Edge Number of 3-Connected Diameter 3 Graphs

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Abstract

Let the decay number, $\varsigma(G)$ be the minimum number of components of a cotree of a connected graph G. Let Ω be the collection of all 3-connected diameter 3 graphs. In this paper, we prove that if k is the minimum number such that $q \geq 2p-k$ for each $(p,q)-$graph $G \in \Omega$, and l is the minimum number such that $\varsigma(H) \leq l-1$ for each graph $H \in \Omega$, then $k=l$. Furthermore, we prove that $k \leq 11$ and we find a 3-connected, diameter 3 graph with $q=2p-8$. So we have that $8 \leq k \leq 11$ and we conjecture that $k=8$.

Keywords: connectivity, diameter, decay number, edge number

1. Introduction

Throughout this paper, a graph may have multiple edges or loops, but a simple graph contains neither multiple edges nor loops.

For interconnection networks, topologies of network usually mean architectures of networks and graph theory provides the theoretical basis. A study of a network with lower cost, shorter delay in transmission and high reliability is equivalent to find a graph with less edges, small diameter and higher connectivity. This is what we focusing in this study.

In [4] Škoviera defined the decay number of G, $\varsigma(G)$, to be the minimum number of components of a cotree of a connected graph G. For a 2-connected, diameter 2 graph G, Škoviera [4] gave a tight upper bound on $\varsigma(G)$.

Theorem 1.1 *If G is a 2-connected, diameter 2 graph, then*

$$\varsigma(G) \leq 4.$$

On the other hand, Nebesky [3] discovered a formula to calculate $\varsigma(G)$.

Theorem 1.2 *For any connected graph G,*

$$\varsigma(G) = \max\{2c(G-A) - |A| - 1 \mid A \subseteq E(G)\}.$$

In [1], they showed that if one take $A = E(G)$ in Theorem 1.2, then Theorem 1.1 can imply the following theorem which was found by Murty [2] in 1969. Furthermore, they proved that the extremal simple graphs of Theorem 1.1 and Theorem 1.3 are the same.

Theorem 1.3 *If G is a 2-connected, diameter 2 $(p,q)-$graph, then*

$$q \geq 2p-5.$$

In Theorem 1.2, if we define a new graph G/A by contracting each component of $G-A$ into a vertex, then it is easy to see that G/A is also 2-connected, diameter 2 if G is 2-connected, diameter 2, and then Theorem 1.3 can also imply Theorem 1.1. That is, Theorem 1.1 and Theorem 1.3 are equivalent. This concept intrigues us to

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consider 3-connected graphs of diameter 3. Thus our purpose in this paper is to find the minimum number k such that if G is a 3-connected, $(p,q)-$graph diameter 3, then $q \geq 2p-k$.

2. The Main Results

Let G be a connected graph. Fu, Tsai and Xuong first introduced in [1] that a subset A of $E(G)$ is $\varsigma-$minimal if $\varsigma(G)=2c(G-A)-|A|-1$ and for each $B \subset A$, $\varsigma(G)>2c(G-B)-|B|-1$; and a subset A of $E(G)$ is $E-$minimal if any two different components of $G-A$ are joined by at most one edge of A in G. By Remark 3.1 of [1], any $\varsigma-$minimal subset of $E(G)$ is also an $E-$minimal subset of $E(G)$. Furthermore, we let G/A be the graph obtained from G by contracting each component of $G-A$ into a vertex. We are now ready to describe the main results.

Theorem 2.1 *Let Ω be the collection of all 3-connected diameter 3 graphs. If k is the minimum number such that $q \geq 2p-k$ for each $(p,q)-$graph $G \in \Omega$, and l is the minimum number such that $\varsigma(H) \leq l-1$ for each graph $H \in \Omega$, then $k=l$.*

Proof. Let $G \in \Omega$ be a $(p,q)-$graph with $q=2p-k$, and let $A=E(G)$. By Theorem 1.2, we have

$$k-1=2p-q-1 \leq \varsigma(G) \leq l-1.$$

This implies that $k \leq l$.

