Packing Graphs with Graph of Size Three

Zhen-Chun Chen and Hung-Lin Fu
Department of Applied Mathematics
National Chiao Tung University
Hsinchu 300, Taiwan

Kuo-Ching Huang*
Department of Financial and Computational Mathematics
Providence University
Shalu 43301, Taichung, Taiwan

Abstract

An H-packing $\mathcal{F}$ of a graph G is a set of edge-disjoint subgraphs of G in which each subgraph is isomorphic to H. The leave L or the remainder graph L of a packing $\mathcal{F}$ is the subgraph induced by the set of edges of G that does not occur in any subgraph of the packing $\mathcal{F}$. If a leave L contains no edges, or simply $L = \phi$, then G is said to be H-decomposable, denoted by $H|G$. In this paper, we prove a conjecture made by Chartrand, Saba and Mynhardt[11]: *If G is a graph of size $q(G) \equiv 0 (\text{mod } 3)$ and $\delta(G) \geq 2$, then G is H-decomposable for some graph H of size* 3.

Keywords: Graph decomposition, H-decomposition, packing, H-packing, Maximum packing, Minimum leave

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, simple and undirected graph. The *order*, *size*, *maximum* and *minimum degree* of G are denoted by $p(G)$, $q(G)$, $\Delta(G)$ and $\delta(G)$, respectively. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices

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adjacent to v. The graphs P_n and C_k are a path of order n and a cycle of order $k \geq 3$, respectively. The graph $G_1 \bigcup G_2$ is the vertex disjoint union of G_1 and G_2. For more graph theoretic terminologies we refer to [10].

A graph G is said to be *H-decomposable*, denoted by $H|G$, if the edge set $E(G)$ of G can be partitioned into subsets such that the edge-induced subgraph of each subset is isomorphic to H. Graph decomposition is one of the most important topics in the study of both graph theory and combinatorial designs, don't mention their applications on many fields. Quite a few research results are obtained in considering the decomposition of complete graphs or complete multipartite graphs into complete subgraphs or cycles. See [1-9, 12-26] for references. Theoretically if the graph to be decomposed has better structure, then the decomposition can be carried out through difference method or applying combinatorial objects such as Latin squares. On the other hand, if we consider the decomposition, packing or covering of a general graph, it is getting more complicate. In [11], Chartrand, Saba and Mynhardt study prime graphs and proposed the following:

Conjecture 1 [11] *Suppose G is a graph of size $q(G) \equiv 0 (\text{mod } 3)$ and $\delta(G) \geq 2$. Then G is H-decomposable for some graph H of size 3.*

Conjecture 2 [11] *Suppose G is a 2-connected graph of order $p(G) \geq 2$ and of size $q(G) \equiv 0 (\text{mod } 3)$. Then G is P_4-decomposable.*

These conjectures motivate our study of decomposing a graph of size $3k$ into k copies of isomorphic graphs of size 3. If $q(H) = 3$, then $H = K_3,\ P_4,\ K_{1,3},\ P_3 \bigcup P_2$ or M_3(a matching of size 3). Since the graph $D = \{x_1x_2x_3x_4x_5x_6x_1\} \cup \{x_1y_1x_2, x_3y_2x_4, x_5y_3x_6\}$ disproves the Conjecture 2, we will focus on the Conjecture 1. In order to prove the Conjecture 1, for each given graph G such that $q(G) \equiv 0 (\text{mod } 3)$, we have to find a graph H of size 3 and prove that $H|G$. It is not difficult to see that $G|G$ if $q(G) = 3$ and the complete graph K_4 is P_4-decomposable. Moreover, it can be argued that the complete bipartite graph $K_{2,3}$ and the complete 3-partite graph $K_{1,1,4}$ both are P_4-decomposable. Since the graph $K_{1,1,3c+1} = K_{1,1,4} \cup (c-1)K_{2,3}$, we have $P_4|K_{1,1,3c+1}, c \geq 1$. In this paper, we claim that if G is of size $q(G) \equiv 0 (\mathrm{mod}\ 3)$ and $\delta(G) \geq 2$, then G is $(P_3 \bigcup P_2)$-decomposable if and only if G is different from K_4 and $K_{1,1,3c+1}, c \geq 0$. Therefore, the Conjecture 1 is affirmative.

2 Main results

We start this section with the study of $(P_3 \bigcup P_2)$-packings of graphs. An *H-packing* of a graph G is a set of edge-disjoint subgraphs of G in which each subgraph is isomorphic to H. An H-packing $\mathcal{F}$ is *maximum* if $|\mathcal{F}| \geq |\mathcal{F}'|$ for all other H-packings $\mathcal{F}'$ of G. The *leave* L of an H-packing $\mathcal{F}$ is the subgraph induced by the set of edges of G that does not occur in any subgraph of the H-packing $\mathcal{F}$. Therefore, a maximum packing has a minimum leave. In what follows, all the leaves we consider are minimum. It is easy to see that $H|G$ if and only if G has an H-packing with empty leave L, i.e., L contains no edge, or simply $L = \phi$.

