Prime Labellings of Trees

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Abstract

Let G be a simple and finite graph. A bijection from its vertex set onto $\{1, 2, \ldots, |G|\}$ is called a prime labelling of G if any two adjacent vertices are labelled by coprime integers. Entringer conjectured that every tree has a prime labelling. In this paper, we show that a tree $T_n = (A, B)$ of order $n \geq 105$ with bipartition (A, B) satisfying $|A| \leq \pi(n)$ has a prime labelling, where $\pi(n)$ is the number of primes at most n.

Keywords: Prime labelling, Hall Theorem, System of distinct representatives (SDR), Tree, Matching.

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1 Introduction

Let m and n be positive integers with $m \leq n$. The set $\{m, m+1, \ldots, n\}$ of integers from m to n is denoted by $[m, n]$. If $m = 1$, we use $[n]$ to replace $[1, n]$ for short. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ be a collection of subsets of $[n]$. A system of distinct representatives (SDR) for $\mathcal{S}$ is a subset $\{q_1, q_2, \ldots, q_k\}$ such that $q_i \in S_i$ and $q_i \neq q_j$ for all $i \neq j$.

Let $G = (V, E)$ be a simple and finite graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. The number of vertices of G is called the order of G and denoted by $|G|$. For a vertex $x \in V$, a neighbor of x is a vertex adjacent to x. The set of neighbors of x is called the neighborhood of x and denoted by $N(x)$. The number of neighbors of x is called the degree of x and denoted by $\deg(x)$. A pendent vertex is a vertex of degree one. For a set $S \subseteq V$, the neighborhood of S is the set of vertices adjacent to some vertex of S and denoted by $N(S)$. A tree is a connected graph without cycles. A matching is a subset of $E(G)$ in which any two edges have no endpoints in common.

A bijection $\varphi : V(G) \to \{1, 2, \ldots, |G|\}$ is called a prime labelling of G if any two adjacent vertices are labelled by coprime integers. Around 1980, Entringer conjectured that every tree has a prime labelling. So far, the conjecture is still unsolved. In 1994, Fu and Huang [4] showed that every tree of order $n \leq 15$ has a prime labelling. In 1998, Lin [8] extended Fu and Huang's result to a tree of order $n \leq 105$. Recently, Haxell, Pikhurko and Taraz [6] proved that the Entringer Conjecture is true provided the order n is sufficiently large. On the other hand, the conjecture has been verified for some classes of trees (complete binary trees, caterpillars, star-like trees, spider trees, spider colonies, binomial trees, palm trees, banana trees, etc in [4, 7, 9]). In this paper, we show that a tree $T_n(A, B)$ of order $n \geq 105$ with bipartition (A, B) satisfying $|A| \leq \pi(n)$ has a prime labelling, where $\pi(n)$ is the number of primes at most n.

First, we consider that $\pi(n) = |A| \leq |B| = n - \pi(n)$. At the beginning, we label the vertices of $A = \{a_1, a_2, \ldots, a_{\pi(n)}\}$ with $\deg(a_1) \geq \deg(a_2) \geq \cdots \geq \deg(a_{\pi(n)})$ by $p_1 = \varphi(a_1) = 1, p_2 = \varphi(a_{\pi(n)}) = 3, p_3 = \varphi(a_{\pi(n)-1}), \ldots, p_{\pi(n)} = \varphi(a_2)$, where $p_2 < p_3 < \cdots < p_{\pi(n)}$ are odd primes in $[n]$. Let $L(A) = \{\varphi(a_i) \mid a_i \in A\}$. For each $b_j \in B$, there is a corresponding set $S_j \subseteq [n] \setminus L(A)$ consisting of the integers which can be used to label b_j. We can argue that the collection $\{S_1, S_2, \ldots, S_{|B|}\}$ has an SDR $\{q_1, q_2, \ldots, q_{|B|}\}$. Label each b_j by q_j. Then the tree $T_n = (A, B)$ has a prime labelling. By the same argument, we can deal with the case that $|A| < \pi(n)$.

