ON THE NUMBER OF RAINBOW SPANNING TREES IN EDGE-COLORED COMPLETE GRAPHS

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Abstract. A spanning tree of a properly edge-colored complete graph, K_n, is rainbow provided that each of its edges receives a distinct color. In 1996, Brualdi and Hollingsworth conjectured that if K_{2m} is properly $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m rainbow spanning trees except when $m = 2$. In 2000, Krussel et al. proved the existence of 3 edge-disjoint rainbow spanning trees for sufficiently large m. In this paper, we use an inductive argument to construct Ω_m rainbow edge-disjoint spanning trees recursively, the number of which is approximately $\sqrt{m}$.

Key words. edge-coloring, complete graph, rainbow spanning tree

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1. Introduction.

A spanning tree T of a graph G is an acyclic subgraph of G for which $V(T) = V(G)$. A proper k-edge-coloring of a graph G is a mapping from $E(G)$ into a set of colors, $\{1, 2, ..., k\}$, such that adjacent edges of G receive distinct colors. Since all edge-colorings considered in this paper are proper, if G has a proper k-edge-coloring, then G is said to be k-edge-colored. The chromatic index $\chi'(G)$ of a graph G is the minimum number k such that G is k-edge-colorable. It is well known that $\chi(K_{2m}) = 2m - 1$ and thus, if K_{2m} is properly $(2m-1)$-edge-colored, each color appears at every vertex exactly once.

A subgraph in an edge-colored graph is said to be rainbow (sometimes called multicolored or poly-chromatic) if all of its edges receive distinct colors. Observe that with any $(2m-1)$-edge-coloring of K_{2m}, it is not hard to find a rainbow spanning tree by taking the spanning star, S_v, with center $v \in V(K_{2m})$. Further, K_{2m} has $m(2m-1)$ edges and it is well known that these edges can be partitioned into m spanning trees. This led Brualdi and Hollingsworth [3] to make the following conjecture in 1996.

Conjecture A. [3] If K_{2m} is $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m rainbow spanning trees except when $m = 2$.

Based on Brualdi and Hollingsworth's concept, Constantine [5] proposed two related conjectures in 2002.

Conjecture B. (Weak version) [5] K_{2m} can be edge-colored with $2m - 1$ colors in such a way that the edges can be partitioned into m isomorphic rainbow spanning trees except when $m = 2$.

Conjecture B was proved to be true by Akbari, Alipour, Fu, and Lo in 2006 [1].

Conjecture C. (Strong version) [5] If K_{2m} is $(2m-1)$-edge-colored, then the edges of K_{2m} can be partitioned into m isomorphic rainbow spanning trees except when $m = 2$.

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Concerning Conjecture A, in [3], Brualdi and Hollingsworth proved that there exist two edge-disjoint rainbow spanning trees for $m > 2$, and in 2000, Krussel, Marshall, and Verrall [9] improved this result to three spanning trees. Recently, Carraher, Hartke, and Horn [4] submitted a paper with the result that if $m \geq 500,000$ then an edge-colored K_{2m} in which each color appears on at most m edges contains at least $\left\lfloor \frac{m}{500\log(2m)} \right\rfloor$ edge-disjoint rainbow spanning trees.

Essentially, not much has been done on Conjecture C. The best result so far is by Fu and Lo [6]. They proved that three isomorphic rainbow spanning trees exist in any $(2m-1)$-edge-colored K_{2m} for each $m \geq 14$.

In this paper, we focus on Conjecture A. We further improve the lower bound of three in [9] by proving that in any $(2m-1)$-edge-coloring of K_{2m}, $m \geq 1$, there exist at least $\left\lfloor \frac{\sqrt{6m+9}}{3} \right\rfloor$ mutually edge-disjoint rainbow spanning trees. Asymptotically, this is not as good as the bound submitted in [4], but our result applies to all values of m and it is better until m is extremely large (over 5.7×10^7).

It is worth mentioning here that the above conjectures will play important roles in certain applications if they are true. Notice that a rainbow spanning tree is orthogonal to the 1-factorization of K_{2m} (induced by any $(2m-1)$-edge-coloring). An application of parallelisms of complete designs to population genetics data can be found in [2]. Parallelisms are also useful in partitioning consecutive positive integers into sets of equal size with equal power sums [8]. In addition, the discussions of applying colored matchings and design parallelisms to parallel computing appeared in [7].

2. The Main Result.

Here is the main theorem of this paper.

THEOREM 2.1. *Let K_{2m} be a properly $(2m-1)$-edge-colored graph. Then the edges of K_{2m} can be partitioned into at least $\left\lfloor \frac{\sqrt{6m+9}}{3} \right\rfloor$ rainbow edge-disjoint spanning trees.*

Proof. Let K_{2m}, $m \geq 1$, be any properly $(2m-1)$-edge-colored complete graph. We will use induction on the number of trees to prove this result. Let $\Omega_m = \left\lfloor \frac{\sqrt{6m+9}}{3} \right\rfloor$. We can assume $m \geq 5$ since for $1 \leq m \leq 4$, $\Omega_m = 1$ and the spanning star, S_r, in which $r \in V(K_{2m})$ and r is joined to every other vertex, is clearly a rainbow spanning tree of K_{2m}. When the value of m is clear, it will cause no confusion to simply refer to this value as Ω. It is worth noting that the following induction proof can be used as a recursive construction to create Ω rainbow edge-disjoint spanning trees, $T_1, T_2, ..., T_\Omega$.

For $1 \leq \psi \leq \Omega$ and rainbow edge-disjoint spanning trees, $T_1, T_2, ..., T_\psi$, let $f(\psi)$ be the proposition consisting of the three following degree and structure characteristics:

Each tree has a designated distinct root. (2.1)

The root of T_1 has degree $(2m-1) - 2(\psi - 1)$ and has at least $(2m-1) - 4(\psi - 1)$ adjacent leaves. (2.2)

For $2 \leq i \leq \psi$, The root of T_i has degree $(2m-1) - i - 2(\psi - i)$ and has at least $(2m-1) - 2i - 4(\psi - i)$ adjacent leaves. (2.3)

In particular, note here that by (2.3), if $\psi > 1$, then the root of T_2 has degree $(2m-1)-2-2(\psi-2) = (2m-1)-2(\psi-1)$ and at least $(2m-1)-4-4(\psi-2) = (2m-1)-4(\psi-1)$ adjacent leaves, sharing these characteristics with T_1 (as stated in (2.2)).

It is useful in our construction to ensure that the rainbow edge-disjoint spanning trees have suitable characteristics that allow the properties (2.1), (2.2), and (2.3) to be established. Thus, the trees $T_1, T_2, ..., T_\Omega$ will eventually satisfy $f(\Omega)$.

We begin with some necessary notation. All vertices defined in what follows are in $V(K_{2m})$, the given edge-colored complete graph.

The proof proceeds inductively, producing a list of j edge-disjoint rainbow spanning trees from a list of $j-1$ edge-disjoint rainbow spanning trees; so for $1 \leq i \leq j \leq \Omega$, let T_i^j be the i^{th} rainbow spanning tree of the j^{th} induction step and let r_i be the designated root of T_i^j. Notice that r_i is independent of j.

