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Linear 3-arboricity of the Balanced Complete Multipartite Graph

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Abstract

A *linear k-forest* is a graph whose components are paths with lengths at most k. The minimum number of linear k-forests needed to decompose a graph G is the *linear k-arboricity* of G and denoted by $la_k(G)$. In this paper, we study the linear 3-arboricity of balanced complete multipartite graphs and we obtain some substantial results.

Keywords: Linear k-forest; Linear k-arboricity; Balanced complete multipartite graph

1 Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges.

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An *independent set* in a graph is a set of pairwise nonadjacent vertices. A graph G is *m-partite* if its vertex set $V(G)$ can be partitioned into m (possibly empty) independent sets called *partite sets* of G. A *complete m-partite graph* G is a m-partite graph having the additional property that the edge $uv \in E(G)$ if and only if u and v belong to different partite sets. When $m \geq 2$, we write $K_{n_1,n_2,\ldots,n_m}$ for the complete m-partite graph with partite sets of sizes $n_1, n_2, \ldots, n_m$. Moreover, if $n_1 = n_2 = \cdots = n_m = n$, then it is called a *balanced complete m-partite graph* and denoted by $K_{m(n)}$. For $m = 2$, such a graph is called a *balanced complete bipartite graph* and denoted by $K_{n,n}$.

A *balanced complete multipartite graph* is a balanced complete m-partite graph with $m \geq 2$. A *complete graph* is a graph whose vertices are pairwise adjacent; the complete graph with m vertices is denoted K_m. We can also view K_m as $K_{m(n)}$ with $n = 1$.

A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \ldots, G_d$, then we say G can be decomposed into $G_1, G_2, \ldots, G_d$ or $G_1, G_2, \ldots, G_d$ decompose G. A *linear k-forest* is a graph whose components are paths with lengths at most k. The *linear k-arboricity* of a graph G, denoted by $la_k(G)$, is the minimum number of linear k-forests needed to decompose G.

The notion of linear k-arboricity was defined by Habib and Peroche in [7]. It is a natural generalization of *edge coloring*. Clearly, a linear 1-forest is induced by a matching and $la_1(G) = \chi'(G)$ which is the *chromatic index* of a graph G. It is also a refinement of the concept of *linear arboricity*, introduced earlier by Harary in [9], in which no length constraints are needed. In 1982, Habib and Peroche [6] made the following conjecture on linear k-arboricity.

Conjecture 1.1. *If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then*

$$la_k(G) \leq \begin{cases} \left\lceil \frac{\Delta(G)\cdot|V(G)|}{2\left\lfloor \frac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) = |V(G)| - 1 \text{ and} \\ \left\lceil \frac{\Delta(G)\cdot|V(G)|+1}{2\left\lfloor \frac{k\cdot|V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) < |V(G)| - 1. \end{cases}$$

So far, quite a few results have been obtained, mainly on the cases where k is small and the graphs we consider are special, such as trees [2, 7], cubic graphs [1] and complete graphs [1, 2, 5] when $k = 2, 3$. Chen and Huang [3] also determined $la_k(K_m)$ for $k \geq \lceil \frac{m}{2} \rceil - 1$ and $la_k(K_{n,n})$ for $k \geq n-1$. As for small k for $la_k(K_{n,n})$, only $k = 2$ and $k = 3$ were considered, see [4, 5, 12].

In this paper, we determine $la_3(K_{m(n)})$ when $mn \equiv 0 \pmod 4$. The result is coherent with the corresponding case of Conjecture 1.1.

2 Preliminary lemmas

Assume that G and H are graphs. A spanning subgraph F of G is called an *H-factor* if each component of F is isomorphic to H. If G is expressible as an edge-disjoint sum of H-factors, then this sum is called an *H-factorization* of G. Let P_λ be a path on λ vertices. From the meanings of P_k-factorization and linear $(k-1)$-arboricity of a graph, we know that if a graph G has a P_k-factorization then $la_{k-1}(G)$ is equal to $\frac{k\cdot|E(G)|}{(k-1)\cdot|V(G)|}$, which is the number of P_k-factors required to decompose G.