2 Main Results

In this section, we first assume that $T_n = (A, B)$ is a tree of order $n \geq 105$, $A = \{a_1, a_2, \ldots, a_{\pi(n)}\}$, $\deg(a_1) \geq \deg(a_2) \geq \cdots \geq \deg(a_{\pi(n)})$, $B = \{b_1, b_2, \ldots, b_{|B|}\}$, $\deg(b_1) \geq \deg(b_2) \geq \cdots \geq \deg(b_{|B|})$, $|A| = \pi(n)$ and $|B| = n - \pi(n)$. Moreover, let the vertices of A be labelled by $p_1 = \varphi(a_1) = 1, p_2 = \varphi(a_{\pi(n)}) = 3, p_3 = \varphi(a_{\pi(n)-1}) = 5, \ldots, p_{\pi(n)} = \varphi(a_2)$, where $p_2 < p_3 < \cdots < p_{\pi(n)}$ are odd primes in $[n]$, $L(A) = \{\varphi(a_i) \mid a_i \in A\}$, $L(A') = \{\varphi(a) \mid a \in A'\}$ for $A' \subseteq A$, $R = [n] \setminus L(A)$ and for each $b_j \in B$, define $S_j = \{k \mid k \in R \setminus \{tp \mid p \in L(N(b_j)), t \geq 1\}\}$. Note that $\gcd(k, \varphi(a)) = 1$ for all $k \in S_j$ and $a \in N(b_j)$. We use the label $\varphi(a_i)$ to replace the vertex a_i if no confusion occurs.

The following lemmas are essential for main results.

Lemma 1. *[1, 2, 3] Let $\pi(n)$ be the number of primes in $[n]$. Then the following hold.*

(1) $\pi(n) > n/\ln n$ *if* $n > 16$.

(2) $\pi(n) < \frac{n}{\ln n}(1 + \frac{2}{3\ln n})$ *if* $n > 1$.

(3) $\pi(n) < 2 \cdot \pi(n/2)$ *if* $n > 21$.

(4) $\pi(n) \leq 2 \cdot \pi(n/2)$ *if* $n > 1$.

(5) $\pi(n) < n/3$ *if* $n > 33$.

Lemma 2. *[5] An SDR for a collection $\{S_1, S_2, \ldots, S_t\}$ exists if and only if the Hall's condition $|\bigcup_{j=1}^{k} S_{i_j}| \geq k$ holds for subcollections $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$, $k \geq 1$.*

Lemma 3. *Let $T = (X, Y)$ be a tree with $|X| \leq |Y|$. Then Y contains at least $|Y| - |X| + 1$ pendent vertices.*

Proof. Suppose not, that is, Y has at most $|Y| - |X|$ pendent vertices. Then $|E(T)| = \sum_{y \in Y} \deg(y) \geq (|Y| - |X|) + 2(|Y| - (|Y| - |X|)) = |X| + |Y| = |T| = |E(T)| + 1$. A contradiction. Hence, the assertion holds. □

For checking the Hall's condition $|\bigcup_{j=1}^{k} S_{i_j}| \geq k$ for all $k \geq 1$, we need to study the cardinality $|\bigcup_{j=1}^{k} S_{i_j}|$. If $\deg(b_j) = 1$ and $L(N(b_j)) = \{p\}$, then either $|S_j| = |R \setminus \{tp \mid t \geq 1\}| = |R| - \lfloor n/p \rfloor + 1 = n - \pi(n) - \lfloor n/p \rfloor + 1$ if $p > 1$, or $|S_j| = |R| = n - \pi(n)$ if $p = 1$. For $i \neq j$, let $L(N(b_i)) = \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}$ and $L(N(b_j)) = \{p_{j_1}, p_{j_2}, \ldots, p_{j_\ell}\}$. If $N(b_i) \cap N(b_j) = \emptyset$, since $\gcd(p_{i_\alpha}, p_{j_\beta}) = 1$, $2^r p_{i_\alpha} \in S_j, 2^r p_{j_\beta} \in S_i$ and $2^r q \in S_i \cap S_j$ for $1 \leq \alpha \leq k, 1 \leq \beta \leq \ell, q \in L(A) \setminus (L(N(b_i) \cup L(N(b_j))$ and $r \geq 1$. Hence, $\bigcup_{r\geq 1}\{2^r p | p \in L(A) \text{ and } 2^r p \leq n\} \subseteq S_i \cup S_j$. If $N(b_i) \cap N(b_j) \neq \emptyset$, since T_n is a tree, $N(b_i) \cap N(b_j) = \{q\}$ for some $q \in L(A)$. By the same argument as above, either $\bigcup_{r\geq 1}\{2^r p | p \in L(A) \text{ and } 2^r p \leq n\} \subseteq S_i \cup S_j$ if $q = 1$, or $\bigcup_{r\geq 1}\{2^r p | p \in L(A) \setminus \{q\} \text{ and } 2^r p \leq n\} \subseteq S_i \cup S_j$ if $q > 1$. Note that $|\{2^r p | p \in L(A) \setminus \{q, 1\} \text{ and } 2^r p \leq n\}| = \pi(n/2^r) - 1$ for $r \geq 1$.

