

# Optimal equi-difference conflict-avoiding codes of prime length and weight 4

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**Abstract** A conflict-avoiding code (CAC)  $\mathcal{C}$  of length  $n$  and weight  $k$  is a collection of  $k$ -subsets of  $\mathbb{Z}_n$  such that  $\Delta(x) \cap \Delta(y) = \emptyset$  for any  $x, y \in \mathcal{C}$  and  $x \neq y$ , where  $\Delta(x) = \{a - b : a, b \in x, a \neq b\}$ . Let  $\text{CAC}(n, k)$  denote the class of all CACs of length  $n$  and weight  $k$ . A CAC  $\mathcal{C} \in \text{CAC}(n, k)$  is said to be equi-difference if any codeword  $x \in \mathcal{C}$  has the form  $\{0, i, 2i, \dots, (k-1)i\}$ . A CAC with maximum size is called optimal. In this paper we propose a graphical characterization of an equi-difference  $\text{CAC}(p, 4)$  where  $p$  is a prime, and then provide an infinite number of optimal equi-difference CACs for weight four by finding the independence number of a circulant graph.

**Keywords** conflict-avoiding code · equi-difference · circulant graph · prime length

**Mathematics Subject Classification (2010)** 94B25 · 94C15 · 05C70

## 1 Introduction

Let  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$  denote the ring of residues modulo  $n$  and  $\mathcal{P}(n, k)$  denote the set of all  $k$ -subsets of  $\mathbb{Z}_n$ . Given a  $k$ -subset  $x \in \mathcal{P}(n, k)$ , we define the *difference set* of  $x$  as  $\Delta(x) = \{a - b : a, b \in x, a \neq b\}$ . Note that  $|\Delta(x)| \leq k(k-1)$ . A subset  $\mathcal{C} \subset \mathcal{P}(n, k)$  is said to be a *conflict-avoiding code*, CAC for short, of length  $n$  and weight  $k$  if

$$\Delta(x) \cap \Delta(y) = \emptyset \text{ for any } x, y \in \mathcal{C} \text{ with } x \neq y, \quad (1)$$

and each element in  $\mathcal{C}$  is called a *codeword*. Since  $\Delta(x)$  is symmetric with respect to  $\frac{n}{2}$ , it is natural to define the *half-difference set* of  $x$  as  $\Delta_2(x) =$

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$\Delta(x) \cap \Omega(n)$ , where  $\Omega(n) = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ . Then (1) can be rewritten as

$$\Delta_2(x) \cap \Delta_2(y) = \emptyset \text{ for any } x, y \in \mathcal{C} \text{ with } x \neq y. \quad (2)$$

Without loss of generality, we assume that any codeword contains the element 0. For given  $n$  and  $k$ , let  $\text{CAC}(n, k)$  denote the class of all CACs of length  $n$  and weight  $k$ . The maximum size of some code in  $\text{CAC}(n, k)$  will be denoted by  $M(n, k)$ . A code  $\mathcal{C} \in \text{CAC}(n, k)$  is said to be *optimal* if  $|\mathcal{C}| = M(n, k)$ . For instance,  $x = \{0, 3, 6, 9\}$  and  $y = \{0, 4, 8, 12\}$  form a CAC of length 16 and weight 4, where  $\Delta_2(x) = \{3, 6, 7\}$  and  $\Delta_2(y) = \{4, 8\}$ . It is easy to verify that  $\mathcal{C} = \{x, y\}$  is optimal in  $\text{CAC}(16, 4)$ ; that is,  $M(16, 4) = 2$ .

A codeword  $x \in \mathcal{P}(n, k)$  is said to be *equi-difference* with *generator*  $i \in \mathbb{Z}_n \setminus \{0\}$  if  $x$  is of the form  $\{0, i, 2i, \dots, (k-1)i\}$ . We denote by  $x(i)$  the equi-difference codeword with generator  $i$ . Note here that  $|\Delta_2(x(i))| \leq k-1$ . A code  $\mathcal{C} \in \text{CAC}(n, k)$  is called *equi-difference* if it entirely consists of equi-difference codewords. Let  $\text{CAC}^e(n, k)$  denote the class of all the equi-difference codes in  $\text{CAC}(n, k)$  and  $M^e(n, k)$  be the maximum size among  $\text{CAC}^e(n, k)$ . Notice that  $M^e(n, k)$  provides a natural lower bound for  $M(n, k)$ .