In 1999, Muthusamy and Paulraja [11] showed that for $k = p+1 > 3$, p is a prime, $K_{m(n)}$ has a P_k-factorization if and only if $mn \equiv 0 \pmod k$ and $2(k-1) \mid k(m-1)n$. Hence we obtain the following result on linear 3-arboricity of $K_{m(n)}$.

Corollary 2.1. *$la_3(K_{m(n)}) = \frac{2(m-1)n}{3}$ when $mn \equiv 0\ (mod\ 4)$ and $(m-1)n \equiv 0\ (mod\ 3)$.*

Furthermore, we say that a *1-factor* of a graph G is a spanning 1-regular subgraph of G. A 1-factor and a *perfect matching* are almost the same thing. The precise distinction is that "1-factor" is a spanning 1-regular subgraph of G, while "perfect matching" is the set of edges in such a subgraph. A decomposition of a regular graph G into 1-factors is a *1-factorization* of G. A graph with a 1-factorization is *1-factorable.* For complete graphs K_m, the following results are well-known.

Lemma 2.2. *[8] K_m has a K_4-factorization if and only if $m \equiv 4\ (mod\ 12)$.*

Lemma 2.3. *A complete graph with even order K_{2v} has a 1-factorization in which there are $2v-1$ 1-factors.*

Proof. See for instance [10]. □

Let $G(A, B)$ be a balanced bipartite graph with $A = \{a_j \mid j \in Z_n\}$ and $B = \{b_j \mid j \in Z_n\}$. In [5], Fu et al. defined the *bipartite difference* of an edge $a_p b_q$ in $G(A, B)$ by the value $q - p$ (mod n). It is not difficult to see that an edge subset in $G(A, B)$ containing the edges of the same bipartite difference must be a matching. In particular, the edge subset is also a perfect matching if $G(A, B)$ is $K_{n,n}$. Hence we can partition the edge set $E(K_{n,n})$ into n perfect matchings. Each perfect matching can be labelled by the bipartite difference of its own edges. For convenience, the perfect matching in $K_{n,n}$ consisting of the edges with bipartite difference ℓ is called "perfect matching ℓ", where $\ell \in \{0, 1, \ldots, n-1\}$. Note that the index of each vertex is modulo n.

Fu et al. [5] also observed that if n is even, then the edges of every three perfect matchings of $K_{n,n}$ with consecutive labels can generate two linear 3-forests. Otherwise, if n is odd, then the edges of every three perfect matchings of $K_{n,n}$ with consecutive labels can generate two linear 3-forests and one isolated edge. At last, they obtained the following theorem.

Theorem 2.4. *[5]*

$$la_3(K_{n,n}) = \left\lceil \frac{n^2}{\lfloor \frac{3n}{2} \rfloor} \right\rceil \quad and \quad la_3(K_m) = \left\lceil \frac{m(m-1)}{2\lfloor \frac{3m}{4} \rfloor} \right\rceil .$$

For example, Fig. 1 and Fig. 2 show that the edges of perfect matchings 0, 1, 2 in $K_{6,6}$ and $K_{7,7}$ can construct two linear 3-forests respectively except the edge a_6b_0 in $K_{7,7}$ is not used.

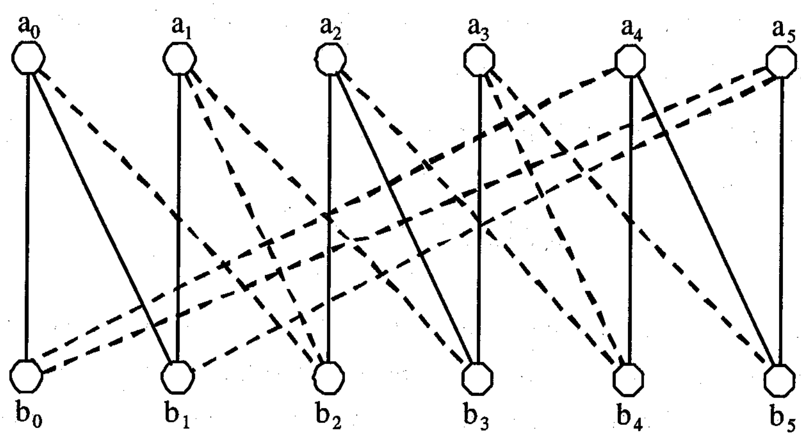

Figure 1: Two linear 3-forests in $K_{6,6}$.