Let $Y = \{y_1, y_2, \ldots, y_t\}$ be the set of pendent vertices in B and corresponding to the collection $\{S'_1, S'_2, \ldots, S'_t\}$ for some $t \geq 2$. Suppose $N(Y) = \{x_1, x_2, \ldots, x_s\}$, $\deg(x_i) \leq m$ for some m and $1 \leq i \leq s$ and $|Y| > f \cdot m$ for some $f \geq 1$. Since $|N(A') \cap Y| \leq f \cdot m$ for any f-set $A' \subseteq N(Y)$ and $(N(x'_i) \cap Y) \cap (N(x'_j) \cap Y) = \emptyset$ for all x'_i and x'_j in A', T_n contains a matching $M = \{x_{j_1}y_{j_1}, x_{j_2}y_{j_2}, \ldots, x_{j_{f+1}}y_{j_{f+1}}\}$. Suppose $\varphi(x_{j_k}) = p'_{j_k}$ for $1 \leq k \leq f+1$. Since $\deg(y_{j_k}) = 1$, $S'_{j_k} = R \setminus \{rp'_{j_k} | r \geq 1\}$. Since $\gcd(p'_{j_\alpha}, p'_{j_\beta}) = 1$, $\bigcup_{k=1}^{f+1} S'_{j_k} = R \setminus \bigcap_{k=1}^{f+1}\{rp'_{j_k} | r \geq 1\} = R \setminus \{rp'_{j_1}p'_{j_2}\cdots p'_{j_{f+1}} | r \geq 1\}$ and then $|\bigcup_{k=1}^{f+1} S'_{j_k}| = n - \pi(n) - \lfloor n/p_{j_1}p_{j_2}\cdots p_{j_{f+1}}\rfloor$. Note that if $p_{j_1}p_{j_2}\cdots p_{j_{f+1}} > n$, then $\bigcup_{k=1}^{f+1} S'_{j_k} = R$ and $|\bigcup_{k=1}^{f+1} S'_{j_k}| = n - \pi(n)$.

Lemma 4. *Let $T_n = (A, B)$ be a tree defined as above with $\pi(n) = |A| \leq |B|$. Then $|S_i \cup S_j| \geq \pi(n) - 1$ for all $1 \leq i < j \leq |B| = n - \pi(n)$.*