On the other hand, let $H \in \Omega$ be a graph with $\varsigma(H)=l-1$, and let A be an $\varsigma-$minimal subset of $E(G)$. We have that $\varsigma(H)=2c(H-A)-|A|-1$ and A is also an $E-$minimal subset of $E(G)$. Since H is a 3-connected graph of diameter 3, H/A is also a 3-connected graph of diameter 3. This implies that

$$l-1=\varsigma(H)=2p'-q'-1 \leq k-1,$$

where $p'=V(H/A)$ and $q'=E(H/A)$. Thus $l \leq k$ and we complete the proof.

This theorem implies that to find the minimum number k such that $q \geq 2p-k$ for each 3-connected, diameter 3 $(p,q)-$graph G is equivalent to find the minimum number l such that $\varsigma(H) \leq l-1$ for each 3-connected, diameter 3 graph H.

Theorem 2.2 *If G is a diameter 3 $(p,q)-$graph with minimum degree $\delta(G) \geq 3$, then*

$$q \geq 2p-11.$$

Proof. If $\delta(G) \geq 4$, then

$$q \geq \frac{1}{2}\sum_{u \in V(G)} \deg u \geq \frac{1}{2} \times 4p = 2p > 2p-11.$$

So it is sufficient to consider that $\delta(G)=3$. Accordingly, let z be a vertex of G with $\deg z=3$. And we denote $N_0(z)=\{z\}$ and $N_i=N_i(z)=\{u \in V(G) \mid d(z,u)=i\}$, $i=1,2,3$. Thus if $\deg u \geq 4$ for each $u \in N_3(z)$, then

$$\begin{aligned}
2q &= \sum_{u \in V(G)} \deg u \\
&= \deg z + \sum_{u \in N_1} \deg u + \sum_{u \in N_2} \deg u + \sum_{u \in N_3} \deg u \\
&\geq 3+(3+|N_2|)+3|N_2|+4|N_3| \\
&= 6+4(|N_2|+|N_3|) \\
&= 6+4(p-4) \\
&= 4p-10
\end{aligned}$$

This implies $q \geq 2p-5 > 2p-11$.

Therefore we only need to consider that $\deg v=3$ for some $v \in N_3$. Let $N_0(v)=\{v\}$ and $N_i(v)=\{u \in V(G) \mid d(v,u)=i\}$ $i=1,2,3$. Clearly, $|N_1(v)|=3$ and $|N_1(v) \cap N_2| \geq 1$. Next, let $D=\{u \in N_3 \cap N_3(v) \mid \deg u=3\} =\{u_1,u_2,\cdots,u_n\}$. If $D=\phi$, then let $E_0=\phi$ and $D'=\phi$. Otherwise, we apply Algorithm 1 to find an edge subset E_0 of $E(G)$ and a vertex subset D' of D.

Algorithm 1.

1. Set $i=1$, $E_0=\phi$ and $D'=\phi$.
2. Select an edge $u_iv_i \in E(G)$, where v_i satisfies one of the following properties:
   (1) $v_i \in N_2 \cap N_2(v)$ and $\deg_{G-E_0} v_i \geq 3$,
   (2) $v_i \in N_2 \cup N_2(v)$ and $\deg_{G-E_0} v_i \geq 4$.
   (3) $v_i \in N_3 \cap N_3(v)$ and $\deg_{G-E_0} v_i \geq 5$.
   Then replace E_0 by $E_0 \cup \{u_iv_i\}$. If there is no such edge, replace D' by $D' \cup \{u_i\}$.
3. If $i=n$, then stop. Otherwise, replace i by $i+1$ and go to step 2.

Now in order to figure out the edge number of G, we define a direct graph $\vec{G}$ corresponding to G as follows.