The above statements are necessary to obtain our results. Furthermore, we also need some properties of $la_k(G)$.

Lemma 2.5. *If H is a subgraph of G, then $la_k(H) \leq la_k(G)$.*

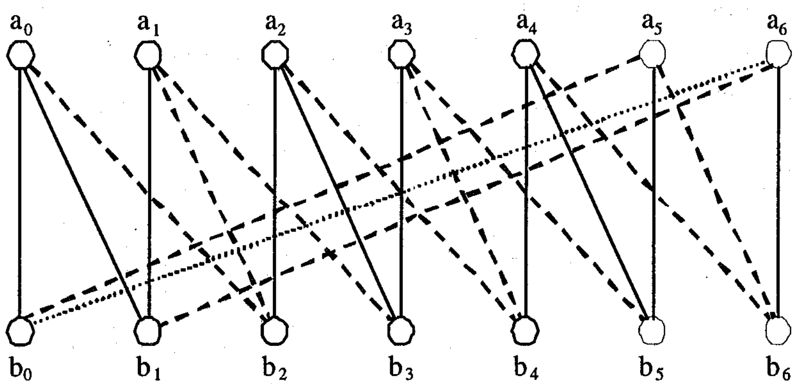

Figure 2: Two linear 3-forests and one isolated edge in $K_{7,7}$.

Lemma 2.6. *If a graph G is the edge-disjoint union of two graphs G_1 and G_2, then $la_k(G) \leq la_k(G_1) + la_k(G_2)$.*

Lemma 2.7. *If a graph G has an H-factorization with t H-factors, then $la_k(G) \leq t \cdot la_k(H)$.*

Lemma 2.8. $la_k(G) \geq \max\left\{\left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{|E(G)|}{\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor} \right\rceil\right\}$.

Lemmas 2.5 and 2.6 are evident by the definition of linear k-arboricity. Lemma 2.7 can be obtained from Lemma 2.6. We shall use Lemmas 2.5 $\sim$ 2.7 frequently and without an explicit reference. Since any vertex in a linear k-forest of a graph G has degree at most 2 and a linear k-forest of G has at most $\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor$ edges, we have Lemma 2.8.

3 The main results

Let $P_{\alpha(\beta)}$ be an α-partite graph such that each partite set V_i has β vertices for all $i \in \{0, 1, \ldots, \alpha - 1\}$ and the edge $uv \in E(P_{\alpha(\beta)})$ if and only if $u \in V_w$

and $v \in V_{w+1}$ where $w \in \{0, 1, \ldots, \alpha - 2\}$.

Lemma 3.1. *$la_k(P_{k+1(s)}) = s$.*

Proof. For all $i \in \{0, 1, \ldots, k\}$, assume that the vertices of the partite set V_i of $P_{k+1(s)}$ are $v_{i[0]}, v_{i[1]}, \ldots, v_{i[s-1]}$. Then, let the ℓth linear k-forest be the set of P_{k+1}'s $\{v_{0[j]}v_{1[j+(\ell-1)]} \cdots v_{k[j+k(\ell-1)]} | \, j \in \{0, 1, \ldots, s-1\}\}$ for all $\ell \in \{1, 2, \ldots, s\}$. Note that the index y of each vertex $v_{x[y]}$ is modulo s. It is not difficult to check that the edges of all linear k-forests are distinct and that their union is equal to the edge set $E(P_{k+1(s)})$. Thus $la_k(P_{k+1(s)}) = s$. □

Lemma 3.2. *$la_k(K_{m(tn)}) \leq t \cdot la_k(K_{m(n)})$.*

Proof. We can obtain $K_{m(tn)}$ from $K_{m(n)}$ by replacing each edge of $K_{m(n)}$ with $K_{t,t}$. Hence a path P_r in a linear k-forest of $K_{m(n)}$ corresponds to a r-partite subgraph $P_{r(t)}$ of $K_{m(tn)}$, where $2 \leq r \leq k+1$. From Lemma 2.5, $la_k(P_{r(t)}) \leq la_k(P_{k+1(t)})$ for all $2 \leq r \leq k+1$. Therefore, $la_k(K_{m(tn)}) \leq la_k(P_{k+1(t)}) \cdot la_k(K_{m(n)}) = t \cdot la_k(K_{m(n)})$ by Lemma 3.1. □