Proof. If $\deg(b_i) = 1$, then b_i is adjacent to 1 or an odd prime p. Hence, $|S_i| \geq |R| - \lfloor n/p \rfloor \geq n - \pi(n) - n/3 \geq \pi(n)$ by Lemma 1(5) and then $|S_i \cup S_j| \geq \pi(n)$. Suppose $\deg(b_i) \geq \deg(b_j) \geq 2$. Let $h = \lfloor \log_2 n \rfloor \geq 6$. If $N(b_i) \cap N(b_j) \neq \emptyset$, since T_n is a tree, $N(b_i) \cap N(b_j) = \{p\}$, where $p = 1$ or an odd prime. For $1 \leq r \leq h-2$, let $X_r = \{2^r q \mid q \text{ is an odd prime with } q < n/2^r \text{ and } q \neq p\}$. Then $X_1, X_2, \ldots, X_r$ are mutually disjoint, $|X_k| \geq \pi(n/2^k) - 2$ and $X_k \subseteq S_i \cup S_j$ for all $1 \leq k \leq r$. Moreover, if $p > 3$, then $\{2, 2^2, \ldots, 2^h, 9, 18, 27\} \subseteq S_i \cup S_j$, since at most one of b_i and b_j is adjacent to 3; otherwise, $\{2, 2^2, \ldots, 2^h, 25, 49, 50\} \subseteq S_i \cup S_j$ if $p = 3$. Note that $2 \leq \pi(n/2^{h-2}) \leq 4$ since $4 \leq n/2^{h-2} < 8$. Hence, $|S_i \cup S_j| \geq |\bigcup_{r=1}^{h-2} X_r| + h + 3 = \sum_{r=1}^{h-2} |X_r| + h + 3 \geq \sum_{r=1}^{h-2}(\pi(n/2^r) - 2) + h + 3 = \sum_{r=1}^{h-4}(\pi(n/2^r) - 1) + \pi(n/2^{h-3}) + \pi(n/2^{h-2}) + 3 \geq \sum_{r=1}^{h-4}(\pi(n/2^r) - 1) + \pi(n/2^{h-3}) + 2\pi(n/2^{h-2}) - 1 \geq \sum_{r=1}^{h-4}(\pi(n/2^r) - 1) + 2\pi(n/2^{h-3}) - 1 \geq \cdots \geq \pi(n) - 1$, by Lemma 1(3) and (4).

Suppose $N(b_i) \cap N(b_j) = \emptyset$. For $1 \leq r \leq h-2$, let $Y_r = \{2^r p \mid p \text{ is an odd prime with } p < n/2^r\}$. Then $Y_1, Y_2, \ldots, Y_{h-2}$ are mutually disjoint, $|Y_r| \geq \pi(n/2^r) - 2$ and $Y_r \subseteq S_i \cup S_j$ for all $1 \leq r \leq h-2$. Moreover, $\{2, 2^2, \ldots, 2^h, 9, 18, 27\} \subseteq S_i \cup S_j$. By the same argument as above,

$|S_i \cup S_j| \geq |\bigcup_{r=1}^{h-2} Y_r| + h + 3 \geq \pi(n) - 1$. Therefore, $|S_i \cup S_j| \geq \pi(n) - 1$ for all $1 \leq i < j \leq |B|$. $\square$

Lemma 5. *Let $T_n = (A, B)$ be a tree defined as above with $\pi(n) = |A| \leq |B|$. If $k \in [\pi(n), n - \pi(n) - \lfloor n/3 \rfloor + 1]$, then $|\bigcup_{j=1}^{k} S_{i_j}| \geq k$ for any $1 \leq i_1 < i_2 < \cdots < i_k \leq |B|$.*

Proof. Let $Y = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$ and be corresponding to the collection $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$. By Lemma 3, Y contains at least $|Y| - |A| + 1 \geq 1$ pendent vertices. Let $N(b_{i_t}) = \{p\}$ for some pendent vertex b_{i_t}. Then $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_t}| \geq n - \pi(n) - (\lfloor n/p \rfloor - 1) \geq n - \pi(n) - \lfloor n/3 \rfloor + 1 \geq k$ if $p > 1$. Otherwise, for $p = 1$, we have $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_t}| = |[n] \setminus L(A)| = n - \pi(n) \geq k$. We complete the proof. $\square$

Lemma 6. *Let $T_n = (A, B)$ be a tree defined as above with $\pi(n) = |A| \leq |B|$. If $k \in [n - \pi(n) - \lfloor n/3 \rfloor + 2, n - \pi(n) - \lfloor n/15 \rfloor]$, then $|\bigcup_{j=1}^{k} S_{i_j}| \geq k$ for any $1 \leq i_1 < i_2 < \cdots < i_k \leq |B|$.*