The estimation of  $M(n, k)$  or  $M^e(n, k)$  has been investigated for years. In the case of weight  $k = 3$ , the characterization of even length was completely settled by [1, 3, 4, 9], and some optimal (equi-difference) CACs of odd length were studied in [2, 5, 8, 10, 15]. Several optimal constructions for weight  $k = 4, 5$  can be found in [11]. For general weights, please refer to [13, 14]. As to  $M^e(n, 4)$ , an infinite family of  $n$ 's has been obtained by Lo et al. in [7]. Mainly, they prove that

$$M^e(2^c \times 3^d, 4) = \begin{cases} 2^{c-3}(3^d - 1) + M^e(2^c, 4) & \text{if } d \text{ is even, and} \\ 2^{c-3}(3^d - 3) + M^e(3 \times 2^c, 4) & \text{if } d \text{ is odd.} \end{cases}$$

where the closed form of the values  $M^e(2^c, 4)$  and  $M^e(2^c \times 3, 4)$  are given as well. A further improvement of the above result was obtained recently by Lin et al. in [6],  $M^e(2^c \times 3^d \times m, 4)$  is studied where  $\Sigma$ ,  $\gcd(m, 6) = 1$ .

In this paper, we characterize an equi-difference CAC of prime length  $n > 3$  and weight 4 in terms of an independent set of a circulant graph defined on  $\{1, 2, \dots, \frac{n-1}{2}\}$ . As a consequence, finding the optimal equi-difference  $\text{CAC}(p, 4)$  is equivalent to finding the independence number of its corresponding graph. Note that a similar idea has been utilized in [6] by Lin et al. But, we use an undirected graph in this paper instead of a directed graph. Moreover, due to the order we consider is a prime, the graphs obtained do have better structure which we shall describe in what follows.

## 2 Preliminaries

Let  $m$  be a positive integer and  $D \subseteq \{1, 2, \dots, \lfloor \frac{m}{2} \rfloor\}$  be a non-empty set. Then a circulant graph  $G_m(D)$  is the graph defined on  $\{1, 2, \dots, m\}$  such that two vertices  $x$  and  $y$  are adjacent if and only if  $\min\{|x-y|, m-|x-y|\} \in D$ . Clearly,

if  $m$  is odd and  $D$  is a singleton set, then  $G_m(D)$  is a 2-factor. Moreover, if  $|D| = k$ , then  $G_m(D)$  is a  $2k$ -regular graph. For convenience, if  $D = \{i, j, k\}$ , then we write  $G_m(i, j, k)$  instead of  $G_m(\{i, j, k\})$ . Throughout of this paper, we shall mainly consider the cases where  $2m + 1$  is a prime and  $|D| = 3$ .

A set of vertices in a graph  $G$  is an independent set if no two vertices of them are adjacent in  $G$ . The size of an independent set in  $G$  of maximum size is called the independence number of  $G$ , denoted by  $\alpha(G)$ . It is folklore that for a general graph  $G$ , finding  $\alpha(G)$  is an NP-hard problem.

The chromatic number of  $G$ , denoted by  $\chi(G)$ , is the minimum number of independent sets which partition the vertex-set of  $G$ . As a matter of fact, finding  $\chi(G)$  is also a very difficult task. But, for special graphs, there are great results. For example, if  $G$  is a planar graph, then  $\chi(G) \leq 4$ . (Four-color Theorem) The following results are useful in providing lower bounds of  $\alpha(G)$ .

**Proposition 2.1**  $\alpha(G) \geq |G|/\chi(G)$ .

**Proposition 2.2** If  $m \not\equiv 0 \pmod{3}$ , then  $\chi(G_m(1, a, a + 1)) \geq 4$ .

*Proof* Clearly, if  $a = 2$ , then  $G_m(1, 2, 3)$  contains a subgraph  $K_4$  and thus we have the assertion. On the other hand, if  $a > 2$ , then the vertex subset  $\{1, a + 1, a + 2\}$  of  $V(G_m)$  induces a  $K_3$ . Therefore, if  $\pi$  is a proper vertex coloring of  $G_m$ , we may let  $\pi(1) = 1$ ,  $\pi(a + 1) = 2$  and  $\pi(a + 2) = 3$ . Now, the other vertices can be colored accordingly. This implies that  $\pi(2) = 2$ ,  $\pi(3) = 3$ ,  $\dots$ . So,  $\chi(G_m) = 3$  occurs only if  $m \equiv 0 \pmod{3}$ . This concludes the proof.