Suppose T is any spanning tree of the complete graph K_{2m} with root r containing vertices y, v, w, and v', where ry and rv are distinct pendant edges in T (so y and v are leaves of T). Then define $T' = T[r; y, v; w, v']$ to be the new graph formed from T with edges ry and rv removed and edges yw and vv' added. Formally, $T' = T[r; y, v; w, v'] = T - ry - rv + yw + vv'$. We note here that T' is also a spanning tree of K_{2m} because y and v are leaves in T, and thus adding edges yw and vv' does not create a cycle in T'.

Our inductive strategy will be to assume we have $k-1$ (where $1 < k \leq \Omega$) edge-disjoint rainbow spanning trees with suitable characteristics satisfying proposition $f(k-1)$ that yield properties (2.1), (2.2), and (2.3) with $\psi = k-1$. From those trees we will construct k edge-disjoint rainbow spanning trees with suitable characteristics that allow properties (2.1), (2.2), and (2.3) to be eventually established when $\psi = k$, thus satisfying $f(k)$.

For this construction, given any T_i^{j-1} with root r_i and distinct pendant edges $r_i y_i^j$ and $r_i v_i^j$, we define T_i^j in the following way:

$$T_i^j = T_i^{j-1}[r_i; y_i^j, v_i^j; w_i^j, v_i^{j'}] = T_i^{j-1} - r_i y_i^j - r_i v_i^j + y_i^j w_i^j + v_i^j v_i^{j'} \tag{2.4}$$

The choice of the vertices defined in (2.4) will eventually be made precise, based on the discussion which follows.

When the value of j is clear, it will cause no confusion to refer to the vertices $y_i^j, v_i^j; w_i^j, v_i^{j'}$ by omitting the superscript and instead writing $T_i^j = T_i^{j-1}[r_i; y_i, v_i; w_i, v_i']$. We now make the following remarks about the definition of T_i^j above. Recall that for $1 \leq i \leq j \leq \Omega$, r_i is independent of j, and thus is the root of both T_i^{j-1} and T_i^j. The following is easily seen to be true.

(2.5) If φ is any proper edge-coloring of K_{2m} and T_i^{j-1} is a rainbow spanning tree of K_{2m} with root r_i and distinct pendant edges $r_i y_i$ and $r_i v_i$, then T_i^j as defined in (2.4) is also a rainbow spanning tree of K_{2m} if $\varphi(r_i y_i) = \varphi(v_i v_i')$ and $\varphi(r_i v_i) = \varphi(y_i w_i)$.

Next, for $1 \leq i \leq j \leq \Omega$, let $L_i^j = \{x \mid x r_i \text{ is a pendant edge in } T_i^j\}$ (so x is a leaf adjacent to r_i in T_i^j). Define

$$L_j = \bigcap_{i=1}^{j} L_i^j. \tag{2.6}$$

Notice that if $x \in L_j$, then for $1 \leq i \leq j$, xr_i is a pendant edge in T_i^j .

We now begin our inductive proof with induction parameter k. Specifically we will prove that for $1 \leq k \leq \Omega$ there exist k edge-disjoint rainbow spanning trees, $T_1^k, T_2^k, ..., T_k^k$ satisfying $f(k)$, which for convenience we explicitly state in terms of the inductive parameter k:

1. Each tree T_i^k has a designated distinct root r_i,
2. The root of T_1^k has degree $(2m-1)-2(k-1)$ and has at least $(2m-1)-4(k-1)$ adjacent leaves,
3. For $2 \leq i \leq k$, the root of T_i^k has degree $(2m-1)-i-2(k-i)$ and has at least $(2m-1)-2i-4(k-i)$ adjacent leaves.

Base Step. The case $k = 1$ is seen to be true for all properly edge-colored complete graphs, K_{2m}, by letting r_1 be any vertex in $V(K_{2m})$ and defining $T_1^1 = S_{r_1}$, the spanning star with root r_1. It is also clear that S_{r_1} satisfies $f(1)$ since r_1 has degree $2m-1$ and has $2m-1$ adjacent leaves, as required in (2.2). Property (2.3) is vacuously true.

Induction Step. Suppose that φ is a proper edge-coloring of K_{2m} and that for some k with $1 < k \leq \Omega$, K_{2m} contains $k-1$ edge-disjoint rainbow spanning trees, $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$, satisfying $f(k-1)$:

1. r_i is the root of tree T_i^{k-1} and $r_i \neq r_c$ for $1 \leq i, c < k$, $i \neq c$,
2. $d_{T_1^{k-1}}(r_1) = (2m-1)-2(k-2)$ and r_1 is adjacent to at least $(2m-1)-4(k-2)$ leaves in T_1^{k-1}, and
3. For $2 \leq i \leq k-1$, $d_{T_i^{k-1}}(r_i) = (2m-1)-i-2(k-1-i)$ and r_i is adjacent to at least $(2m-1)-2i-4(k-1-i)$ leaves in T_i^k.

It thus remains to construct k edge-disjoint rainbow spanning trees satisfying $f(k)$.

We note here that $f(k-1)$ and the definition of L_{k-1} in (2.6) guarantee that a lower bound for $|L_{k-1}|$ can be obtained by starting with a set containing all $2m$ vertices, then removing the $k-1$ roots of $T_1^{k-1}, T_2^{k-2}, ..., T_{k-1}^{k-1}$, the (at most $4(k-2)$) vertices in $V(T_1^{k-1} \backslash \{r_1\})$ which are not leaves adjacent to r_1, and for $2 \leq i < k$, the (at most $2i + 4(k-1-i)$) vertices in $V(T_i^{k-1} \backslash \{r_i\})$ which are not leaves adjacent to r_i. Formally,

$$\begin{aligned} |L_{k-1}| &\geq 2m-(k-1)-4(k-2)-\sum_{i=2}^{k-1}(2i+4(k-1-i)) \\ &= 2m-(k-1)-4(k-2)-(3k^2-11k+10) \\ &= 2m-3k^2+6k-1. \end{aligned} \tag{2.7}$$

Knowing $|L_{k-1}|$ is useful because later we will show that if $|L_{k-1}| \geq 6k-7$, then from $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ we can construct k rainbow edge-disjoint spanning trees which satisfy proposition $f(k)$. As the reader might expect, it is from here that the bound on Ω is obtained: it actually follows that since $k \leq \Omega$, $|L_{k-1}| \geq 6k-7$.

First select any two distinct vertices $r_k, w_k^k \in L_{k-1}$; since it will cause no confusion, we will write w_k for w_k^k. Set r_k equal to the root of the k^{th} tree, T_k^k. Later, $r_k w_k$ will be an edge removed from T_k^k. For now, the two special vertices r_k and w_k play a role in the construction of T_i^k from T_i^{k-1} for $1 \leq i < k$. For convenience we explicitly state and observe the following

Since r_k and w_k are distinct vertices in L_{k-1} (defined in (2.6)), r_k and w_k are leaves adjacent to r_i for $1 \leq i < k$. (2.8)

For the sake of clarity, having selected r_k and w_k, we now discuss how to construct the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ before returning to our discussion of the construction of T_k^k (though in actuality T_k^k is formed recursively as we are constructing $T_1^k, T_2^k, ..., T_{k-1}^k$).