Lemma 3.3. *If $n \equiv 0 \pmod{2^\sigma}$ where $\sigma \geq 1$, then $K_{m(n)}$ has a $K_{\frac{n}{2^\sigma},\frac{n}{2^\sigma}}$-factorization and there are $2^\sigma(m-1)$ $K_{\frac{n}{2^\sigma},\frac{n}{2^\sigma}}$-factors in it.*

Proof. We prove this lemma by using induction on the number σ. Assume $\sigma = 1$. First, by partitioning each partite set of $K_{m(n)}$ into two subsets of $\frac{n}{2}$ vertices, we can find that $K_{2m(\frac{n}{2})}$ is the union of a $K_{\frac{n}{2},\frac{n}{2}}$-factor of $K_{2m(\frac{n}{2})}$ and $K_{m(n)}$. Then, from Lemma 2.3 (by replacing each edge of K_{2m} by $K_{\frac{n}{2},\frac{n}{2}}$), $K_{2m(\frac{n}{2})}$ has a $K_{\frac{n}{2},\frac{n}{2}}$-factorization in which there are $2m-1$ $K_{\frac{n}{2},\frac{n}{2}}$-factors. Therefore, $K_{m(n)}$ has a $K_{\frac{n}{2},\frac{n}{2}}$-factorization and there are $2m-2 = 2(m-1)$ $K_{\frac{n}{2},\frac{n}{2}}$-factors in it. This provides the basis.

For the induction step, suppose $\sigma = h+1 \geq 2$. The induction hypothesis is that $K_{m(n)}$ has a $K_{\frac{n}{2^h},\frac{n}{2^h}}$-factorization in which there are $2^h(m-1)$ $K_{\frac{n}{2^h},\frac{n}{2^h}}$-factors. Since a $K_{\frac{n}{2^h},\frac{n}{2^h}}$-factor can be decomposed into two $K_{\frac{n}{2^{h+1}},\frac{n}{2^{h+1}}}$-factors, then $K_{m(n)}$ has a $K_{\frac{n}{2^{h+1}},\frac{n}{2^{h+1}}}$-factorization and there are $2 \cdot 2^h(m-1) = 2^{h+1}(m-1)$ $K_{\frac{n}{2^{h+1}},\frac{n}{2^{h+1}}}$-factors in it. Hence, by mathematical induction, the assertion holds. □

Now, we are ready to prove our main results.

Proposition 3.4. *$la_3(K_{m(n)}) \le \left\lceil \frac{2(m-1)n}{3} \right\rceil$ when $m \equiv 0,4,6,8 \pmod{12}$ and $n \equiv 4 \pmod{6}$.*

Proof. From Lemma 2.3 (by replacing each edge of K_m by $K_{n,n}$), $K_{m(n)}$ has a $K_{n,n}$-factorization in which there are $m-1$ $K_{n,n}$-factors. Moreover, the edge set of $K_{n,n}$ can be partitioned into n perfect matchings whose labels are from 0 to $n-1$. Then the edges of perfect matchings $1, 2, \ldots, n-1$ can construct $\frac{2(n-1)}{3}$ linear 3-forests. Note that perfect matching 0 has not been used.