**Corollary 2.3** If  $m \equiv 0 \pmod{3}$  and  $a \equiv 1 \pmod{3}$ , then  $\chi(G_m) = 3$ .

*Proof* The proof follows by letting  $\pi : V(G_m) \rightarrow \{1, 2, 3\}$  defined by  $\pi(x) = i$  provided  $x \equiv i \pmod{3}$ ,  $i = 1, 2, 3$ .

Now, we turn our attention to the independence number of  $G_m$ . By Proposition 2.1, we have  $\alpha(G_m(1, a, a + 1)) \geq m/\chi(G_m(1, a, a + 1))$ .

**Proposition 2.4**  $\alpha(G_m(1, a, a + 1)) \leq \lfloor \frac{m}{3} \rfloor$  for each  $1 < a < \frac{m}{2} - 1$ .

*Proof* By greedy algorithm.

### 3 The main results

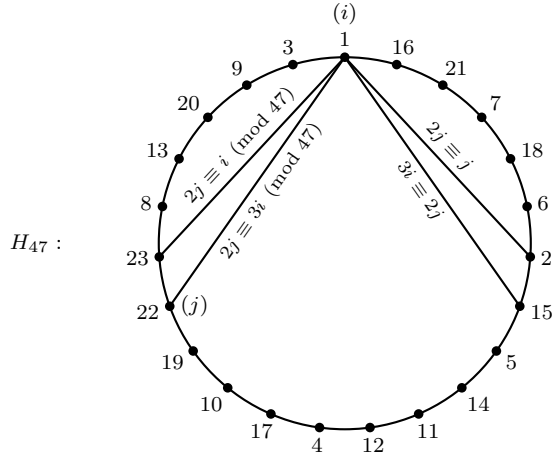
First, we describe the relationship between an optimal equi-difference CAC( $n, 4$ ) and the independence number of a circulant graph defined on  $\{1, 2, \dots, \frac{n-1}{2}\}$  where  $n$  is odd.

As mentioned in section 1, our goal is to pack the set  $\{1, 2, \dots, \frac{n-1}{2}\}$  with disjoint sets (multisets) of the form  $\{i, 2i, 3i\} \pmod{n}$ . For convenience, we use  $\langle i \rangle$  to denote the codeword which is obtained from  $\{i, 2i, 3i\}$  i.e.  $\Delta_2(\langle i \rangle) = \{i, 2i, 3i\}$ . In fact, we may let  $\langle i \rangle = \{0, i, 2i, 3i\} \pmod{n}$ .

Without mention otherwise, let  $n > 3$  be an odd prime in this section. For each  $n$ , we define a graph  $H_n$  where  $V(H_n) = \{1, 2, \dots, \frac{n-1}{2}\}$  and two

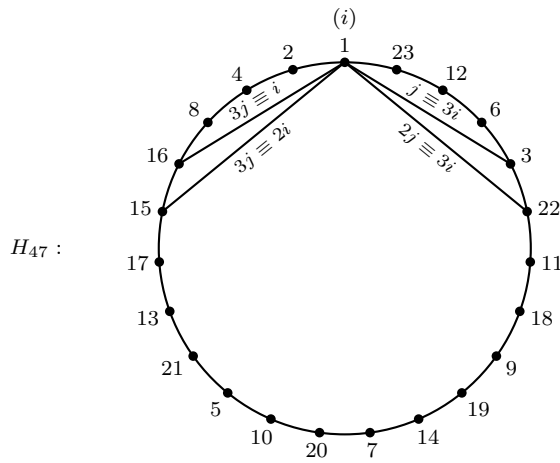
vertices  $i$  and  $j$  are adjacent if one of the following conditions holds: (1)  $j \equiv 2i \pmod{n}$ , (2)  $2j \equiv i \pmod{n}$ , (3)  $j \equiv 3i \pmod{n}$ , (4)  $3j \equiv i \pmod{n}$ , (5)  $2j \equiv 3i \pmod{n}$  and (6)  $3j \equiv 2i \pmod{n}$ .

For clearness, we use  $n = 47$  to depict this graph, see Figure 1. (The outer cycle is obtained by using conditions (3) and (4) mentioned above.