For $1 \leq i < k$, we will find suitable vertices v_i^k, w_i^k, and $v_i^{k'}$, which for convenience we refer to as v_i, w_i, and v_i' respectively, and define T_i^k in the following way:

$$T_i^k = T_i^{k-1}[r_i; r_k, v_i; w_i, v_i'] \quad \text{where } \varphi(r_i r_k) = \varphi(v_i v_i') \text{ and } \varphi(r_i v_i) = \varphi(r_k w_i) \tag{2.9}$$

It is clear by (2.5) that for $1 \leq i < k$, since T_i^{k-1} is a rainbow spanning tree of K_{2m}, if v_i is chosen so that $v_i r_i$ is a pendant edge in T_i^{k-1} with $v_i \neq r_k$, then T_i^k is also a rainbow spanning tree of K_{2m} (recall from (2.8) that $r_k \in L_{k-1}$, so by (2.6) $r_k r_i$ is a pendant edge in T_i^{k-1}).

Further, since $r_k, w_k \in L_{k-1}$, it is clear from (2.9) that (1) $r_k, v_i \notin L_k$, and (2) all leaves adjacent to r_i in T_i^k are leaves adjacent to r_i in T_i^{k-1}. Therefore $|L_k| < |L_{k-1}|$. (2.10)

Lastly, since the trees $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ satisfy $f(k-1)$, it can be seen that $T_1^k, T_2^k, ..., T_{k-1}^k$ satisfy $f(k)$, as the following shows.

First, clearly (2.1) is satisfied. Further, for $1 \leq i < k$, when T_i^k is formed from T_i^{k-1} (see (2.9)), it can easily be seen that the degree of r_i is decreased by 2 and the number of leaves adjacent to r_i is decreased by at most 4.

(i.) T_1^k
By our induction hypothesis, we have that $d_{T_1^{k-1}}(r_1) = (2m-1)-2(k-2)$ and that r_1 is adjacent to at least $(2m-1)-4(k-2)$ leaves in T_1^{k-1}. From (2.9) we have that $d_{T_1^k}(r_1) = d_{T_1^{k-1}}(r_1)-2 = (2m-1)-2(k-2)-2 = (2m-1)-2(k-1)$ and that r_1 is adjacent to at least $(2m-1)-4(k-2)-4 = (2m-1)-4(k-1)$ leaves in T_1^k. So (2.2) of $f(k)$ is satisfied.

(ii.) T_i^k, $2 \leq i < k$
By our induction hypothesis, we have that $d_{T_i^{k-1}}(r_i) = (2m-1)-i-2(k-1-i)$ and that r_i is adjacent to at least $(2m-1)-2i-4(k-1-i)$ leaves in T_i^k. From (2.9) we have that $d_{T_i^k}(r_i) = d_{T_i^{k-1}}(r_i) - 2 = (2m-1) - i - 2(k-1-i) - 2 = (2m-1) - i - 2(k-i)$ and that r_i is adjacent to at least $(2m-1) - 2i - 4(k-1-i) - 4 = (2m-1) - 2i - 4(k-i)$ leaves in T_i^k. So (2.3) of $f(k)$ is satisfied except possibly when $i = k$.

Lastly, we can observe that once v_i is selected, vertices w_i and v_i' are determined by the required property from (2.9) that $\varphi(r_ir_k) = \varphi(v_iv_i')$ and $\varphi(r_iv_i) = \varphi(r_kw_i)$.

It remains to ensure that the trees, $T_1^k, T_2^k, ..., T_{k-1}^k$, are all edge-disjoint. This is also proved using the induction hypothesis that $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all edge-disjoint, which allows us to show that $T_1^k, T_2^k, ..., T_{k-1}^k$ are all edge-disjoint.

Now, while forming the rainbow edge-disjoint spanning trees, $T_1^k, T_2^k, ..., T_{k-1}^k$, we simultaneously construct the k^{th} rainbow spanning tree, T_k^k, from a sequence of inductively defined graphs, $T_k^k(1), T_k^k(2), ..., T_k^k(k) = T_k^k$ where at the i^{th} induction step, the formation of $T_k^k(i)$ depends on the choice of v_i used in the construction of T_i^k: for $2 \leq i \leq k$ define

$$\begin{array}{l} T_k^k(i) = S_{r_k} - r_kw_1 - ... - r_kw_i + w_1w_1' + ... + w_iw_i', \\ \text{where } \varphi(w_1w_1') = \varphi(r_kw_k) \text{ and } \varphi(w_iw_i') = \varphi(r_kw_{i-1}) \text{ for } 2 \leq i \leq k. \end{array} \tag{2.11}$$

Note that for $1 \leq i \leq k-1$, the choice of v_i determines $T_k^k(i)$; the formation of $T_k^k(k)$ is dictated by $T_k^k(k-1)$ since w_k' is determined by requiring that $\varphi(w_kw_k') = \varphi(r_kw_{k-1})$. It is worth explicitly stating that

$$\begin{array}{l} T_k^k = T_k^k(k) = S_{r_k} - r_kw_1 - ... - r_kw_k + w_1w_1' + ... + w_kw_k', \\ \text{where } \varphi(w_1w_1') = \varphi(r_kw_k) \text{ and } \varphi(w_cw_c') = \varphi(r_kw_{c-1}) \text{ for } 2 \leq c \leq k \end{array} \tag{2.12}$$

Observe that T_k^k is a rainbow graph since each edge removed from S_{r_k} is replaced by a corresponding edge of the same color. Also, one can easily see that: T_k^k has $2m-1$ edges; $d_{T_k^k}(r_k) = (2m-1) - k$ since $r_k \notin \{w_1', w_2', ..., w_k'\}$; and r_k has at least $(2m-1) - 2k$ adjacent leaves. Therefore, condition (2.3) of $f(k)$ is satisfied. So it remains to show that T_k^k is acyclic and contains no edges in the trees T_i^k for $1 \leq i \leq k-1$.