The Lemmas 3, 4 and 5 are essential for proving the main theorem. Since they are easy to be proved, we omit the proofs.

Lemma 3 *If $G \cong G_i$, $1 \leq i \leq 18$, given in Figure 1, then $P_3 \bigcup P_2|G$.*

Lemma 4 *If $G \cong G_i$, $19 \leq i \leq 26$, given in Figure 2, then G has a $(P_3 \bigcup P_2)$-packing with leave an edge.*

Lemma 5 *If $G \cong G_i$, $27 \leq i \leq 40$, given in Figure 3, then G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.*

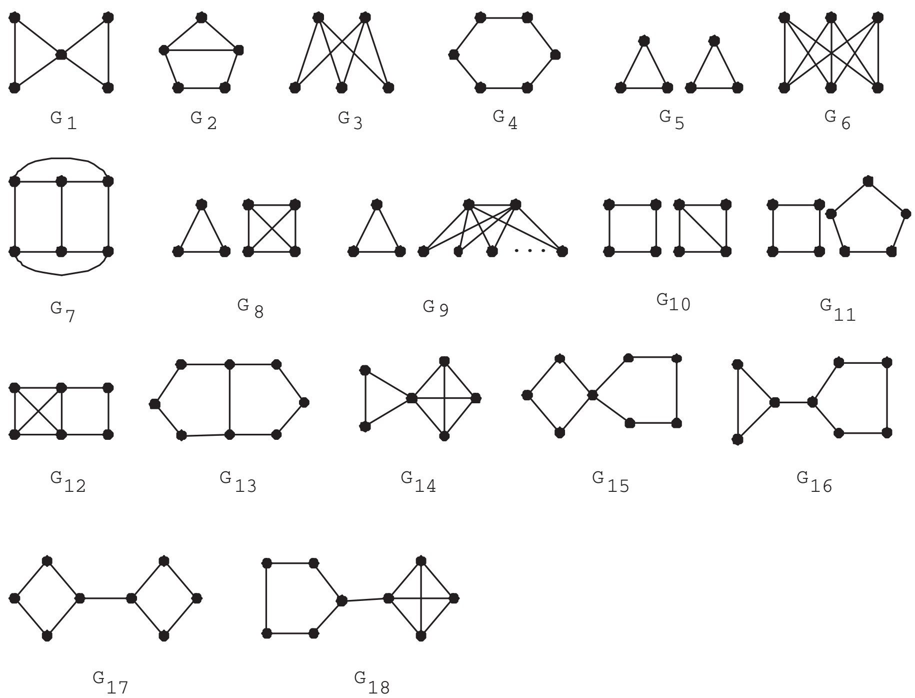

Figure 1.

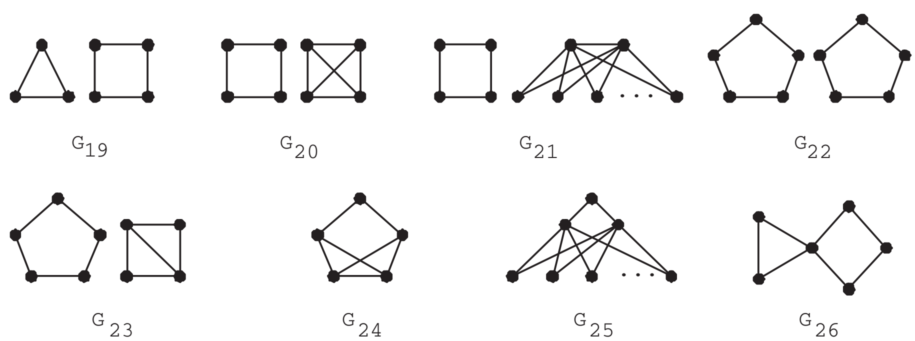

Figure 2.