However, it is not difficult to see that the subgraph induced by the union of perfect matching 0 in those $K_{n,n}$ of $K_{n,n}$-factors in $K_{m(n)}$ is just a K_m-factor. Hence, $la_3(K_{m(n)}) \le (m-1) \cdot \frac{2(n-1)}{3} + la_3(K_m)$. By Theorem 2.4, $la_3(K_m) = \left\lceil \frac{m(m-1)}{2\lfloor \frac{3m}{4} \rfloor} \right\rceil = \left\lceil \frac{2m-2}{3} \right\rceil$ when $m \equiv 0,4,6,8 \pmod{12}$. Therefore, $la_3(K_{m(n)}) \le (m-1) \cdot \frac{2(n-1)}{3} + \left\lceil \frac{2m-2}{3} \right\rceil = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.5. *$la_3(K_{m(n)}) \le \left\lceil \frac{2(m-1)n}{3} \right\rceil$ when $m \equiv 2 \pmod{6}$ and $n \equiv 0 \pmod{2}$.*

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{2}$ disjoint pairs of two partite sets shows that $K_{m(n)}$ is the union of a $K_{n,n}$-factor of $K_{m(n)}$ and $K_{\frac{m}{2}(2n)}$. Therefore, $la_3(K_{m(n)}) \le la_3(K_{n,n}) + la_3(K_{\frac{m}{2}(2n)})$. Since $\frac{m}{2} \equiv 1 \pmod{3}$ and $2n \equiv 0 \pmod{4}$, from Corollary 2.1, $la_3(K_{\frac{m}{2}(2n)}) \le \frac{2(\frac{m}{2}-1)(2n)}{3} = \frac{(m-2)(2n)}{3}$. Thus, $la_3(K_{m(n)}) \le \left\lceil \frac{2n}{3} \right\rceil + \frac{(m-2)2n}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$ by Theorem 2.4. □

Proposition 3.6. $la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ *when* $m \equiv 0$ *(mod* 6) *and* $n \equiv 2$ *(mod* 6).

Proof. From Lemma 2.3 (by replacing each edge of K_m by $K_{n,n}$), $K_{m(n)}$ has a $K_{n,n}$-factorization and there are $m-1$ $K_{n,n}$-factors in it. Moreover, the edge set of $K_{n,n}$ can be partitioned into n perfect matchings whose labels are from 0 to $n-1$. Then we obtain $\frac{2(n-2)}{3}$ linear 3-forests which are constructed by the edges of perfect matchings $2, \ldots, n-1$. Assume that the vertices of $K_{n,n} = G(A,B)$ are $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_{n-1}$. The edges of perfect matchings 0 and 1 also produce a linear 3-forest $\{b_j a_j b_{j+1} a_{j+1} \mid j = 0, 2, \ldots, n-2\}$. But, the edges of the matching $\{a_j b_{j+1} \mid j = 1, 3, \ldots, n-1\}$ of $K_{n,n}$ have not been used. Thus we have to estimate the number of linear 3-forests induced by the union of the above edges which are not used in those $K_{n,n}$ of $K_{n,n}$-factors in $K_{m(n)}$.

First, for all $i \in \{0, 1, \ldots, m-1\}$, let the vertices of partite set V_i of $K_{m(n)}$ be denoted by $v_{i[0]}, v_{i[1]}, \ldots, v_{i[n-1]}$. Without loss of generality, we can assume that the set of all edges not used of $K_{m(n)}$ is the union of $\frac{m}{2} - 1$ perfect matchings $U_1, U_2, \ldots, U_{\frac{m}{2}-1}$, and a matching $X_{\frac{m}{2}}$, where

$$U_\ell = \left\{ v_{i[j]} v_{i+\ell[j+1]} \mid i \in \{0, 1, \ldots, m-1\}, j \in \{1, 3, \ldots, n-1\} \right\}$$

for all $\ell \in \{1, \ldots, \frac{m}{2} - 1\}$ and

$$X_{\frac{m}{2}} = \left\{ v_{i[j]} v_{i+\frac{m}{2}[j+1]} \mid i \in \{0, 1, \ldots, \tfrac{m}{2} - 1\}, j \in \{1, 3, \ldots, n-1\} \right\}.$$

Then the edges of $U_1, U_2, \ldots, U_{\frac{m}{2}-3}$ can generate $\frac{2(\frac{m}{2}-3)}{3}$ linear 3-forests. Besides, the edges of $U_{\frac{m}{2}-2}$, $U_{\frac{m}{2}-1}$, and $X_{\frac{m}{2}}$ also produce two linear 3-forests. Hence, $la_3(K_{m(n)}) \leq (m-1) \cdot \left(\frac{2(n-2)}{3} + 1 \right) + \left(\frac{2(\frac{m}{2}-3)}{3} + 2 \right) = \frac{2(m-1)n+1}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.7. *$la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ when $m \equiv 3 \pmod 6$ and $n \equiv 4$ $\pmod{12}$.*