Finally, we have noted previously, but restate here because of its importance,

For $1 \leq i < k$, once v_i is chosen, T_i^k and $T_k^k(i)$ are completely determined by the constructions described in (2.9) and (2.11) respectively. (2.13)

Due to the fact highlighted above in (2.13), our strategy will be to select a suitable v_i and construct T_i^k from T_i^{k-1}, while simultaneously constructing $T_k^k(i)$ from $T_k^k(i-1)$. In doing so, we restrict the choices for each v_i in order to achieve the following three properties:

(C1) The edges in $T_a^k, 1 \leq a < i$ do not appear in T_i^k,
(C2) The edges in T_k^k do not appear in T_i^k, $1 \leq i < k$, and
(C3) T_k^k is acyclic

To that end, we let

$$L_{k-1}^* = L_{k-1} \backslash \{r_k, w_k\} \tag{2.14}$$

and let v_i be any vertex for which the following properties are satisfied (so by (2.13), this choice completes the formation of T_i^k and $T_k^k(i)$ for $1 \leq i < k$):

(R1) $v_i \in L_{k-1}^*$,

(R2) For $1 \leq c < k$, $c \neq i$, $\varphi(v_i r_c) \neq \varphi(r_i r_k)$,
(R3) For $1 \leq a < i$, $\varphi(v_i r_i) \neq \varphi(r_a v_a)$,
(R4) For $i < b < k$, $\varphi(v_i r_i) \neq \varphi(r_k r_b)$,
(R5) $\varphi(v_i r_i) \neq \varphi(r_k w_k)$,
(R6) For $1 \leq a < i$, $\varphi(v_i r_i) \neq \varphi(r_k w'_a)$,
(R7) For $2 \leq i < k$, $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$,
where α is the vertex such that $\varphi(w_k \alpha) = \varphi(r_k w_{i-1})$,
(R8) For $i = 1$ and for $1 \leq c < k$, $\varphi(v_1 r_1) \neq \varphi(r_k \alpha)$,
for each vertex α incident with the edge of color $\varphi(r_k w_k)$ in T_c^{k-1},
(R9) For $2 \leq i < k$, $1 \leq a < i$, and for $i \leq b < k$, $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$,
for each vertex α incident with the edge of color $\varphi(r_k w_{i-1})$ in T_a^k and in T_b^{k-1},
(R10) For $1 \leq i < k$, $\varphi(v_i w_k) \neq \varphi(r_i r_k)$,
(R11) For $1 \leq d \leq k-2$, $\varphi(v_{k-1} r_{k-1}) \neq \varphi(w_k r_d)$.

From the observation in (2.7), we know that $\left|L_{k-1}^*\right| \geq 2m - 3k^2 + 6k - 3$.

An upper bound for the number of vertices eliminated through items (R2 - R11) as candidates for v_i is achieved when $i = k-1$. In this case, the number of vertices eliminated by R2, R3, ..., R11 is $(k-2), (k-2), 0, 1, (k-2), 1, 0, 2(k-1), 1, (k-2)$ respectively, the sum of which is $6k-7$. Now, since the induction hypothesis includes the condition $k \leq \Omega$, we can observe the following.

First, from $f(\Omega)$ and the definition of $L_{\Omega-1}$, we can follow the same steps as we did in (2.7) to see that $|L_{\Omega-1}| \geq 2m - 3\Omega^2 + 6\Omega - 1$ and further, that $\left|L_{\Omega-1}^*\right| \geq 2m - 3\Omega^2 + 6\Omega - 3$. Now, since by the induction hypothesis $k \leq \Omega$ and by (2.10) and (2.14) $\left|L_{i-1}^*\right| > |L_i^*|$ for $2 \leq i \leq k-1$, we have the following:

$$\begin{aligned}
\left|L_{k-1}^*\right| &\geq \left|L_{\Omega-1}^*\right| \\
&\geq 2m - 3\Omega^2 + 6\Omega - 3 \\
&= 2m - 3\left(\left\lfloor \frac{\sqrt{6m+9}}{3} \right\rfloor\right)^2 + 6\left\lfloor \frac{\sqrt{6m+9}}{3} \right\rfloor - 3 \\
&\geq 2m - (2m+3) + 2\sqrt{6m+9} - 3 \\
&= 2\sqrt{6m+9} - 6 \\
&= \frac{6}{3}\sqrt{6m+9} - 6 \\
&\geq 6\Omega - 6 \\
&> 6\Omega - 7 \\
&\geq 6k - 7.
\end{aligned} \tag{2.15}$$

In summary, we have that $\left|L_{k-1}^*\right| \geq \left|L_{\Omega-1}^*\right| > 6\Omega - 7 \geq 6k-7$. Therefore, $\left|L_{k-1}^*\right| > 6k-7$, and so such a vertex v_i meeting the restrictions in (R1 - R11) exists. The following cases show that this choice of v_i ensures that (C1), (C2), and (C3) hold.

2.1. Case 1. (C1) Edges in $T_a^k, 1 \leq a < i$ do not appear in T_i^k.

First, by the induction hypothesis we know that the trees $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all rainbow edge-disjoint and spanning. Inductively, we also assume for some i with $2 \leq i < k$ the trees $T_1^k, T_2^k, ..., T_{i-1}^k$ are edge-disjoint rainbow spanning trees as well. By (2.9), regardless of the choice of v_i, the only edges in T_i^k $(1 \leq i < k)$ that are not in T_i^{k-1} are $v_i v'_i$ and $r_k w_i$. Thus, if we can prove that the edges in $(E(T_i^{k-1}) \backslash \{r_i v_i, r_i r_k\}) \cup \{v_i v'_i, r_k w_i\}$ are not in T_a^k, $1 \leq a < i$, we will have shown that the trees $T_1^k, T_2^k, ..., T_i^k$ are all edge-disjoint rainbow and spanning; so by induction, $T_1^k, T_2^k, ..., T_{k-1}^k$ are edge-disjoint rainbow spanning trees.

To that end, for the remainder of Case 1 suppose that $2 \leq i < k$, $1 \leq a < i$, and $i < b < k$ and define the following sets of edges.

1. $E_{old}(T_a^k) = \{xy \mid xy \in E(T_a^{k-1}) \cap E(T_a^k)\}$
2. $E_{new}(T_a^k) = E(T_a^k) \backslash E(T_a^{k-1}) = \{v_a v_a', \, r_k w_a\}$
3. $E_{old}(T_i^k) = \{xy \mid xy \in E(T_i^{k-1}) \cap E(T_i^k)\}$
4. $E_{new}(T_i^k) = E(T_i^k) \backslash E(T_i^{k-1}) = \{v_i v_i', \, r_k w_i\}$

Observe that by (2.9), $E_{old}(T_a^k) \cap E_{new}(T_a^k) = \emptyset$ and $E(T_a^k) = E_{old}(T_a^k) \cup E_{new}(T_a^k)$. Similarly, $E_{old}(T_i^k) \cap E_{new}(T_i^k) = \emptyset$ and $E(T_i^k) = E_{old}(T_i^k) \cup E_{new}(T_i^k)$.

Since the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ are formed sequentially, it is clearly necessary to prohibit edges $v_i v_i'$ and $r_k w_i$ from appearing in T_a^k. It is also very useful to prohibit edges $v_i v_i'$ and $r_k w_i$ from appearing in T_b^{k-1}.

Consequently, when v_i was selected to satisfy (R1 - R11) it was done in such a way that ensures the following six properties are satisfied:

(P1) $v_i v_i', \, r_k w_i \notin E_{old}(T_a^k)$,
(P2) $v_i v_i', \, r_k w_i \notin E_{new}(T_a^k)$,
(P3) $v_i v_i', \, r_k w_i \notin E(T_b^{k-1})$,
(P4) $E_{old}(T_i^k) \cap E_{old}(T_a^k) = \emptyset$,
(P5) $E_{old}(T_i^k) \cap E_{new}(T_a^k) = \emptyset$,
(P6) $E_{old}(T_i^k) \cap E(T_b^{k-1}) = \emptyset$.