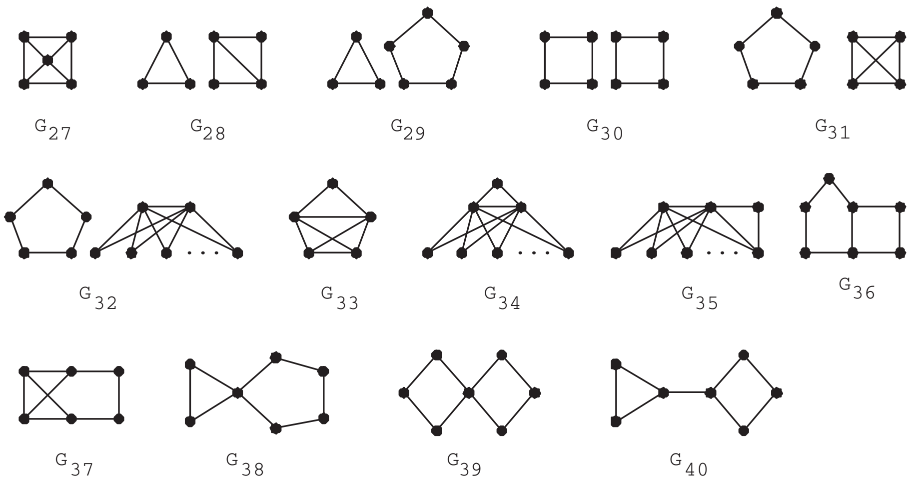

Figure 3.

The next result is our main theorem.

Theorem 6 *Suppose G is a graph different from $K_{1,1,3c+1}$ with $p(G) \geq 5$, $q(G) \geq 6$ and $\delta(G) \geq 2$. Then G has a $(P_3 \bigcup P_2)$-packing with leave L, where*

$$L = \begin{cases} \phi & if\ q(G) \equiv 0 (\text{mod } 3); \\ P_2 & if\ q(G) \equiv 1 (\text{mod } 3); \\ P_3 & if\ q(G) \equiv 2 (\text{mod } 3). \end{cases}$$

Proof. By induction on $q(G)$.

If $q(G) = 6$, then $G = G_i$, $1 \leq i \leq 5$, given in Figure 1. By Lemma 3, we have $P_3 \bigcup P_2 | G$.

Suppose the assertion holds for any graph G' different from $K_{1,1,3c+1}$ with $p(G') \geq 5$, $\delta(G') \geq 2$ and $q(G') < q$, where $q \geq 7$. Let G be a graph different from $K_{1,1,3c+1}$ with $p(G) \geq 5$, $q(G) = q$ and $\delta(G) \geq 2$. There are three cases to be considered.

Case 1. $\Delta(G) \geq 4$ and $\delta(G) \geq 3$.

By degree-sum formula, $q(G) = \frac{1}{2} \sum_{x \in V(G)} d(x) \geq \frac{1}{2}(4 + 3 \times 4) = 8$. If $q(G) = 8$, then $G = G_{27}$. By Lemma 5, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

Now, suppose $q(G) > 8$. Let v be a vertex with $d(v) = \Delta(G)$ and $N(v) =$

$\{v_1, v_2, \cdots, v_{\Delta(G)}\}$. If v_1 is adjacent to some v_i for $i \geq 2$, say $v_1v_2 \in E(G)$, let $G' = G - \{v_3vv_4, v_1v_2\}$; otherwise, let u be a neighbor of v_1 which is different from v and $G' = G - \{v_2vv_3, v_1u\}$. Then G' satisfies the induction hypothesis. Since $G = G' \bigcup (P_3 \bigcup P_2)$, the assertion holds for the graph G.

Case 2. G is 3-regular.

First, Suppose G is connected. If $p = p(G) = 6$, then $G = G_6$ or G_7. By Lemma 3, $P_3 \bigcup P_2 | G$.

Suppose $P_3 \bigcup P_2 | G'$ for any connected 3-regular graph G' of order less than p, where $p \geq 8$. Let G be a connected 3-regular graph of order p. It can be argued that G has an edge xy with $N(x) = \{x_1, x_2, y\}$, $N(y) = \{y_1, y_2, x\}$, $N(x) \bigcap N(y) = \phi$, $x_1y_1 \notin E(G)$ and $x_2y_2 \notin E(G)$. Let $G' = (G - \{x, y\}) \bigcup \{x_1y_1, x_2y_2\}$. Then G' is a connected 3-regular graph of order $p-2$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Without loss of generality, we may consider the following cases.

(1) If there is an $F = \{x_1y_1v_1, x_2y_2\}$ in $\mathcal{F}$, then G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F} - F)$ $\bigcup \{x_1xx_2, yy_1\} \bigcup \{xyy_2, y_1v_1\}$ with empty leave.

(2) If there are $F_1 = \{v_1v_2v_3, x_1y_1\}$ and $F_2 = \{u_1u_2u_3, x_2y_2\}$ in $\mathcal{F}$, then G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \bigcup \{x_1xx_2, yy_1\} \bigcup \{v_1v_2v_3, xy\} \bigcup \{u_1u_2u_3, yy_2\}$ with empty leave.

(3) If there are $F_1 = \{v_1v_2v_3, x_1y_1\}$ and $F_2 = \{x_2y_2u_1, u_2u_3\}$ in $\mathcal{F}$, then G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \bigcup \{x_1xx_2, yy_1\} \bigcup \{v_1v_2v_3, xy\} \bigcup \{yy_2u_1, u_2u_3\}$ with empty leave.