(1) $V(\vec{G})=V(G)$,

(2)if $e=xy \in E_1=\{xy \in E(G) \mid x \in N_i$ and $y \in N_{i+1}, i=0$ or $1\}$, then we join two arcs from x to y,

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(3)if $e = xy \in E_2 = \{xy \in E(G) \mid x \in N_i(v)$ and $y \in N_{i+1}(v), i = 0$ or $1\}$, then we join two arcs from x to y,

(4)if $e = xy \in E_0$ and $x \in D_3$, then we join two arcs from x to y,

(5)if $e = xy \in E_3 = E(G) \setminus (E_0 \cup E_1 \cup E_2)$, then we let (x, y) and (y, x) be arcs of $\vec{G}$.

By the definition of $\vec{G}$, it is easy to see that each edge of G is corresponding to two arcs of $\vec{G}$ and

$$\sum_{x \in V(G)} \deg x = \sum_{x \in V(\vec{G})} \deg^-(x)$$

where $\deg^- x$ denote the indegree of x in $\vec{G}$. Therefore, the indegree sum of $\vec{G}$ equals to $2q$.

Now we count the indegree sum of $N_i(z), i = 0, 1, 2, 3$ as follows.

(1) $\deg^-(z) = 0$

(2)For any $u \in N_1$, $\deg^-(u) \geq 2$. This implies that

$$\sum_{u \in N_1(z)} \deg^-(u) \geq |N_1| \cdot 2 = 6.$$

(3)For any $u \in N_2$ there exists at least a vertex $u' \in N_1$ such that $uu' \in E_2$ and $\deg u \geq 3$, hence $\deg^-(u) \geq 4$. This implies that

$$\sum_{u \in N_2(z)} \deg^-(u) \geq |N_2| \cdot 4 = 4|N_2|$$

(4)For any $u \in N_3$,

(i)if $u \in N_3 \cap N_1(v)$, then $uv \in E_2$ and there exists at least a vertex $u' \in N_2(v)$ and a vertex $u'' \in N_2$ such that $uu', uu'' \in E_2$. Thus $\deg^- u \geq 3$.

(ii)if $u \in N_3 \cap N_2(v)$, then there exists at least a vertex $u' \in N_1(v)$ such that $uu' \in E_2$ and $\deg u \geq 3$. Thus $\deg^- u \geq 4$.

(iii)if $u \in N_3 \cap N_3(v)$ and there exists a vertex v such that $uv \in E_0$, then it is clear that $\deg^- u \geq 4$.

By the above counting, we have that $\deg^-(u) \geq 4$ for each vertex $v \in V(G) \setminus D'$. Now we claim that $|D'| \leq 6$.

Let $w \in D'$. For any vertex $x \in N(w) = \{y \mid xy \in E(G)\}$, $x \notin N_2 \cap N_2(v)$. Thus we can let $N(w) = \{x_1, x_2, x_3\}$ and $x_1 \in N_2$, $x_2 \in N_2(v)$ and let $D'(x_i) = \{u \in D' \mid d(x_i, u) < d(w, u) \leq 3\}$. Now we count $|D'(x_i)|$ for $i = 1,2,3$. First, since $x_1 \in N_2$ and $d(x_1, v) = 3$, the neighbors of x_1 must be w, one of N_1 (let this vertex be z_1) and one of $N_2(v)$ (let this vertex be v_1). It is clear that $N(z_1) \cap D' = \phi$. On the other hand v_1 must be adjacent to x_1 and one of $N_1(v)$. Hence there is at most one vertex in $N(v_1) \cap D'$. Accordingly, $|D'(x_1)| \leq 1$. Similarly, $|D'(x_2)| \leq 1$. Next, if $x_3 \in N_2$ or $N_2(v)$, then $|D'(x_3)| \leq 1$. Otherwise, $x_3 \in N_3 \cap N_3(v)$. Thus we can let $N(x_3) = \{w, y_1, y_2\}$ and $y_1 \in N_2$. Since y_1 must be adjacent to x_3 and one of N_1. Hence there is at most one vertex in $(N(y_1) \setminus \{x_3\}) \cap D'$. On the other hand, y_2 must be adjacent to x_3 and one of N_1 or N_2. Hence there is at most one vertex in $(N(y_2) \setminus \{x_3\}) \cap D'$. Thus

$$\begin{aligned} |D'(x_3)| &\leq |\{x_3\}| + |(N(y_1) \setminus \{x_3\}) \cap D'| \\ &\quad + |(N(y_2) \setminus \{x_3\}) \cap D'| \\ &\leq 3 \end{aligned}.$$