**Fig. 1**  $H_{47} \simeq G_{23}(1, 6, 7)$ .

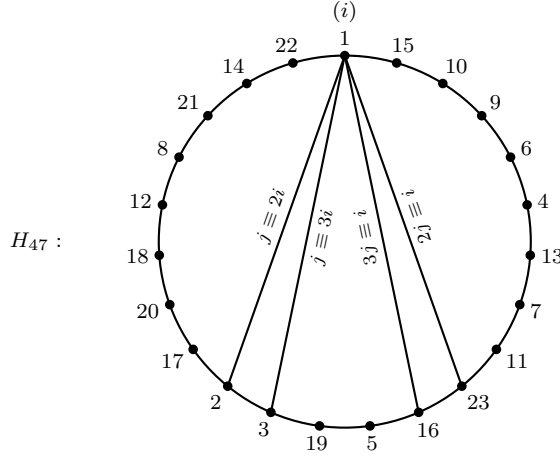
For the same example  $n = 47$ , we can also use conditions (1) and (2) to obtain the outer cycle, see Figure 2.



**Fig. 2**  $H_{47} \simeq G_{23}(1, 4, 5)$ .

Notice that both of the above two graphs are vertex-transitive and their independence numbers (respectively) are equal to  $M^e(47, 4)$ . Hence, we conclude  $\alpha(G_{23}(1, 6, 7)) = \alpha(G_{23}(1, 4, 5))$  without knowing the exact value  $M^e(47, 4)$ . In fact  $G_{23}(1, 6, 7) \simeq G_{23}(1, 4, 5)$  by taking the isomorphism  $\varphi(2^i) \equiv 3^i \pmod{47}$ ,  $i = 1, 2, 3, \dots, 23$ .

Clearly, conditions (5) and (6) can also be applied to obtain the outer cycle if indeed it is a cycle which contains all 23 vertices.



**Fig. 3** From (5) and (6):  $H_{47} \simeq G_{23}(1, 9, 10)$ .

Based on the above results we have the following lemma.

**Lemma 3.1** *If we can use conditions (1) and (2), or (3) and (4), or (5) and (6) to obtain a Hamilton cycle in  $H_n$ , then  $H_n \simeq G_{\frac{n-1}{2}}(1, a, a+1)$  for some  $1 < a < \frac{n-1}{4}$ .*

*Proof* We prove the second case and the other two cases are similar. Let the Hamilton cycle be ordered as  $(v_1, v_2, \dots, v_{\frac{n-1}{2}})$  where  $v_1 = 1$  and  $3 \cdot v_{\frac{n-1}{2}} \equiv 1 \pmod{n}$ . For convenience, let  $k = \frac{n-1}{2}$ . Observe that  $v_i \equiv 3^{i-1} \pmod{n}$ . Now, consider the neighbors of  $v_i$ .  $v_{i-1}$  and  $v_{i+1}$  are of distance 1 from  $v_i$  which is easy to see. By Condition (1)  $v_i$  is adjacent to  $v_j \equiv 2v_i \equiv 2 \cdot 3^{i-1} \pmod{n}$ . Since 2 is on the Hamilton cycle, let  $3^a \equiv 2 \pmod{n}$ . Hence,  $v_j \equiv 2v_i \equiv 3^a \cdot 3^{i-1} \equiv 3^{a+i-1} \pmod{n}$ . By condition (2)  $v_i$  is adjacent to  $v_j$  where  $2v_j \equiv v_i \pmod{n}$ . This implies that  $v_j \equiv 3^{-a} \cdot 3^{i-1} \equiv 3^{i-1-a} \pmod{n}$ , and therefore the distance between  $v_i$  and  $v_j$  is  $a$ . Finally, conditions (5) and (6) tell us the vertices which are adjacent to  $v_i$  are  $3^{(i-1)+a+1}$  and  $3^{i-1-(a+1)} \pmod{n}$ . This concludes the proof that  $H_n \simeq G_{\frac{n-1}{2}}(1, a, a+1)$  where  $3^a \equiv 2 \pmod{n}$  in the second case.

**Lemma 3.2** *Let  $S = \{s_1, s_2, \dots, s_t\}$  be an independent set of  $H_n$ . Then  $\mathbb{C} = \{\langle s_1 \rangle, \langle s_2 \rangle, \dots, \langle s_t \rangle\}$  is an equi-difference CAC( $n, 4$ ).*

*Proof* It follows by the fact that for  $1 \leq i \neq j \leq t$ ,  $\Delta(\langle s_i \rangle) \cap \Delta(\langle s_j \rangle) = \emptyset$  where  $\langle s_i \rangle = \{0, s_i, 2s_i, 3s_i\} \pmod{n}$  and  $\langle s_j \rangle = \{0, s_j, 2s_j, 3s_j\} \pmod{n}$ .