(4) Suppose there are $F_1 = \{x_1y_1v_1, v_2v_3\}$ and $F_2 = \{x_2y_2u_1, u_2u_3\}$(or $F_2 = \{y_2x_2u_1, u_2u_3\}$)

in $\mathcal{F}$. If $x_1 \notin \{u_2, u_3\}$, then G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \bigcup \{x_1xy, u_2u_3\} \bigcup \{yy_1v_1, v_2v_3\} \bigcup \{yy_2u_1, xx_2\}$(or $\{xx_2u_1, yy_2\}$) with empty leave.

If $x_1 = u_2$ or u_3(say $x_1 = u_4$) and $u_3 \neq v_1$, then G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \bigcup \{xx_1u_3, y_1v_1\} \bigcup \{xyy_1, v_2v_3\} \bigcup \{yy_2u_1, xx_2\}$(or $\{xx_2u_1, yy_2\}$) with empty leave.

If $x_1 = u_2$ and $u_3 = v_1$, then G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \bigcup \{x_1xy, y_2u_1$(or $x_2u_1)\} \bigcup \{x_1v_1y_1, v_2v_3\} \bigcup \{y_1yy_2, xx_2\}$ with empty leave.

Hence, by induction, $P_3 \bigcup P_2|G$ for any connected 3-regular graph G.

Secondly, let $G = (mK_4) \bigcup H_1 \bigcup \cdots \bigcup H_n$ be a disconnected 3-regular graph, where $m \geq 0$ and $H_i \neq K_4$ for $1 \leq i \leq n$. Since $P_3 \bigcup P_2|H_i$, $G - mK_4$ has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave.

If $m = 1$, choose an F in $\mathcal{F}$. It can be argued that $K_4 \bigcup F = 3(P_3 \bigcup P_2)$. Hence, $P_3 \bigcup P_2|G$.

If $m \neq 1$, then $G = \frac{m}{2}(2K_4) \bigcup G_1 \bigcup \cdots \bigcup G_n$ when m is even and $G = \frac{m-3}{2}(2K_4) \bigcup (3K_4) \bigcup H_1 \bigcup \cdots \bigcup H_n$ when m is odd. It can be argued that $P_3 \bigcup P_2|(tK_4)$ for $t = 2$ or 3. Hence, $P_3 \bigcup P_2|(mK_4)$ for $m \geq 2$ and then $P_3 \bigcup P_2|G$.

Case 3. $\delta(G) = 2$.

Suppose G has a cycle-component. Let $C_n = x_1x_2 \cdots x_nx_1$ be the minimum cycle-component. If $3 \leq n \leq 5$, let $G' = G - C_n$.

Suppose $n = 3$ and $C_n = x_1x_2x_3x_1$. If $G = G_8, G_9, G_{19}, G_{28}$ or G_{29}, by Lemmas 3, 4 and 5, the assertion holds for these graphs G. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave L. Choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. Hence, G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F}-F)\bigcup\{x_1x_2x_3, v_4v_5\}\bigcup\{v_1v_2v_3, x_1x_3\}$ with leave L.

Suppose $n = 4$ and $C_n = x_1x_2x_3x_4x_1$. If $G = G_{10}, G_{11}, G_{20}, G_{21}$ or G_{30}, by Lemmas 3, 4 and 5, the assertion holds for these graphs G. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave L. For $L = \phi$, choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. Then G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F}-F)\bigcup\{x_1x_2x_3, v_4v_5\}\bigcup\{v_1v_2v_3, x_3x_4\}$ with leave x_1x_4. For $L = v_1v_2$, G has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}\bigcup\{x_1x_2x_3, v_1v_2\}$ with leave $x_3x_4x_1$. For $L = v_1v_2v_3$, G has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}\bigcup\{x_1x_2x_3, v_1v_2\}\bigcup\{x_3x_4x_1, v_2v_3\}$ with empty leave.

Suppose $n = 5$ and $C_n = x_1x_2x_3x_4x_5x_1$. If $G = G_{22}, G_{23}, G_{31}$ or G_{32}, by Lemmas 4 and 5, the assertion holds for these graphs G. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave L. Choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. For $L = \phi$, G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F}-F)\bigcup\{x_1x_2x_3, v_4v_5\}\bigcup\{v_1v_2v_3, x_3x_4\}$ with leave $x_4x_5x_1$. For $L = u_1u_2$, G has a $(P_3 \bigcup P_2)$-packing $(\mathcal{F}-F)\bigcup\{x_1x_2x_3, v_4v_5\}$ $\bigcup\{x_3x_4x_5, u_1u_2\}\bigcup\{v_1v_2v_3, x_1x_5\}$ with empty leave. For $L = u_1u_2u_3$, G has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}\bigcup\{x_1x_2x_3, u_1u_2\}\bigcup\{x_3x_4x_5, u_2u_3\}$ with leave x_1x_5.