It is clear that if properties (P1 - P6) are satisfied, then T_i^k is edge-disjoint from the trees, T_a^k and T_b^{k-1}. We consider edges $v_i v_i'$ and $r_k w_i$ in turn for properties (P1 - P3), then address properties (P4 - P6).

2.1.1. Property (P1) for $v_i v_i'$.
Since $E_{old}(T_a^k) \subset E(T_a^{k-1})$, we can prove $v_i v_i'$ is not an edge in $E_{old}(T_a^k)$ by showing that $v_i v_i' \notin E(T_a^{k-1})$.

Recall from (R1) and (2.14) that because $v_i \in L_{k-1}^*$, v_i is a leaf adjacent to the root r_c in T_c^{k-1}, for $1 \leq c < k$. Therefore, to show that $v_i v_i' \notin E(T_a^{k-1})$, we need only prove that $v_i' \neq r_a$. The following argument shows that (R2) guarantees this property.

Suppose to the contrary that $v_i' = r_a$. Then $v_i v_i' = v_i r_a$ and by (2.9), $\varphi(v_i r_a) = \varphi(v_i v_i') = \varphi(r_i r_k)$, contradicting (R2). It follows that $v_i' \neq r_a$ so $v_i v_i' \notin E_{old}(T_a^k)$, as required.

2.1.2. Property (P2) for $v_i v_i'$.
Recall that $E_{new}(T_a^k) = \{v_a v_a', r_k w_a\}$. Thus, to prove that $v_i v_i' \notin E_{new}(T_a^k)$ for $1 \leq a < i$, we need only show that $v_i v_i' \neq v_a v_a'$ and $v_i v_i' \neq r_k w_a$. We consider each in turn.

(i.) $v_i v_i' \neq v_a v_a'$
By (2.9), we have that $\varphi(v_i v_i') = \varphi(r_i r_k)$ and $\varphi(v_a v_a') = \varphi(r_a r_k)$. But, by property (1) of $f(\psi)$ when $\psi = k - 1$ we know $r_i \neq r_a$ and so $\varphi(r_i r_k) \neq \varphi(r_a r_k)$. It follows that $\varphi(v_i v_i') \neq \varphi(v_a v_a')$ and, therefore, $v_i v_i' \neq v_a v_a'$.

(ii.) $v_i v_i' \neq r_k w_a$
Assume that $v_i v_i' = r_k w_a$ and recall from (2.14) that because $v_i \in L_{k-1}^*$, $v_i \neq r_k$. Therefore, $v_i = w_a$. By (2.9), $\varphi(v_i v_i') = \varphi(r_k r_i)$, so since we are assuming that $v_i v_i' = r_k w_a$, clearly $\varphi(r_k r_i) = \varphi(r_k w_a)$ and so $w_a = r_i = v_i$. But because $v_i \in L_{k-1}^*$, $v_i \neq r_i$ and this is a contradiction.

Combining the above two arguments, it is clear that $v_i v_i' \notin E_{new}(T_a^k)$, as required.

2.1.3. Property (P3) for v_iv_i'.
Recall from (2.14) that $v_i \in L^*_{k-1}$, so r_bv_i is a pendant edge with leaf v_i in T_b^{k-1}, for $i < b < k$. Thus, v_iv_i' would only be an edge in T_b^{k-1} if $v_i' = r_b$. As in Section 2.1.1 above, (R2) prevents v_i' from equalling r_b by guaranteeing that $\varphi(v_ir_b) \neq \varphi(r_ir_k)$ and therefore, $v_iv_i' \notin E(T_b^{k-1})$, as required.

2.1.4. Property (P1) for r_kw_i.
Recall from (2.6) that $r_k \in L_{k-1}$, so r_kr_a is a pendant edge in T_a^{k-1} with leaf r_k. Therefore, from (2.9) it is clear that $r_kr_a \notin E(T_a^k)$ since it is removed from T_a^{k-1} in forming T_a^k. So r_k is not incident with any edges in $E_{old}(T_a^k)$ and thus, r_kw_i cannot be an edge in $E_{old}(T_a^k)$, as required.

2.1.5. Property (P2) for r_kw_i.
Recall that $E_{new}(T_a^k) = \{v_av_a', r_kw_a\}$. To show that $r_kw_k \notin E_{new}(T_a^k)$, we prove that $r_kw_i \neq r_kw_a$ and $r_kw_i \neq v_av_a'$ for $1 \leq a < i$. We consider each in turn.

(i.) $r_kw_i \neq r_kw_a$
To show that $r_kw_i \neq r_kw_a$, we need only show that $w_i \neq w_a$.
By (2.9) we have that $\varphi(r_kw_i) = \varphi(r_iv_i)$ and $\varphi(r_kw_a) = \varphi(r_av_a)$. So if $r_kw_i = r_kw_a$, then $\varphi(v_ir_i) = \varphi(r_av_a)$, contradicting (R3). Therefore, $r_kw_i \neq r_kw_a$, as required.

(ii.) $r_kw_i \neq v_av_a'$
Assume that $r_kw_i = v_av_a'$. Recall from (2.14) that because $v_a \in L^*_{k-1}$, $v_a \neq r_k$. Therefore, $v_a = w_i$. By (2.9), $\varphi(v_av_a') = \varphi(r_ar_k)$, so since we are assuming that $r_kw_i = v_av_a'$, then $\varphi(r_kw_i) = \varphi(r_kr_a)$ and it follows that $r_a = w_i = v_a$. But this is a contradiction because $v_a \in L^*_{k-1}$ so by (2.14), $v_a \neq r_a$.

Combining the above two arguments, it is clear that $r_kw_i \notin E_{new}(T_a^k)$, as required.

2.1.6. Property (P3) for r_kw_i.
Recall that by (2.8), because r_k was chosen to be in L_{k-1}, r_k is a leaf adjacent to the root of T_b^{k-1}, $i < b < k$. Thus, to show $r_kw_i \notin E(T_b^{k-1})$, we need only prove that $w_i \neq r_b$.

By (2.9), we have that $\varphi(r_kw_i) = \varphi(v_ir_i)$. So if $w_i = r_b$, then $r_kw_i = r_kr_b$ and $\varphi(v_ir_i) = \varphi(r_kr_b)$, contradicting (R4). Therefore, $r_kw_i \notin E(T_b^{k-1})$, as required.

2.1.7. Properties (P4), (P5), and (P6).
We consider each property, (P4), (P5), and (P6), in turn.

(i.) Property (P4)
By our induction hypothesis, the trees, $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all edge disjoint. So (P4) follows because $E_{old}(T_i^k) \subset E(T_i^{k-1})$ and $E_{old}(T_a^k) \subset E(T_a^{k-1})$.