**Corollary 3.3**  $M^e(n, 4) = \alpha(H_n)$ .

Finding the independence number of a graph  $G$  is not an easy task especially if we have no idea about the structure of  $G$ . Therefore, it is important to discover the graph structure of  $H_n$  in order to find  $M^e(n, 4)$  eventually.

**Lemma 3.4** *If  $n > 3$  is a prime, then  $H_n$  is a 6-regular graph.*

*Proof* This is a direct conclusion from the definition of  $H_n$ .

**Lemma 3.5** *If  $\frac{n-1}{2}$  is a prime, then  $H_n \simeq G_{\frac{n-1}{2}}(1, a, a+1)$  for some  $1 < a < \frac{n-1}{4}$ .*

*Proof* The multiplicative subgroup generated by conditions (1), (3) and (5) respectively is a subgroup of  $\mathbb{Z}_n^*/x \sim y$  where  $x+y \equiv 0 \pmod{n}$ . Hence, their orders must be a divisor of  $\frac{n-1}{2}$ . Since  $\frac{n-1}{2}$  is a prime, we conclude the proof of Lemma 3.1.

We remark here that “ $\frac{n-1}{2}$  is a prime” is a sufficient condition only. There are examples that  $\frac{n-1}{2}$  is not a prime but  $H_n$  is indeed isomorphic to a circulant graph of the form  $G_{\frac{n-1}{2}}(1, a, a+1)$ . For examples,  $H_{31} \simeq G_{15}(1, 6, 7)$  and  $H_{37} \simeq G_{18}(1, 7, 8)$ . Now, if we take a close look at  $H_{37}$ , it is not difficult to see that  $\alpha(H_{37}) \geq 6$  where the six vertices of the independent set can be selected as  $\{1, 8, 10, 6, 11, 14\}$ . Therefore,  $M^e(37, 4) = 6$  since  $\{1, 2, 3\}$ ,  $\{8, 16, 5\}$ ,  $\{10, 17, 7\}$ ,  $\{6, 12, 18\}$ ,  $\{11, 15, 4\}$  and  $\{14, 9, 5\}$  are in fact six sets which partition  $\{1, 2, \dots, 18\}$ . A more general result can be stated as follows.

**Proposition 3.6**  $\alpha(G_{3t}(1, a, a+1)) \geq t$  provided  $\frac{3t}{2} > a > 1$  and  $a \equiv 1 \pmod{3}$ .

*Proof* Let the Hamilton cycle (obtained by difference 1) of  $G_{3t}(1, a, a+1)$  be  $(s_1, s_2, \dots, s_{3t})$ . Then, it is not difficult to see that  $S = \{s_i | i \equiv 1 \pmod{3} \text{ and } 1 \leq i \leq 3t\}$  is an independent set. This implies that  $\alpha(G_{3t}(1, a, a+1)) \geq t$ .

**Corollary 3.7** *If  $H_{6t+1} \simeq G_{3t}(1, a, a+1)$  with  $\frac{3t}{2} > a > 1$  and  $a \equiv 1 \pmod{3}$ , then  $M^e(6t+1, 4) = t$ .*

*Proof* By Proposition 3.6,  $\alpha(H_{6t+1}) \geq t$ . This implies that there exist  $t$  mutually disjoint subsets of  $\{1, 2, \dots, 3t\}$  which is of the form  $\{i, 2i, 3i\} \pmod{6t+1}$ . Since  $6t+1$  is a prime ( $n = 6t+1$ ),  $\{i, 2i, 3i\}$  is indeed a set of size 3. Therefore, these  $t$  subsets are the most which we can get for codewords  $\{0, i, 2i, 3i\}$  and the proof follows.

But, it is not that lucky for other cases. We propose an ideal to find lower bounds for  $\alpha(G_m(1, a, a+1))$  and thus equivalently for  $M^e(2m+1, 4)$ . First, we consider the case where  $m \not\equiv 0 \pmod{3}$  and  $a \equiv 1 \pmod{3}$ .