For $n \geq 6$, let $C_n = x_1x_2\cdots x_nx_1$. If $q(G) \equiv 0 \pmod 3$, let $G' = (G - \{x_2, x_3, x_4\})$ $\bigcup\{x_1x_5\}$. Then $q(G') = q(G) - 3 \equiv 0 \pmod 3$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_5 \in F$. Since $F = \{x_1x_5x_6, v_1v_2\}$, $\{x_nx_1x_5, v_1v_2\}$ or $\{v_1v_2v_3, x_1x_5\}$, it can be argued that $(F-x_1x_5)\bigcup\{x_1x_2x_3x_4x_5\} = 2(P_3 \bigcup P_2)$. Hence, G has a $(P_3 \bigcup P_2)$-packing with empty leave.

If $q(G) \equiv 1 \pmod 3$, let $G' = (G-x_2)\bigcup\{x_1x_3\}$. Then $q(G') = q(G)-1 \equiv 0 \pmod 3$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ such that $x_1x_3 \in F$. Since $F = \{x_1x_3x_4, v_1v_2\}$, $\{x_nx_1x_3, v_1v_2\}$ or $\{v_1v_2v_3, x_1x_3\}$, it can be argued that $(F - x_1x_3)\bigcup\{x_1x_2x_3\} = (P_3 \bigcup P_2)\bigcup L$, where $L = x_1x_2$ or x_2x_3. Hence,

G has a $(P_3 \bigcup P_2)$-packing with leave L.

If $q(G) \equiv 2 (\text{mod } 3)$, let $G' = (G - \{x_2, x_3\}) \bigcup \{x_1x_4\}$. Then $q(G') = q(G) - 2 \equiv 0 (\text{mod } 3)$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ such that $x_1x_4 \in F$. Since $F = \{x_1x_4x_5, v_1v_2\}$, $\{x_nx_1x_4, v_1v_2\}$ or $\{v_1v_2v_3, x_1x_4\}$, it can be argued that $(F - x_1x_4) \bigcup \{x_1x_2x_3x_4\} = (P_3 \bigcup P_2) \bigcup L$, where $L = x_1x_2x_3$ or $x_2x_3x_4$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

Suppose G has no cycle-component. Since $\delta(G) = 2$, there is a shortest path $x_0x_1x_2 \cdots x_t$(not necessary open) in G with $d(x_0) \geq 3, d(x_t) \geq 3$ and $d(x_i) = 2$ for $1 \leq i < t$, where $t \geq 2$. Consider the following cases.

(1) $x_0x_t \in E(G)$.

Suppose $q(G) \equiv 2 (\text{mod } 3)$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 0 (\text{mod } 3)$. If $G = G_{33}, G_{34}$ or G_{35}, by Lemma 5, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3. Otherwise, by induction hypothesis, $P_3 \bigcup P_2 | G'$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave $x_0x_1x_2$.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 2 (\text{mod } 3)$. If $G = G_{36}$, by Lemma 5, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave a $L' = P_3$. It can be argued that $L' \bigcup \{x_0x_1x_2x_3\} = (P_3 \bigcup P_2) \bigcup P_3$ except $L' = x_0vx_3$. For $L' = x_0vx_3$, choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. It can be argued that $F \bigcup \{x_0x_1x_2x_3vx_0\} = 2(P_3 \bigcup P_2) \bigcup L$, where $L = x_0x_3x_2$ or $x_1x_0x_3$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

If $t \geq 4$, let $G' = (G - \{x_1, x_2\}) \bigcup \{x_0x_3\}$. Then $q(G') \equiv 0 (\text{mod } 3)$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. It it can be argued that $(F - x_0x_3) \bigcup \{x_0x_1x_2x_3\} = (P_3 \bigcup P_2) \bigcup L$, where $L = x_0x_1x_2$ or $x_1x_2x_3$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

Suppose $q(G) \equiv 1 \pmod 3$. Let $G' = G - x_0x_t$. Then $q(G') \equiv 0 \pmod 3$. Since x_1 is of degree at least two in G' and $x_0x_t \notin E(G')$, G' is neither K_4 nor $K_{1,1,3c+1}$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing with empty leave. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave x_0x_t.

Suppose $q(G) \equiv 0 \pmod 3$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 1 \pmod 3$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave an edge e.

If $\{x_0x_1x_2, e\}$ forms a $P_3 \bigcup P_2$, then $P_3 \bigcup P_2 | G$.