Proof. Let $Y = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$ and be corresponding to the collection $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$. By Lemma 3, Y contains at least $|Y| - |A| + 1 \geq n - 2\pi(n) - \lfloor n/3 \rfloor + 3$ pendent vertices. If some pendent vertex b_{i_t} is adjacent to 1 or $p > n/2$, then $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_t}| = |[n] \setminus L(A)| = n - \pi(n) > n - \pi(n) - \lfloor n/15 \rfloor \geq k$. Suppose all the pendent vertices in Y are adjacent to the set $X = \{a_i \mid 3 \leq \varphi(a_i) \leq n/2\}$. Then $|X| = \pi(n/2) - 1$ and $\deg(a_j) \geq \deg(a_i)$ for all $a_i \in X$ and $a_j \in A \setminus X$ by assumption. If $a \in X$, then $n - 1 = |E(T)| = \sum_{a_i \in A \setminus X} \deg(a_i) + \deg(a) + \sum_{a_i \in X \setminus \{a\}} \deg(a_i) \geq (|A| - |X| + 1) \deg(a) + |X| - 1$, or $\deg(a) \leq \lfloor \frac{n-1-(|X|-1)}{|A|-|X|+1} \rfloor = \lfloor \frac{n-\pi(n/2)+1}{\pi(n)-\pi(n/2)+2} \rfloor = m$. If $|Y| - |A| + 1 > m$, then there are two distinct pendent vertices b_{i_α} and b_{i_β} in Y such that b_{i_α} is adjacent to $p \in X$ and b_{i_β} is adjacent to $q \in X$ with $p \neq q$. In this case, $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_\alpha} \cup S_{i_\beta}| \geq n - \pi(n) - \lfloor n/pq \rfloor \geq n - \pi(n) - \lfloor n/15 \rfloor \geq k$ and we conclude the proof. By Lemma 1,

$$\begin{aligned}
& (|Y| - |A|)(\pi(n) - \pi(n/2) + 2) - (n + 1 - \pi(n/2)) \\
\geq\ & (n - \pi(n) - \lfloor n/3 \rfloor + 2 - \pi(n))(\pi(n) - \pi(n/2) + 2) - (n + 1 - \pi(n/2)) \\
\geq\ & (2n/3 - 2\pi(n) + 2)(\pi(n) - \pi(n/2) + 2) - (n + 1 - \pi(n/2)) \\
=\ & 2(n/3 - \pi(n))(\pi(n) - \pi(n/2)) + n/3 - 2\pi(n) - \pi(n/2) + 3 \\
\geq\ & 12\pi(n) - 15\pi(n/2) + n/3 + 3 \\
& \text{(since } n/3 - \pi(n) \geq 7 \text{ if } n \geq 105 \text{ by elementary Calculus.)} \\
>\ & 12\frac{n}{\ln n} - 15\frac{n/2}{\ln(n/2)}\left(1 + \frac{2}{3\ln(n/2)}\right) + n/3 + 3 \\
>\ & 12\frac{n}{\ln n} - \frac{n}{\ln n} \cdot 15/2 \cdot \frac{1}{1 - \frac{\ln 2}{\ln 105}}\left(1 + \frac{2}{3\ln(105/2)}\right) + n/3 + 3
\end{aligned}$$

$$
\begin{aligned}
&= 12\frac{n}{\ln n} - (10.29\cdots)\frac{n}{\ln n} + n/3 + 3 \\
&> 0.
\end{aligned}
$$

Hence, $|Y| - |A| + 1 > \lfloor \frac{n+1-\pi(n/2)}{\pi(n)-\pi(n/2)+2} \rfloor$ as desired. Thus, $|\bigcup_{j=1}^{k} S_{i_j}| \geq k$. $\square$

Lemma 7. *Let $T_n = (A, B)$ be a tree defined as above with $\pi(n) = |A| \leq |B|$. If $k \in [n - \pi(n) - \lfloor n/15 \rfloor + 1, n - \pi(n)]$, then $|\bigcup_{j=1}^{k} S_{i_j}| = n - \pi(n) \geq k$ for any $1 \leq i_1 < i_2 < \cdots < i_k \leq |B|$.*

