) = \theta(u_i)$, $u_i \in V_1(T)$;
- (2) $\theta^*(v_j, 1) = \theta(v_j) + e - n_1 - n_2 + 1$, $v_j \in V_2(T)$;
- (3) $\theta^*(u_i, 2) = e - n_1 - n_2 - \theta(u_i)$;
- (4) $\theta^*(v_j, 2) = e + n_1 + n_2 - 1 - \theta^*(v_j, 1)$;
- (5) for each $k \geq 3$, and k is odd,

$$\theta^*(u_i, k) = \theta^*(u_i, 1) + (2n_1 + 2n_2 - 1) \cdot (k - 1)/2,$$

and

$$\theta^*(v_l, k) = \theta^*(v_l, 1) - (2n_1 + 2n_2 - 1) \cdot (k - 1)/2;$$

- (6) for each even $k \geq 4$,

$$\theta^*(u_i, k) = \theta^*(u_i, 2) - (2n_1 + 2n_2 - 1) \cdot (k - 2)/2,$$

and

$$\theta^*(v_l, k) = \theta^*(v_l, 2) + (2n_1 + 2n_2 - 1) \cdot (k - 2)/2.$$

It is a routine matter to check that θ^* is an α -labelling on $T \times P_m$.

Q.E.D.

Figure 2.8 is an example which can make the proof of the above proposition more clear.

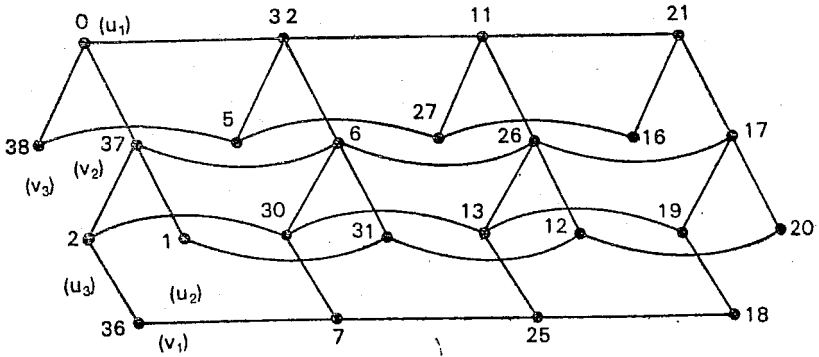


Fig. 2.8

Since a path has an α -labelling, we have the following corollary.

COROLLARY 2.7. [12] $P_n \times P_m$ has an α -labelling for each positive m and n .

We note here that $P_n \times P_m$ can also be called as an (n, m) -net. Fig. 2.9 is an example of $(4, 5)$ -net. For the case $m = 2$, the ladder graph L_m , has been verified in [11], so has a net in general [12].

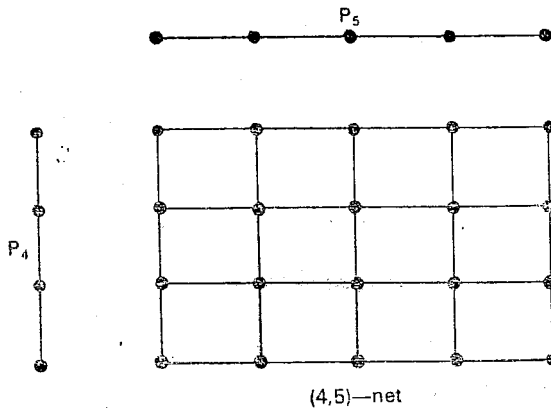


Fig. 2.9

ACKNOWLEDGEMENT

The authors would like to thank the referee's comments and the patience in correcting errors.

REFERENCES

- [1] G. S. Bloom and S. W. Golomb (1977), "Application of number undirect graph", *Proc. IEEE*, 65, 562-570.
- [2] R. Bodendic, H. Schumacher and A. Germa (1978), "systèmes de triplets et différences associées", *Problèmes Combinatoires et Théorie des Graphs* Colloq. International du CRNS, 260 Editions du Centre National de la Recherche Scientifique, Paris, 35-38.
- [3] J. Chhit and R. Cahit (1975), "On the graceful numbering of spanning trees", *Information Processing Letters*, 3, 115-118.
- [4] R. Frucht, (1979), "Graceful numbering of wheels and related graphs", *Ann. N. Y. Acad. of Sci.*, 319, 219-229.
- [5] Hung-Lin Fu and Shung-Liang Wu, "Are books graceful?" (preprint).
- [6] J. A. Gallian (1989), "Recent results, conjectures and open problems in labelling graphs", *Journal of Graph Theory*, 13, 491-504.
- [7] S. W. Golomb (1972), "How to number a graph", in *Graph Theory and Computing*, R. C. Read (ed.), Academic Press, New York, pp. 23-37.
- [8] S. W. Golomb (1974), "The largest graceful subgraph of the complete graph", *Amer. Math. Monthly*, 81, 499-501.
- [9] T. Grace (1983), "On sequential labellings of graphs", *Journal of Graph Theory*, 7, 195-201.
- [10] C. Huang, A. Kotzig and A. Rosa (1982), "Further results on tree labelling", *Utilitas Math.*, 21C, 31-48.
- [11] M. Maheo (1980), "Strongly graceful graphs", *Discrete Math.*, 29, 39-46.
- [12] A. Rosa (personal communication).