If $e = x_0z, z \neq x_2$(similarly if $e = x_2z, z \neq x_0$), choose an F in $\mathcal{F}$ with $x_0x_2 \in F$. It can be argued that $F \bigcup \{zx_0x_1x_2\} = 2(P_3 \bigcup P_2)$ except $F = \{x_0x_2z, v_1v_2\}$. For $F = \{x_0x_2z, v_1v_2\}$, choose an F_1 in $\mathcal{F} - F$. It can be argued that $F_1 \bigcup \{zx_0x_1x_2\} = 2(P_3 \bigcup P_2)$ except $F_1 = \{x_0u_1u_2, zu_3\}$ or $\{x_2u_1z, u_2u_3\}$, where x_0 is neither u_2 nor u_3. If $F_1 = \{x_0u_1u_2, zu_3\}$, then

$$F \bigcup F_1 \bigcup \{zx_0x_1x_2\} = \{x_1x_0x_2, zu_3\} \bigcup \{x_0zx_2, v_1v_2\} \bigcup \{x_0u_1u_2, x_1x_2\}.$$

If $F_1 = \{x_2u_1z, u_2u_3\}\}$, then

$$F \bigcup F_1 \bigcup \{zx_0x_1x_2\} = \{x_0x_1x_2, zu_1\} \bigcup \{x_0zx_2, v_1v_2\} \bigcup \{x_0x_2u_1, u_2u_3\}.$$

Hence, $P_3 \bigcup P_2 | G$.

Suppose $e = x_0x_2$. Since G is different from $K_{1,1,3c+1}$, there is an edge v_1v_2 such that e and v_1v_2 are vertex disjoint edges. Choose an F in $\mathcal{F}$ with $v_1v_2 \in F$. It can be argued that $F \bigcup \{x_0x_1x_2x_0\} = 2(P_3 \bigcup P_2)$ except $F = \{x_0v_3x_2, v_1v_2\}$. For $F = \{x_0v_3x_2, v_1v_2\}$, $F \bigcup \{x_0x_1x_2x_0\} = \{v_3x_0x_1x_2\} \bigcup \{x_0x_2v_3, v_1v_2\}$. Letting $z = v_3$, by the same argument as above, we have $P_3 \bigcup P_2 | G$.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 0 (\mathrm{mod}\ 3)$. If $G = G_{12}$, by Lamma 3, $P_3 \bigcup P_2 | G$. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. Thus, $F \bigcup \{x_0x_1x_2x_3\} = 2(P_3 \bigcup P_2)$ and we have $P_3 \bigcup P_2 | G$.

If $t = 4$, let $G' = G - \{x_1, x_2, x_3\}$. Then $q(G') \equiv 2 (\mathrm{mod}\ 3)$. If $G = G_{13}$, by Lamma 3, $P_3 \bigcup P_2 | G$. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave $v_1v_2v_3$. Since $\{v_1v_2v_3\} \bigcup \{x_0x_1x_2x_3x_4\} = \{v_1v_2v_3, x_2x_3\} \bigcup \{x_0x_1x_2, x_3x_4\}$, $P_3 \bigcup P_2 | G$.

If $t \geq 5$, let $G' = (G - \{x_1, x_2, x_3\}) \bigcup \{x_0x_4\}$. Then $q(G') \equiv 0 (\mathrm{mod}\ 3)$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_4 \in F$. It can be argued that $(F - x_0x_4) \bigcup \{x_0x_1x_2x_3x_4\} = 2(P_3 \bigcup P_2)$. Hence, $P_3 \bigcup P_2 | G$.

(2) $x_0x_t \notin E(G)$ and $x_0 \neq x_t$.

Suppose $q(G) \equiv 2 (\mathrm{mod}\ 3)$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 0 (\mathrm{mod}\ 3)$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Hence, G has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave $x_0x_1x_2$.

If $t \geq 3$, let $G' = (G - \{x_1, x_2\}) \bigcup \{x_0x_3\}$. Then $q(G') \equiv 0 (\mathrm{mod}\ 3)$. If $G = G_{37}$, by Lemma 5, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. It can be argued that $(F - x_0x_3) \bigcup \{x_0x_1x_2x_3\} = (P_3 \bigcup P_2) \bigcup L$, where $L = x_0x_1x_2$ or $x_1x_2x_3$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

Suppose $q(G)\equiv 1 \pmod 3$. Let $G'=(G-x_1)\bigcup\{x_0x_2\}$. Then $q(G')\equiv 0 \pmod 3$. If $G=G_{24}$ or G_{25}, by Lemma 4, G has a $(P_3\bigcup P_2)$-packing with leave a P_2. Otherwise, by induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_2\in F$. It can be argued that $(F-x_0x_2)\bigcup\{x_0x_1x_2\}=(P_3\bigcup P_2)\bigcup L$, where $L=x_0x_1$ or x_1x_2. Hence, G has a $(P_3\bigcup P_2)$-packing with leave a P_2.