Proof. From Lemma 3.3, $K_{m(n)}$ has a $K_{\frac{n}{2},\frac{n}{2}}$-factorization and there are $2m-2$ $K_{\frac{n}{2},\frac{n}{2}}$-factors in it. Since the edge set of $K_{\frac{n}{2},\frac{n}{2}}$ can be partitioned into $\frac{n}{2}$ perfect matchings whose labels are from 0 to $\frac{n}{2}-1$, we obtain $\frac{2(\frac{n}{2}-2)}{3}$ linear 3-forests which are constructed by the edges of perfect matchings $2, \ldots, \frac{n}{2}-1$. Assume that the vertices of $K_{\frac{n}{2},\frac{n}{2}} = G(A,B)$ are $a_0, a_1, \ldots, a_{\frac{n}{2}-1}$ and $b_0, b_1, \ldots, b_{\frac{n}{2}-1}$. The edges of perfect matchings 0 and 1 also produce a linear 3-forest $\left\{b_j a_j b_{j+1} a_{j+1} \middle| \, j = 0, 2, \ldots, \frac{n}{2}-2\right\}$. But, the edges of the matching $\left\{a_j b_{j+1} \middle| \, j = 1, 3, \ldots, \frac{n}{2}-1\right\}$ of $K_{\frac{n}{2},\frac{n}{2}}$ have not been used. Thus we have to estimate the number of linear 3-forests induced by the union of the above edges which are not used in those $K_{\frac{n}{2},\frac{n}{2}}$ of $K_{\frac{n}{2},\frac{n}{2}}$-factors in $K_{m(n)}$. Since $K_{2m(\frac{n}{2})}$ is the union of a $K_{\frac{n}{2},\frac{n}{2}}$-factor of $K_{2m(\frac{n}{2})}$ and $K_{m(n)}$, for convenience, we can consider this question on the graph $K_{2m(\frac{n}{2})}$.

First, for all $i \in \{0, 1, \ldots, 2m-1\}$, let the vertices of partite set V_i of $K_{2m(\frac{n}{2})}$ be denoted by $v_{i[0]}, v_{i[1]}, \ldots, v_{i[\frac{n}{2}-1]}$. Without loss of generality, we can assume that the set of all edges not used in $K_{m(n)}$ is the union of two matchings X_1, X_m, and $m-2$ perfect matchings $U_2, U_3, \ldots, U_{m-1}$ of $K_{2m(\frac{n}{2})}$, where

$$\begin{aligned} X_1 = & \left\{v_{i[j]}v_{i+1[j+1]} \middle| \, i \in \{1, 3, \ldots, 2m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{2}-1\}\right\}, \\ U_\ell = & \left\{v_{i[j]}v_{i+\ell[j+1]} \middle| \, i \in \{0, 1, \ldots, 2m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{2}-1\}\right\} \end{aligned}$$

for all $\ell \in \{2, 3, \ldots, m-1\}$ and

$$X_m = \left\{v_{i[j]}v_{i+m[j+1]} \middle| \, i \in \{0, 2, \ldots, 2m-2\}, j \in \{1, 3, \ldots, \tfrac{n}{2}-1\}\right\}.$$

Then (i) the edges of X_1 and U_2 can produce a linear 3-forest; (ii) the edges of $U_3, U_4, \ldots, U_{m-1}$ can generate $\frac{2(m-3)}{3}$ linear 3-forests; (iii) the edges of X_m

can produce a linear 3-forest. Hence, $la_3(K_{m(n)}) \leq (2m-2)\cdot\left(\frac{2(\frac{n}{2}-2)}{3}+1\right)+\left(2+\frac{2(m-3)}{3}\right)=\frac{2(m-1)n+2}{3}=\left\lceil\frac{2(m-1)n}{3}\right\rceil$. $\square$