(ii.) Property (P5)
Since $a < i$, from (P3) (replacing i with a), it follows that $\{v_av_a', r_kw_a\} \cap E(T_c^{k-1}) = \emptyset$, for $a < c < k$. In particular, since $i > a$, it follows that $E_{new}(T_a^k) \cap E(T_i^{k-1}) = \emptyset$. And lastly, since $E_{old}(T_i^k) \subset E(T_i^{k-1})$, we have that $E_{old}(T_i^k) \cap E_{new}(T_a^k) = \emptyset$.

(iii.) Property (P6)
Again, by our induction hypothesis, the trees, $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ are all edge-disjoint. It follows that $E_{old}(T_i^k) \cap E(T_b^{k-1}) = \emptyset$ because $E_{old}(T_i^k) \subset E(T_i^{k-1})$.

Therefore, properties (P4 - P6) hold for $E_{old}(T_i^k)$.

The above Sections 2.1.1 – 2.1.7 ensure that properties (P1 - P6) hold. As stated above, since these six properties hold, the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ are all edge-disjoint and further, from (2.9), are also rainbow and spanning.

2.2. Case 2. (C2) Edges in T_k^k do not appear in T_i^k.

Recall from (2.11) that T_k^k is defined by a sequence, $T_k^k(1), T_k^k(2), ..., T_k^k(k)$, and from (2.13) that at the i^{th} induction step, $T_k^k(i)$ was determined by the choice of v_i. It is convenient to restate (2.11) and (2.12) here:

$T_k^k(i) = S_{r_k} - r_k w_1 - ... - r_k w_i + w_1 w_1' + ... + w_i w_i'$,
where $\varphi(w_1 w_1') = \varphi(r_k w_k)$ and $\varphi(w_i w_i') = \varphi(r_k w_{i-1})$ for $2 \leq i \leq k$.

$T_k^k = S_{r_k} - r_k w_1 - ... - r_k w_k + w_1 w_1' + ... + w_k w_k'$,
where $\varphi(w_1 w_1') = \varphi(r_k w_k)$ and for $2 \leq c \leq k$, $\varphi(w_c w_c') = \varphi(r_k w_{c-1})$,

For the remainder of Case 2, suppose that $1 \leq i < k$, $1 \leq a < i$, and $i < b < k$.

In order to prevent edges in T_k^k from also appearing in T_i^k, we will now show that T_i^k has been constructed in such a way that $T_k^k(i)$ and T_k^k satisfy the following properties:

(P7) $E(T_k^k(i)) \cap E(T_a^k) = \emptyset$
(P8) $E(T_k^k(i)) \cap E(T_b^{k-1}) = \{r_k r_b\}$
(P9) $E(T_k^k(i)) \cap E_{old}(T_i^k) = \emptyset$
(P10) $E(T_k^k(i)) \cap E_{new}(T_i^k) = \emptyset$
(P11) $w_k w_k' \notin E(T_i^k)$

We note here that by (2.9), when T_b^k was constructed from T_b^{k-1}, edge $r_k r_b$ was removed, so it does not appear in T_b^k. Therefore, it is not necessary to prevent $r_k r_b$ from being an edge in $T_k^k(i)$ nor T_k^k.

Proving the above five properties will be done inductively. We show in the base step that $T_k^k(1)$ satisfies properties (P7 - P10) with $i = 1$, and then show that for $2 \leq i < k$, $T_k^k(i)$ satisfies the same four properties before finally proving property (P11).

The following preliminary result will be useful in proving properties (P7 - P11).

2.2.1. Preliminary Result: $w_i \neq w_k$.

Recall from (2.8) that $w_k \in L_{k-1}$ was selected with r_k before any of the rainbow spanning trees $T_1^{k-1}, T_2^{k-1}, ..., T_{k-1}^{k-1}$ were revised. It will be useful to show that the vertices $w_i \in T_i^k$, $1 \leq i < k$, cannot equal w_k.

From (2.9), we have that $\varphi(v_i r_i) = \varphi(r_k w_i)$. So if $w_i = w_k$, then $\varphi(v_i r_i) = \varphi(r_k w_k)$ contradicting (R5). Therefore, $w_i \neq w_k$.

2.2.2. Base Step: $i = 1$.

Observe that for $2 \leq b < k$, $E(S_{r_k}) \cap E(T_b^{k-1}) = \{r_k r_b\}$ and $E(S_{r_k}) \cap E_{old}(T_1^k) = \emptyset$ since by (2.9), $r_k r_1$ is removed from T_1^{k-1} when forming T_1^k. Further, it is clear from (2.11) that the only edge in $T_k^k(1)$ that is not in S_{r_k} is $w_1 w_1'$.

(i.) (P7)
Since $i = 1$, there do not exist any such trees T_a^k since $1 \leq a < i$ and so property (P7) is vacuously true.

(ii.) (P8) and (P9)
First, recall that $E_{old}(T_1^k) \subset E(T_1^{k-1})$. To establish properties (P8) and (P9), we show that $w_1 w_1' \notin E(T_c^{k-1})$ for $1 \leq c < k$.

Suppose to the contrary that $w_1w_1' \in E(T_c^{k-1})$. Recall from (2.11) that $\varphi(w_1w_1') = \varphi(r_kw_k)$. So if $w_1w_1' \in E(T_c^{k-1})$, then w_1 is a vertex incident to the edge of color $\varphi(r_kw_k)$ in T_c^{k-1}. But this is impossible since from (2.9) we have that $\varphi(v_1r_1) = \varphi(r_kw_1)$ and from (R8) that $\varphi(v_1r_1) \neq \varphi(r_k\alpha)$, where α is a vertex incident to the edge of color $\varphi(r_kw_k)$ in T_c^{k-1}. Therefore, $w_1w_1' \notin E(T_c^{k-1})$ and $T_k^k(1)$ satisfies properties (P8) and (P9).

(iii.) (P10)

Recall that $E_{new}(T_i^k) = \{v_iv_i', r_kw_i\}$. To establish (P10) for $T_k^k(1)$, we need only show that $w_1w_1' \neq v_1v_1'$ and $w_1w_1' \neq r_kw_1$. We consider each in turn.

(a.) $w_1w_1' \neq v_1v_1'$

Recall from (2.9) that $\varphi(v_1v_1') = \varphi(r_kr_1)$ and from (2.11) that $\varphi(w_1w_1') = \varphi(r_kw_k)$. So if $w_1w_1' = v_1v_1'$, then $\varphi(r_kw_k) = \varphi(r_kr_1)$ and so $w_k = r_1$. But this is not possible because by (2.8) $w_k \in L_{k-1}$ and so $w_k \neq r_1$. Therefore, $w_1w_1' \neq v_1v_1'$.

(b.) $w_1w_1' \neq r_kw_1$

Recall from (2.11) that $\varphi(w_1w_1') = \varphi(r_kw_k)$. So if $w_1w_1' = r_kw_1$, then $\varphi(r_kw_k) = \varphi(r_kw_1)$ and so $w_k = w_1$, contradicting the result in Section 2.2.1. Thus, $w_1w_1' \neq r_kw_1$.

Therefore, property (P10) holds for $T_k^k(1)$ and we have established our base step.