Suppose $q(G)\equiv 0 \pmod 3$. If $t=2$, let $G'=G-x_1$. Then $q(G')\equiv 1 \pmod 3$. By induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with leave an edge e. If $\{x_0x_1x_2, e\}$ forms a $P_3\bigcup P_2$, then $P_3\bigcup P_2|G$. Let $e=x_0z$(similarly $e=x_2z$). Choose an F in $\mathcal{F}$ with $x_2\in V(F)$. It can be argued that $F\bigcup\{zx_0x_1x_2\}=2(P_3\bigcup P_2)$ except $F=\{zv_1x_2, v_2v_3\}$ and x_0 is neither v_2 nor v_3. Since $d(x_2)\geq 3$, there is some F_1 in $\mathcal{F}-F$ with $x_2\in V(F_1)$. Similarly, $F_1\bigcup\{zx_0x_1x_2\}=2(P_3\bigcup P_2)$ except $F_1=\{zu_1x_2, u_2u_3\}$ and x_0 is neither u_2 nor u_3. In this case, if v_1 is incident with u_2u_3, say $v_1=u_2$, then $F\bigcup F_1\bigcup\{zx_0x_1x_2\}=\{x_0x_1x_2, v_2v_3\}\bigcup\{x_2v_1u_3, u_1z\}\bigcup\{x_0zv_1, x_2u_1\}$; otherwise,

$$F\bigcup F_1\bigcup\{zx_0x_1x_2\}=\{x_0x_1x_2, v_2v_3\}\bigcup\{u_1x_2v_1, x_0z\}\bigcup\{u_1zv_1, u_2u_3\}.$$

Hence, G has a $(P_3\bigcup P_2)$-packing with empty leave.

If $t=3$, let $G'=G-\{x_1, x_2\}$. Then $q(G')\equiv 0 \pmod 3$. By induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0\in V(F)$. It can be argued that $F\bigcup\{x_0x_1x_2x_3\}=2(P_3\bigcup P_2)$ except $F=\{x_0v_1x_3, v_2v_3\}$. For $F=\{x_0v_1x_3, v_2v_3\}$, by the same argument as above, G has a $(P_3\bigcup P_2)$-packing with empty leave.

If $t\geq 4$, let $G'=G-\{x_1, x_2, x_3\}\bigcup\{x_0x_4\}$. Then $q(G')\equiv 0 \pmod 3$. By induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with

$x_0x_4 \in F$. It can be argued that $(F - x_0x_4)\bigcup\{x_0x_1x_2x_3x_4\} = 2(P_3 \bigcup P_2)$. Hence, G has a $(P_3 \bigcup P_2)$-packing with empty leave.

(3) $x_0 = x_t$ and $t \geq 3$.

Suppose $q(G) \equiv 2(\text{mod } 3)$. For $t = 3$ or 4, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2, \cdots, x_{t-1}\}$. If $G = G_{38}$ or G_{39}, by Lemma 5, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave L. If $t = 3$, then $L = P_3$ and $L\bigcup\{x_0x_1x_2x_0\} = \{L, x_1x_2\}\bigcup\{x_1x_0x_2\}$. If $t = 4$, then $L = P_2$ and $L\bigcup\{x_0x_1x_2x_3x_0\} = \{L, x_1x_2x_3\}\bigcup\{x_1x_0x_3\}$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_{t-1}, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - \{x_0, x_1, \cdots, x_{t-1}\}$. If $G = G_{40}$, by Lemma 5, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave L. If $t = 3$, then $L = P_2$ and $L\bigcup\{x_0x_1x_2x_0\}\bigcup\{x_0z\} = \{L, x_0x_1x_2\}\bigcup\{x_2x_0z\}$. If $t = 4$, then $L = \phi$ and $\{x_0x_1x_2x_3x_0\}\bigcup\{x_0z\} = \{x_1x_2x_3, x_0z\}\bigcup\{x_1x_0x_3\}$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

For $t \geq 5$, let $G' = (G - \{x_2, x_3\})\bigcup x_1x_4$. Then $q(G') \equiv 0(\text{mod } 3)$. By induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_4 \in F$. It can be argued that $(F - x_1x_4)\bigcup\{x_1x_2x_3x_4\} = (P_3 \bigcup P_2)\bigcup L$, where $L = x_1x_2x_3$ or $x_2x_3x_4$. Hence, G has a $(P_3 \bigcup P_2)$-packing with leave a P_3.