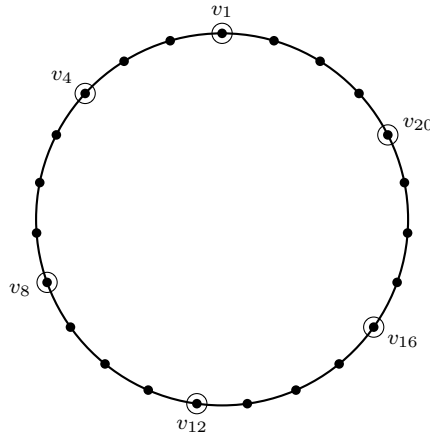
**Proposition 3.8** *If  $m \not\equiv 0 \pmod{3}$  and  $a \equiv 1 \pmod{3}$ , then  $\alpha(G_m(1, a, a + 1)) \geq \lfloor \frac{m}{3} \rfloor - \lfloor \frac{a}{3} \rfloor$ .*

*Proof* Let the Hamilton cycle generated by difference “1” be  $(v_1, v_2, \dots, v_m)$ . Then, let  $S = \{v_1, v_4, \dots, v_{3(\lfloor \frac{m}{3} \rfloor - \lfloor \frac{a}{3} \rfloor - 1) + 1}\}$ . Since  $S$  is an independent set, we have the proof.

Now, consider the case where  $a \not\equiv 1 \pmod{3}$ . We use an example to explain our idea.

**Example** Finding  $\alpha(G_{23}(1, 9, 10))$ .

Let  $(v_1, v_2, \dots, v_{23})$  be the cycle generated by difference “1”.



**Fig. 4**

The idea of the choice of vertices is to combine 3 and 4 to obtain 11. If  $m$  is not divisible by 11, then we need to adjust them slightly. The above example shows that we have consecutive vertices which are of distance 4. This shows that we have  $\lfloor \frac{a+2}{3} \rfloor$  vertices in the proposed independent set for every  $a+2$  vertices. Thus, the independent set we are going to find has size roughly around  $\lfloor \frac{a+2}{3} \rfloor \cdot \lfloor \frac{m}{a+2} \rfloor$ . This value for  $a = 9$  and  $m = 23$  is equal to  $3 \cdot 2 = 6$ .

**Proposition 3.9** *For  $a \geq 3$ ,  $\alpha(G_m(1, a, a + 1)) \approx \lfloor \frac{a+2}{3} \rfloor \cdot \lfloor \frac{m}{a+2} \rfloor$ .*

*Proof* As explained above, we shall use the Hamilton cycle generated by 1 and pick  $\lfloor \frac{a+2}{3} \rfloor$  vertices for the independent set in every consecutive  $a+2$  vertices. The worst case comes from  $m$  is not a multiple of  $a+2$  and we leave the last  $a+2$  vertices or slightly more untouched.

Before we close this section we present another example for reference.

**Example**  $n = 67$ ,  $H_{67} \simeq G_{33}(1, 5, 6)$ .

The cycle generated by difference “1” is (1, 2, 4, 8, 16, 32, 3, 6, 12, 24, 19, 29, 9, 18, 31, 5, 10, 20, 27, 3, 26, 15, 30, 7, 14, 28, 11, 22, 23, 21, 25, 17, 34). So, we choose  $\{\langle 1 \rangle, \langle 8 \rangle, \langle 6 \rangle, \langle 19 \rangle, \langle 31 \rangle, \langle 20 \rangle, \langle 15 \rangle, \langle 28 \rangle, \langle 21 \rangle\}$ . Notice that we adjust the choice of  $\langle 14 \rangle$  to  $\langle 28 \rangle$ . Thus we have 9 vertices in the independent set which is slightly better than the result  $\lfloor \frac{5+2}{3} \rfloor \cdot \lfloor \frac{33}{7} \rfloor = 8$  proposed in Proposition 3.9.

### Concluding remark

The result obtained in Proposition 3.9 is asymptotically good since on the case  $3 \mid (a + 2)$  and  $(a + 2) \mid m$ , the value is around  $\frac{m}{3}$ . But, due to the difficulty in finding the exact value of  $\alpha(G_m(1, a, a + 1))$ , we are not able to determine  $M^e(n, 4)$  for prime  $n$  at this moment.

Another interesting problem raised in this study is the following: For which prime  $n$ ,  $H_n \simeq G_{\frac{n-1}{2}}(1, a, a + 1)$  or a disjoint union of circulant graphs of the same type  $G_{m'}(1, a', a' + 1)$ ? We have to answer this question before we settle the problem of finding  $M^e(n, 4)$ . We note finally, finding the multiplicative order of 2 or 3 in  $\mathbb{Z}_n$  is every difficult problem in Number Theory and Algebra.

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