Proposition 3.8. *$la_3(K_{m(n)}) \leq \left\lceil\frac{2(m-1)n}{3}\right\rceil$ when $m \equiv 5$ (mod 6) and $n \equiv 4$ (mod 12).*

Proof. It is similar to the proof of Proposition 3.7 except the following. (i) The edges of X_1 and X_m can produce a linear 3-forest; (ii) the edges of $U_2, U_3, \ldots, U_{m-1}$ can generate $\frac{2(m-2)}{3}$ linear 3-forests. Hence, $la_3(K_{m(n)}) \leq (2m-2)\cdot\left(\frac{2(\frac{n}{2}-2)}{3}+1\right)+\left(1+\frac{2(m-2)}{3}\right)=\frac{2(m-1)n+1}{3}=\left\lceil\frac{2(m-1)n}{3}\right\rceil$. $\square$

Proposition 3.9. *$la_3(K_{m(n)}) \leq \left\lceil\frac{2(m-1)n}{3}\right\rceil$ when $m \equiv 3$ (mod 6) and $n \equiv 8$ (mod 12).*

Proof. From Lemma 3.3, $K_{m(n)}$ has a $K_{\frac{n}{4},\frac{n}{4}}$-factorization and there are $4m-4$ $K_{\frac{n}{4},\frac{n}{4}}$-factors in it. Since the edge set of $K_{\frac{n}{4},\frac{n}{4}}$ can be partitioned into $\frac{n}{4}$ perfect matchings whose labels are from 0 to $\frac{n}{4}-1$, we obtain $\frac{2(\frac{n}{4}-2)}{3}$ linear 3-forests which are constructed by the edges of perfect matchings $2, \ldots, \frac{n}{4}-1$. Assume that the vertices of $K_{\frac{n}{4},\frac{n}{4}}=G(A,B)$ are $a_0, a_1, \ldots, a_{\frac{n}{4}-1}$ and $b_0, b_1, \ldots, b_{\frac{n}{4}-1}$. The edges of perfect matchings 0 and 1 also produce a linear 3-forest $\{b_j a_j b_{j+1} a_{j+1} \mid j=0,2,\ldots,\frac{n}{4}-2\}$. But, the edges of the matching $\{a_j b_{j+1} \mid j=1,3,\ldots,\frac{n}{4}-1\}$ of $K_{\frac{n}{4},\frac{n}{4}}$ have not been used. Thus we have to estimate the number of linear 3-forests induced by the union of the above edges which are not used in those $K_{\frac{n}{4},\frac{n}{4}}$ of $K_{\frac{n}{4},\frac{n}{4}}$-factors in $K_{m(n)}$. Since $K_{4m(\frac{n}{4})}$ is the union of three $K_{\frac{n}{4},\frac{n}{4}}$-factors of $K_{4m(\frac{n}{4})}$ and $K_{m(n)}$, for convenience, we can consider this question on the graph $K_{4m(\frac{n}{4})}$.

First, for all $i \in \{0,1,\ldots,4m-1\}$, let the vertices of partite set V_i of $K_{4m(\frac{n}{4})}$ be denoted by $v_{i[0]}, v_{i[1]}, \ldots, v_{i[\frac{n}{4}-1]}$. Without loss of generality, we can assume that the set of all edges not used in $K_{m(n)}$ is the union of four matchings X_1, X_2, X_3, X_{2m} and $2m-4$ perfect matchings $U_4, U_5, \ldots, U_{2m-1}$ of $K_{4m(\frac{n}{4})}$, where

$$X_1 = \left\{v_{i[j]}v_{i+1[j+1]} \middle| \, i \in \{3, 7, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\}\right\},$$

$$X_2 = \left\{v_{i[j]}v_{i+2[j+1]} \middle| \, i \in \{2, 3, 6, 7, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\}\right\},$$

$$X_3 = \left\{v_{i[j]}v_{i+3[j+1]} \middle| \, i \in \{1, 2, 3, 5, 6, 7, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\}\right\},$$

$$U_\ell = \left\{v_{i[j]}v_{i+\ell[j+1]} \middle| \, i \in \{0, 1, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\}\right\}$$

for all $\ell \in \{4, 5, \ldots, 2m-1\}$ and

$$X_{2m} = \left\{v_{i[j]}v_{i+2m[j+1]} \middle| \, i \in \{0, 1, 4, 5, \ldots, 4m-3\}, j \in \{1, 3, \ldots, \tfrac{n}{4}-1\}\right\}.$$