2.2.3. Property (P7) for $2 \leq i < k$.
From (2.11), it is clear that the only edge in $T_k^k(i)$ that differs from $T_k^k(i-1)$ is w_iw_i'. Therefore, since by induction we have that $T_k^k(i-1)$ satisfies (P7), in order to prove property (P7) is satisfied for $T_k^k(i)$, we need only show that w_iw_i' is not an edge in T_a^k, $1 \leq a < i$.

To that end, suppose to the contrary that $w_iw_i' \in E(T_a^k)$. Recall from (2.11) that $\varphi(w_iw_i') = \varphi(r_kw_{i-1})$. So if $w_iw_i' \in E(T_a^k)$, then w_i is a vertex incident to the edge of color $\varphi(r_kw_{i-1})$ in T_a^k. But this is impossible since from (2.9) we have that $\varphi(v_ir_i) = \varphi(r_kw_i)$ and from (R9) that $\varphi(v_ir_i) \neq \varphi(r_k\alpha)$, where α is a vertex incident to the edge of color $\varphi(r_kw_{i-1})$ in T_a^k. Therefore, $w_iw_i' \notin E(T_a^k)$ and $T_k^k(i)$ satisfies property (P7).

2.2.4. Properties (P8) and (P9) for $2 \leq i < k$.
Observe again that $E_{old}(T_i^k) \subset E(T_i^{k-1})$. As in Section 2.2.3, to prove properties (P8) and (P9) for $T_k^k(i)$, we can show that $w_iw_i' \notin E(T_d^{k-1})$, $i \leq d < k$.

For $i \leq d < k$, property (R9), which guarantees $\varphi(v_ir_i) \neq \varphi(r_k\alpha)$, where α is a vertex incident to the edge of color $\varphi(r_kw_{i-1})$ in T_d^{k-1}, ensures $w_iw_i' \notin E(T_d^{k-1})$, thus ensuring that (P8) and (P9) hold for $T_k^k(i)$. The argument has been omitted here due to its similarity to the argument used above for (P7) in Section 2.2.3.

2.2.5. Property (P10) for $2 \leq i < k$.
To prove (P10) for $T_k^k(i)$, we need only show that $w_iw_i' \neq v_iv_i'$ and $w_iw_i' \neq r_kw_i$. We consider each in turn.

(i.) $w_iw_i' \neq v_iv_i'$

Recall from (2.9) that $\varphi(v_iv_i') = \varphi(r_kr_i)$ and from (2.11) that $\varphi(w_iw_i') = \varphi(r_kw_{i-1})$. If $w_iw_i' = v_iv_i'$, then $\varphi(r_kw_{i-1}) = \varphi(r_kr_i)$ and so $w_{i-1} = r_i$. But $r_kr_i \in E(T_i^{k-1})$ and $r_kw_{i-1} \in E(T_{i-1}^k)$; so if $w_{i-1} = r_i$, this contradicts property (P3) in the $i-1^{th}$ induction step, which in particular (i.e. when $b = i$) ensures that $r_kw_{i-1} \notin E(T_i^{k-1})$. Therefore, $w_iw_i' \neq v_iv_i'$, as required.

(ii.) $w_iw_i' \neq r_kw_i$

Recall from (2.11) that $\varphi(w_iw'_i) = \varphi(r_kw_{i-1})$. If $w_iw'_i = r_kw_i$, then $\varphi(r_kw_{i-1}) = \varphi(r_kw_i)$ and so $w_{i-1} = w_i$. However, this is impossible by the result in Section 2.1.5 which, in particular, proved that $r_kw_i \neq r_kw_a$ for $1 \leq a < i$. Thus, $w_iw'_i \neq r_kw_i$.

Therefore, property (P10) holds for $T_k^k(i)$, as required.

2.2.6. Property (P11) for $w_kw'_k$.

The above sections of Case 2 ensure that the rainbow spanning trees $T_1^k, T_2^k, ..., T_{k-1}^k$ and the rainbow spanning graph, $T_k^k(k-1)$ are all edge-disjoint. Thus, it remains to show that $T_1^k, T_2^k, ..., T_{k-1}^k$ and T_k^k are all edge-disjoint. As above, recall from (2.11) that the only edge in T_k^k that differs from $T_k^k(k-1)$ is $w_kw'_k$. Therefore, showing property (P11) holds will prove that $T_1^k, T_2^k, ...T_{k-1}^k$ and T_k^k are edge-disjoint.

First, observe from (2.8) that since $w_k \in L_{k-1}$, w_k is a leaf adjacent to the root r_i in T_i^{k-1} for $1 \leq i < k$. So if $w_kw'_k \in E(T_i^k)$, $w_kw'_k = w_ir_k$, $v_iv'_i$, or w_kr_i. We consider each in turn.

(i.) $w_kw'_k \neq w_ir_k$
From (2.8) we know that $w_k \neq r_k$. So if $w_kw'_k = w_ir_k$, then $w_k = w_i$, contradicting the preliminary result in Section 2.2.1. Therefore, $w_kw'_k \neq w_ir_k$, as required.

(ii.) $w_kw'_k \neq v_iv'_i$
Recall from (2.14) that since $v_i \in L_{k-1}^*$, $v_i \neq w_k$. So if $w_kw'_k = v_iv'_i$, then $w_k = v'_i$. From (2.9) we know that $\varphi(v_iv'_i) = \varphi(r_ir_k)$, so if $w_k = v'_i$, then $\varphi(v_iw_k) = \varphi(r_ir_k)$, contradicting (R10). Therefore, $w_kw'_k \neq v_iv'_i$, as required.

(iii.) $w_kw'_k \neq w_kr_i$
Recall from (2.11) that $\varphi(w_kw'_k) = \varphi(r_kw_{k-1})$ and suppose that $w_kw'_k = w_kr_i$. First observe that $i \neq k-1$ since $r_kw_{k-1} \in E(T_{k-1}^k)$ and we know from (2.8) and Section 2.2.1 that $w_k \neq r_k$ and $w_k \neq w_{k-1}$.
Now, for $1 \leq i \leq k-2$, if $w_kw'_k = w_kr_i$ then $r_i = w'_k$. But from (2.9) and (2.11) if $r_i = w'_k$ then $\varphi(w_kw'_k) = \varphi(r_kw_{k-1}) = \varphi(v_{k-1}r_{k-1}) = \varphi(w_kr_i)$, contradicting (R11). Therefore, $w_kw'_k \neq w_kr_i$, as required.

It follows that $w_kw'_k \notin E(T_i^k)$, $1 \leq i < k$.

The above Sections 2.2.1 - 2.2.6 ensure that the trees $T_1^k, T_2^k, ..., T_{k-1}^k$ and the graph T_k^k are all edge-disjoint. Further, from (2.9) it is clear that $T_1^k, T_2^k, ..., T_{k-1}^k$ are all rainbow spanning trees and from (2.12) that T_k^k is a spanning rainbow graph (since for every leaf, w_c, $1 \leq c \leq k$, which is adjacent to r_k and for which r_kw_c is removed from T_k^k, there exists w'_c such that the edge $w_cw'_c$ is added to T_k^k and edge $w_dw'_d$ in T_k^k such that $\varphi(w_dw'_d) = \varphi(r_kw_c)$, where $d \equiv c+1 \mod k$.)