Suppose $q(G) \equiv 1(\text{mod } 3)$. For $t = 3$, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2\}$. If $G = G_{26}$, by Lemma 4, G has a $(P_3 \bigcup P_2)$-packing with leave a P_2. Otherwise, by induction hypothesis, G' has a $(P_3 \bigcup P_2)$-packing $\mathcal{F}$ with leave a P_2. Choose an F in $\mathcal{F}$. It can be argued that $F\bigcup\{x_0x_1x_2x_0\} = 2(P_3\bigcup P_2)$. Hence, G has a $(P_3\bigcup P_2)$-packing with leave a P_2.

Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_2, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - x_0z$. Then $q(G') \equiv 0(\text{mod } 3)$. By induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Hence, G has a $(P_3\bigcup P_2)$-packing with leave x_0z.

For $t \geq 4$, let $G' = (G - x_2)\bigcup\{x_1x_3\}$. Then $q(G') \equiv 0(\text{mod } 3)$. By induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_3 \in F$. It can be argued that $(F - x_1x_3)\bigcup\{x_1x_2x_3\} = (P_3\bigcup P_2)\bigcup L$, where $L = x_1x_2$ or x_2x_3. Hence, G has a $(P_3\bigcup P_2)$-packing with leave a P_2.

Suppose $q(G) \equiv 0(\text{mod } 3)$. For $3 \leq t \leq 5$, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2, \cdots, x_{t-1}\}$. If $G = G_{14}$ or G_{15}, by Lemma 3, G has a $(P_3\bigcup P_2)$-packing with empty leave. Otherwise, by induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with leave L. If $t = 3$, then $L = \phi$. Choose an F in $\mathcal{F}$. It can be argued that $F\bigcup\{x_0x_1x_2x_0\} = 2(P_3\bigcup P_2)$. If $t = 4$, then $L = P_3$. It can be argued that $L\bigcup\{x_0x_1x_2x_3x_0\} = 2(P_3\bigcup P_2)$. If $t = 5$, then $L = uv$. If x_0 is incident with uv, say $x_0 = u$, then $\{x_0x_1x_2x_3x_4x_0\}\bigcup\{uv\} = \{x_0x_1x_2, x_3x_4\}\bigcup\{vx_0x_4, x_2x_3\}$. Otherwise, choose an $F = \{z_1z_2z_3, z_4z_5\}$ in $\mathcal{F}$ with $x_0 \in V(F)$. If $x_0 = z_4$ or z_5, then $F\bigcup\{x_0x_1x_2x_3x_4x_0\}\bigcup\{uv\} = \{x_0x_1x_2, uv\}\bigcup\{x_2x_3x_4, z_4z_5\}\bigcup \{z_1z_2z_3, x_4x_0\}$. If $x_0 = z_1, z_2$ or z_3, then $F\bigcup\{x_0x_1x_2x_3x_4x_0\}\bigcup\{uv\} = \{x_0x_1x_2, uv\}\bigcup \{x_3x_4x_0, z_4z_5\}\bigcup\{z_1z_2z_3, x_2x_3\}$. Hence, G has a $(P_3\bigcup P_2)$-packing with empty leave.

Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_{t-1}, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - \{x_0, x_1, \cdots, x_{t-1}\}$. If $G = G_{16}, G_{17}$ or G_{18}, by Lemma 3, G has a $(P_3\bigcup P_2)$-packing with empty leave. Otherwise, by induction hypothesis, G' has a $(P_3\bigcup P_2)$-

packing $\mathcal{F}$ with leave L. It can be argued that $L\bigcup\{x_0x_1\cdots x_tx_0\}\bigcup\{x_0z\} = 2(P_3\bigcup P_2)$ for $3 \leq t \leq 5$. Hence, G has a $(P_3\bigcup P_2)$-packing with empty leave.

Finally, for $t \geq 6$, let $G' = (G - \{x_2, x_3, x_4\})\bigcup\{x_1x_5\}$. Then $q(G') \equiv 0 (\text{mod } 3)$. By induction hypothesis, G' has a $(P_3\bigcup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_5 \in F$. It can be argued that $(F - x_1x_5)\bigcup\{x_1x_2x_3x_4x_5\} = 2(P_3\bigcup P_2)$. Hence, G has a $(P_3\bigcup P_2)$-packing with empty leave.

Therefore, the proof concludes by induction. □

Now, we are ready to prove the Conjecture 1.

Theorem 7 *If G is a graph with $q(G) \equiv 0(\text{mod } 3)$ and $\delta(G) \geq 2$, then $H|G$ for some graph H of size* 3.

Proof. If $q(G) = 3$, then it is trivial that $G|G$. It have been argued that $P_4|G$ if $G = K_4$ or $K_{1,1,3c+1}$. By Theorem 6, we have $P_3\bigcup P_2|G$. Therefore, we complete the proof. □

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