Then (i) the edges of X_1, a subset $\{v_{i[j]}v_{i+3[j+1]} | \, i \in \{2, 6, \ldots, 4m-2\}, j \in \{1, 3, \ldots, \frac{n}{4}-1\}\}$ of X_3 and U_4 can produce a linear 3-forest; (ii) the edges of X_2, a subset $\{v_{i[j]}v_{i+3[j+1]} | \, i \in \{1, 3, \ldots, 4m-1\}, j \in \{1, 3, \ldots, \frac{n}{4}-1\}\}$ of X_3 and X_{2m} can produce a linear 3-forest; (iii) the edges of $U_5, U_6, \ldots, U_{2m-2}$ can generate $\frac{2(2m-6)}{3}$ linear 3-forests; (iv) the edges of U_{2m-1} can produce a linear 3-forest. Hence, $la_3(K_{m(n)}) \le (4m-4) \cdot \left(\frac{2(\frac{n}{4}-2)}{3} + 1\right) + \left(3 + \frac{2(2m-6)}{3}\right) = \frac{2(m-1)n+1}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.10. *$la_3(K_{m(n)}) \le \left\lceil \frac{2(m-1)n}{3} \right\rceil$ when $m \equiv 5$ (mod 6) and $n \equiv 8$ (mod 12).*

Proof. It is similar to the proof of Proposition 3.9 except the following. (i) The edges of X_1 and X_3 can produce a linear 3-forest; (ii) the edges of X_2 and U_4 can produce a linear 3-forest; (iii) the edges of $U_5, U_6, \ldots, U_{2m-1}$, and X_{2m} can generate $\frac{2(2m-4)}{3}$ linear 3-forests. Hence, $la_3(K_{m(n)}) \le (4m-4) \cdot \left(\frac{2(\frac{n}{4}-2)}{3} + 1\right) + \left(2 + \frac{2(2m-4)}{3}\right) = \frac{2(m-1)n+2}{3} = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

Proposition 3.11. $la_3(K_{m(n)}) \leq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ *when* $m \equiv 0$ *or* 8 (*mod* 12) *and* $n \equiv 1$ *or* 5 (*mod* 6).

Proof. Dividing all m partite sets of $K_{m(n)}$ into $\frac{m}{4}$ disjoint collections of four partite sets shows that $K_{m(n)}$ is the union of a $K_{4(n)}$-factor of $K_{m(n)}$ and $K_{\frac{m}{4}(4n)}$. Since $\frac{m}{4} \equiv 0$ or 2 (mod 3) and $4n \equiv 4$ or 8 (mod 12), from Corollary 2.1 and Propositions 3.4 $\sim$ 3.10, $la_3(K_{m(n)}) \leq la_3(K_{4(n)}) + la_3(K_{\frac{m}{4}(4n)}) \leq \frac{2(4-1)n}{3} + \left\lceil \frac{2(\frac{m}{4}-1)(4n)}{3} \right\rceil = \left\lceil \frac{2(m-1)n}{3} \right\rceil$. □

On the other hand, from Lemma 2.8, $la_3(K_{m(n)}) \geq \left\lceil \frac{2(m-1)n}{3} \right\rceil$ when $mn \equiv 0$ (mod 4). Hence, by combining Corollary 2.1 and the above propositions, we determine the linear 3-arboricity of $K_{m(n)}$ when $mn \equiv 0$ (mod 4) and conclude the work of this paper with the following main theorem.

Theorem 3.12. $la_3(K_{m(n)}) = \left\lceil \frac{2(m-1)n}{3} \right\rceil$ *when* $mn \equiv 0$ (*mod* 4).

Concluding Remark. By using the ideas in this paper, we can also find $la_3(K_{m(n)})$ for quite a few other cases when $mn \equiv 2$ (mod 4). But, we are not able to finish the whole part at this moment due to several stubborn subcases. As for the cases when mn is odd, they are expected to be more difficult.

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