By the above counting, we have

$$|D'| = |\{w\}| + \sum_{i=1}^{3} |D'(x_i)| \leq 6.$$

By this claim, we have

$$\begin{aligned} \sum_{u \in N_3(z)} \deg^-(u) &= \deg^-(v) + \sum_{u \in N_3(z) \cap N_1(v)} \deg^-(u) \\ &\quad + \sum_{u \in N_3(z) \cap N_2(v)} \deg^-(u) + \sum_{u \in N_3(z) \cap N_3(v)} \deg^-(u) \\ &\geq 0 + 3|N_3 \cap N_1(v)| + 4|N_3 \cap N_2(v)| \\ &\quad + 4|N_3 \cap N_3(v)| - |D'| \\ &\geq 4|N_3| - 4 - |N_3 \cap N_1(v)| - 6 \\ &\geq 4|N_3| - 4 - 3 - 6 \\ &= 4|N_3| - 13 \end{aligned}$$

Combine (1), (2), (3) and (4),

$$\begin{aligned} 2q = \sum_{u \in V(\vec{G})} \deg(u) &= \deg^-(z) + \sum_{u \in N_1(z)} \deg^- u \\ &\quad + \sum_{u \in N_2(z)} \deg^- u + \sum_{u \in N_3(z)} \deg^- u \\ &\geq 6 + 4|N_2| + 4|N_3| - 13 \\ &= 4(p-4) + 6 - 13 \\ &= 4p - 23 \end{aligned}$$

This implies that $q \geq 2p - 11$.

By Theorem 2.2, we have Corollary 2.3 and Corollary 2.4.

Corollary 2.3 *Let Ω be the collection of all 3-connected diameter 3 graphs. If k is the minimum*

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number such that $q \geq 2p-k$ *for each* $(p,q)-$ *graph* $G \in \Omega$*, then* $k \leq 11$.

Corollary 2.4 *Let* Ω *be the collection of all* 3*-connected diameter* 3 *graphs. If* l *is the minimum number such that* $\varsigma(H) \leq l-1$ *for each graph* $H \in \Omega$*, then* $l \leq 11$.

In Corollary 2.3 and Corollary 2.4, we want to improve it to a sharp bound. Now we find a 3-connected, diameter 3 graph with $q = 2p-8$ as the following figure.

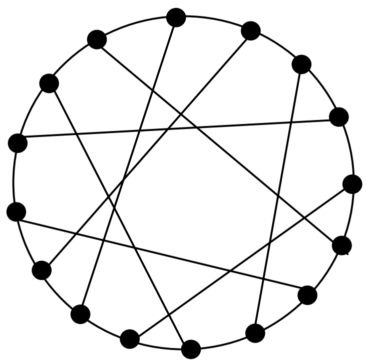

Figure 1.

So we have the following result.

Theorem 2.5 *Let* Ω *be the collection of all* 3*-connected diameter* 3 *graphs. If* k *is the minimum number such that* $q \geq 2p-k$ *for each* $(p,q)-$ *graph* $G \in \Omega$*, then*

$$8 \leq k \leq 11.$$

3. Concluding Remark

We have been trying to find a 3-connected, diameter 3 graph with $2p-k$ edges where $12 > k > 8$, but not successful in constructing one at this moment. It seems to us that $k = 8$ in Theorem 2.5 will be the right one. If this is determined, then we have an analog of the previous work on 2-connected and diameter 2 graphs stated in Theorem 1.3.

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