Proof. Let $Y = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$ and be corresponding to the collection $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$. Note that $\bigcup_{j=1}^{k} S_{i_j} \subseteq [n] \setminus L(A)$ for all $k \geq 1$. Hence, $|\bigcup_{j=1}^{k} S_{i_j}| \leq |[n] \setminus L(A)| = n - \pi(n)$. Set $X_1 = \{a_i \mid 3 \leq \varphi(a_i) \leq n/3\}$ and $X_2 = \{a_i \mid n/3 < \varphi(a_i) \leq n/2\}$. By Lemma 3, Y contains at least $|Y| - |A| + 1 \geq n - 2\pi(n) - \lfloor n/15 \rfloor + 2$ pendent vertices. By the proof as in Lemma 6, Y contains two pendent vertices b_{i_α} and b_{i_β} such that b_{i_α} is adjacent to p and b_{i_β} is adjacent to q with $p \neq q$. If b_{i_α} or b_{i_β} is adjacent to 1 or a prime $p' > n/2$, then $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_\alpha} \cup S_{i_\beta}| \geq |[n] \setminus L(A)| = n - \pi(n) \geq k$. If b_{i_α} or b_{i_β} is adjacent to a vertex in X_2, then $pq > 3 \cdot n/3 = n$ and then $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_\alpha} \cup S_{i_\beta}| \geq n - \pi(n) - \lfloor \frac{n}{pq} \rfloor = n - \pi(n) \geq k$. Suppose the neighborhood of the pendent vertices in Y are contained in X_1. By the same argument in Lemma 6, $\deg(a) \leq \lfloor \frac{n-1-(|X_1|-1)}{\pi(n)-(|X_1|-1)} \rfloor = \lfloor \frac{n+1-\pi(n/3)}{\pi(n)-\pi(n/3)+2} \rfloor$ for all vertices $a \in X_1$. Let f be the maximum number of distinct odd primes whose product is less than n. Then $n > p_2 p_3 \cdots p_{f+1} = 3 \cdot 5 \cdot 7 \cdots p_{f+1} > 10^2 \cdot 10^{f-3} = 10^{f-1}$, or $f < 1 + \log_{10} n$. If $|Y| - |A| + 1 > f \lfloor \frac{n+1-\pi(n/3)}{\pi(n)-\pi(n/3)+2} \rfloor$, then there are $f + 1$ pendent vertices $y_1, y_2, \ldots, y_{f+1}$ in Y such that each y_j is adjacent to $x_j \in X_1$ and corresponding to S_{y_j} and $t = \varphi(x_1)\varphi(x_2)\cdots\varphi(x_{f+1}) > n$. In this case, $|\bigcup_{j=1}^{k} S_{i_j}| \geq |\bigcup_{j=1}^{f+1} S_{y_j}| \geq n - \pi(n) - \lfloor n/t \rfloor = n - \pi(n) \geq k$ and we conclude the proof. By Lemma 1,

$$
\begin{aligned}
d &= (|Y| - |A|)(\pi(n) - \pi(n/3) + 2) - f(n + 1 - \pi(n/3)) \\
&\geq (n - \pi(n) - \lfloor n/15 \rfloor + 1 - \pi(n))(\pi(n) - \pi(n/3) + 2) \\
&\quad -(1 + \log_{10} n)(n + 1 - \pi(n/3)) \\
&\geq \left(\frac{14}{15}n - 2\pi(n)\right)(\pi(n) - \pi(n/3)) - n(1 + \log_{10} n).
\end{aligned}
$$