2.3. Case 3. (C3) Preventing cycles from appearing in T_k^k.

Properties (C1) and (C2) in the previous sections guarantee that the rainbow spanning trees $T_1^k, T_2^k, ..., T_{k-1}^k$ and the rainbow spanning graph T_k^k are all edge-disjoint. Thus, it remains to prove that T_k^k is acyclic and, therefore, a tree. This is proved inductively, showing that for $1 \leq i \leq k$, $T_k^k(i)$ is acyclic. Formally, we will show the following two properties:

(P12) $T_k^k(i)$ is acyclic for $1 \leq i < k$, and

(P13) T_k^k is acyclic

We consider each in turn.

2.3.1. Property (P12).

Proving $T_k^k(i)$ is acyclic will also be done inductively. For our base step, we let

$T_k^k(0) = S_{r_k}$ and observe that this graph is clearly acyclic.

It is clear from (2.11) that for $1 \leq i < k$, $T_k^k(i) = T_k^k(i-1) - r_k w_i + w_i w_i'$. Therefore, since by induction we have that $T_k^k(i-1)$ satisfies (P12), in order to prove $T_k^k(i)$ is acyclic, we need only show that adding $w_i w_i'$ to $T_k^k(i-1) - r_k w_i$ does not create a cycle. Let $T_k^k(i-1)^* = T_k^k(i-1) - r_k w_i$.

Now, from (2.11) observe that all of the edges in $T_k^k(i-1)$ are of the form $r_k x$, $r_k w_a'$, and $w_a w_a'$, where $1 \leq a < i$ and $x \in V(K_{2m}) \backslash (\{\bigcup_{a=1}^{i-1} w_a, w_a'\} \cup \{r_k\})$. Thus, $w_i \in \{r_k, x, w_a, w_a'\}$. We now show that $w_i = x$ and, further, that since $w_i = x$, $T_k^k(i)$ is acyclic. We consider each claim in turn.

(i.) $w_i = x$

First observe that $w_i \neq r_k$ since $r_k w_i$ is an edge in T_i^k. Also, $w_i \neq w_a$ (this property is established by (R3) and was discussed in Section 2.1.5). Lastly, recall from (2.9) that $\varphi(v_i r_i) = \varphi(r_k w_i)$. So if $w_i = w_a'$ then $\varphi(v_i r_i) = \varphi(r_k w_a')$, contradicting (R6). Therefore, $w_i \neq w_a'$ and it follows that $w_i = x$.

(ii.) $T_k^k(i)$ is acyclic

Observe that since $w_i = x$, $w_i \in V(K_{2m}) \backslash (\{\bigcup_{a=1}^{i-1} w_a, w_a'\} \cup \{r_k\})$ and w_i is a leaf adjacent to r_k in $T_k^k(i-1)$. Now, in order for $w_i w_i'$ to create a cycle in $T_k^k(i)$, there would have to exist a path from w_i to w_i' in $T_k^k(i-1)^*$. But, as we just observed, w_i is a leaf in $T_k^k(i-1)$ and since $T_k^k(i-1)^* = T_k^k(i-1) - r_k w_i$, w_i is an isolated vertex in $T_k^k(i-1)^*$ so it follows that no such path exists. Therefore, $T_k^k(i)$ is acyclic, as required.

The above two arguments show that (P12) holds for $T_k^k(i)$.

2.3.2. Property (P13).

In Section 2.3.1 above, we showed that $T_k^k(i)$ is acyclic for $1 \leq i < k$. Recall from (2.11) that $T_k^k = T_k^k(k-1) - r_k w_{k-1} + w_k w_l'$. Thus, in order to prove T_k^k is acyclic, we need only show that adding $w_k w_k'$ to $T_k^k(k-1) - r_k w_k$ does not create a cycle. As in Section 2.3.1, let $T_k^k(k-1)^* = T_k^k(k-1) - r_k w_k$.

Observe from (2.11) that all of the edges of $T_k^k(k-1)$ are of the form $r_k x$, $r_k w_i'$ and $w_a w_a'$, where $1 \leq i < k$ and $x \in V(K_{2m}) \backslash (\{\bigcup_{a=1}^{k-1} w_i, w_i'\} \cup \{r_k\})$. Thus, $w_k \in \{r_k, x, w_i, w_i'\}$. We claim that $w_k = x$ and, further, that since $w_k = x$, T_k^k is acyclic. We consider each claim in turn.

(i.) $w_k = x$

Begin by observing that $w_k \neq r_k$ (since by (2.8) w_k and r_k were chosen to be distinct vertices) and, for $1 \leq i < k$, $w_k \neq w_i$ (this property was established by (R5) and discussed in Section 2.2.1). The following argument shows $w_k \neq w_i'$.

First, observe that $w_k \neq w_1'$ since $\varphi(w_1 w_1') = \varphi(r_k w_k)$, so if $w_k = w_1'$ then $w_1 = r_k$, which we know from (2.9) cannot be the case.

Now, for $2 \leq i < k$, let $\alpha \in V(K_{2m})$ be the vertex such that $\varphi(w_k \alpha) = \varphi(r_k w_{i-1})$ and recall from (2.12) that $\varphi(w_i w_i') = \varphi(r_k w_{i-1})$. Suppose that $w_k = w_i'$. Then since $\varphi(w_k \alpha) = \varphi(r_k w_{i-1}) = \varphi(w_i w_i') = \varphi(w_i w_k)$, α must equal w_i. But from (2.9), we have that $\varphi(v_i r_i) = \varphi(r_k w_i)$, so if $w_i = \alpha$ then $\varphi(v_i r_i) = \varphi(r_k \alpha)$, contradicting (R7) which ensures that $\varphi(v_i r_i) \neq \varphi(r_k \alpha)$, where α is the vertex such that $\varphi(w_k \alpha) = \varphi(r_k w_{i-1})$. Therefore, $w_k \neq w_i'$,

$2 \leq i < k$.

Combining the above arguments, it is clear that $w_k = x$.

(ii.) T_k^k is acyclic

Observe that since $w_k = x$ where $x \in V(K_{2m}) \backslash (\{\bigcup_{a=1}^{k-1} w_i, w_i'\} \cup \{r_k\})$, w_k is a leaf adjacent to r_k in $T_k^k(k-1)$. In order for $w_k w_k'$ to form a cycle in T_k^k, there would have to exist a path from w_k to w_k' in $T_k^k(k-1)^*$. But because w_k is a leaf adjacent to r_k in $T_k^k(k-1)$, w_k is an isolated vertex in $T_k^k(k-1)^*$ since $T_k^k(k-1)^* = T_k^k(k-1) - r_k w_k$. It follows that no such path from w_k to w_k' exists in $T_k^k(k-1)^*$ and, consequently, T_k^k must be acyclic, as required.

It follows that T_k^k is acyclic, satisfying (P13).

The above Sections 2.3.1 and 2.3.2 show that properties (P12) and (P13) hold, thus completing the proof of the theorem.

□

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