Since $\frac{14}{15}n - 2\pi(n) \geq \frac{14}{15}n - 2\frac{n}{\ln n}\left(1 + \frac{2}{3\ln n}\right) \geq \frac{14}{15}n - 2n\frac{1}{\ln 105}\left(1 + \frac{2}{3\ln 105}\right) = \frac{14}{15}n - (0.49\ldots)n > \frac{14}{15}n - \frac{1}{2}n = \frac{13}{30}n$ and $\pi(n) - \pi(n/3) \geq \frac{n}{\ln n} - \frac{n/3}{\ln(n/3)}(1 + \frac{2}{3\ln(n/3)}) \geq \frac{n}{\ln n} \cdot \frac{1}{3}\frac{1}{1-\ln 3/\ln 105}\left(1 + \frac{2}{3\ln 35}\right) = \frac{n}{\ln n}(1 - 0.51\cdots) > (0.48)\frac{n}{\ln n}$, we have $d > (13/30)n \cdot (0.48)\frac{n}{\ln n} - n(1 + \log_{10} n) > 0$ by elementary Calculus. Hence, $|Y| - |A| + 1 > f\lfloor \frac{n+1-\pi(n/3)}{\pi(n)-\pi(n/3)+2} \rfloor$ as desired. Therefore, $|\bigcup_{j=1}^{k} S_{i_j}| \geq n - \pi(n) \geq k$. $\square$

Now, we are ready to prove the main results.

Theorem 8. *Let $T_n = (A, B)$ be a tree defined as above with $\pi(n) = |A| \leq |B|$. Then T_n has a prime labelling.*

Proof. Let $Y = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\}$ and be corresponding to the collection $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$ for some $k \in [1, |B|] = [1, n - \pi(n)]$. If $k = 1$, since $2 \in S_{i_j}$ for all j, $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_1}| \geq 1$. If $k \in [2, \pi(n) - 1]$, by Lemma 4, $|\bigcup_{j=1}^{k} S_{i_j}| \geq |S_{i_1} \cup S_{i_2}| \geq \pi(n) - 1 \geq k$. If $k \in [\pi(n), n - \pi(n)]$, by Lemma 5, 6 and 7, $|\bigcup_{j=1}^{k} S_{i_j}| \geq k$. Hence, the Hall's condition holds. By Lemma 2, the collection $\{S_1, S_2, \ldots, S_{|B|}\}$ has an SDR $\{q_1, q_2, \ldots, q_{|B|}\}$. Label the vertex $b_j \in B$ by $\varphi(b_j) = q_j$. Combining $\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_{\pi(n)})$, the bijection $\varphi : V(T_n) = A \cup B \to \{1, 2, \ldots, n\}$ is a prime labelling of T_n as desired. □

For the remaining, suppose $T_n = (A, B)$ is a tree with $|A| < \pi(n)$. By the same argument in Theorem 8, we may label the vertices of A by $\varphi(a_1) = 1, \varphi(a_2) = p_{\pi(n)}, \varphi(a_3) = p_{\pi(n)-1}, \cdots$, and $\varphi(a_{|A|}) = p_{\pi(n)-|A|+2}$. For each $b_j \in B$, the corresponding S_j will have more members with respect to the S_j defined in Theorem 8. Hence, it can be argued that the Hall's condition holds. Therefore, we have the following.

Theorem 9. *Let $T_n = (A, B)$ be a tree of order $n \geq 105$ with bipartition (A, B) satisfying $|A| \leq \pi(n)$. Then T_n has a prime labelling.*

References

[1] J. Barkley Rosser, Schoenfeld and Lowell, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6 (1962), no.1, 64–94.

[2] P. Dusart, Autour de la fonction qui compte le nombre de nombres premiers, Diss. Université de Limoges, 1998.

[3] E. Erhart, On prime numbers, Fib. Quart. 26 (1988), 271-274.

[4] H. L. Fu and K. C. Huang, On prime labellings, Discrete Math. 127 1994, 181-186.

[5] P. Hall, On representatives of subsets, J. London Math. Soc. 10 (1935), 26-30.

[6] P. Haxell, O. Pikhurko and A. Taraz, Primality of trees, J. Combinatorics 2 (2011): 481500.

[7] S. M. Lee, I. Wui and J. Yeh, On the amalgamation of prime graphs, Bull Malaysian Math. Soc. (2) 11 (1988) 59-67.

[8] S. H. Lin, A study of prime labelling, M. Sc. Thesis, National Chao-Tung University, 1998.

[9] L. Robertson and B. Small, On Newman's Conjecture and Prime Trees, Integers, 9(2)(2009), 117-128.