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## On the Existence of Whim Domino Squares

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**Abstract.** The idea of a *domino square* was first introduced by J. A. Edwards *et al* in [1]. In the same paper, they posed some problems on this topic. One problem was to find a general construction for a whim domino square of side $n \equiv 3 \pmod 4$. In this paper, we solve this problem by using a direct construction. It follows that a *whim domino square* exists for each odd side [1].

### 1. Introduction.

Given an $n \times n$ chessboard (an $n \times n$ matrix of cells), can it be covered by a set of distinct dominoes ($1 \times 2$ or $2 \times 1$ matrices) on the numbers $0, 1, \ldots, n$, so that the numbers appearing in each row and each column are all distinct? Clearly one must discard dominoes which have the same number at each end. Given a complete set of dominoes on the numbers $0, 1, 2, \ldots, n$, there are $\binom{n+1}{2} = \frac{1}{2}(n^2+n)$ dominoes from which one may choose. We call an $n \times n$ matrix covered in this way with dominoes based on the numbers $0, 1, 2, \ldots, n$ a domino latin square of side $n$ (or, briefly, a domino square of side $n$). Since each domino covers two cells, and the number of cells in a domino square of side $n$ is $n^2$, it is clear that domino squares of side $n$ can only exist if $n$ is even. It has been shown in [1] that there exists a domino square for each even side. If $n$ is odd we adapt the definition slightly. This time we cover all cells of $n \times n$ matrix except for the central cell. We call such a square a domino latin square *with* a *hole in* the *middle*, or acronimically, a *whim* domino square, or a whimsy for short. In Figure 1.1, we present two known whimsy of side 3 and 5 respectively. A general construction for the whimsy of side $n \equiv 1 \pmod 4$ can be found in [1], but when $n \equiv 3 \pmod 4$, the problem remained open. In this paper, we solve this case, and thus complete the proof of the existence of whim domino squares of odd side.

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 1 | | 3 |
| 3 | 2 | 1 |

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 3 | 2 | 4 |
| 1 | 2 | 0 | 3 | 5 |
| 4 | 5 | | 0 | 2 |
| 2 | 3 | 5 | 4 | 0 |
| 3 | 4 | 2 | 5 | 1 |

Figure 1.1

## 2. The main results.

Before we construct whim domino squares, we need a lemma.

**Lemma 2.1.** *[2] If $m \equiv 0,1 \pmod 4$, then we can arrange the numbers $1,2,\dots,2m$ into $m$ pairs (A-system) $(a_1,b_1),(a_2,b_2),\dots,(a_m,b_m)$ such that $b_i - a_i = i$, $i = 1,2,\dots,m$. If $m \equiv 2,3 \pmod 4$, then the numbers $1,2,\dots,2m-1,2m+1$ can be arranged into $m$ pairs $(c_1,d_1),\dots,(c_m,d_m)$ (B-system) such that $d_i - c_i = i$, $i = 1,2,\dots,m$.*

Let $A$ be the partial latin square given in Figure 2.1.

$$A: \begin{array}{|ccccc|} \hline 0 & 1 & 2 & \dots & n-1 \\ 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & n-1 & n & & n-4 \\ n-1 & n & 0 & \dots & n-3 \\ \hline \end{array}$$

Figure 2.1

Construction: $n = 4k+3$ and $k$ is even.

$1^0$. We start with the latin rectangle $R = [r_{i,j}]$ obtained from $A$ in Figure 2.1 by deleting the first column.

$2^0$. Permute the columns of $R$ to obtain $R'$ such that $r_{1,1} = a_1$, $r_{1,2} = b_1$, $r_{1,3} = a_2$, $r_{1,4} = b_2$, $\dots$, $r_{1,n-2} = a_{2k+1}$, $r_{1,n-1} = b_{2k+1}$, where the pairs $(a_1,b_1),(a_2,b_2),\dots,(a_{2k+1},b_{2k+1})$ form an A-system of the set $\{1,2,\dots,4k+2\}$.

$3^0$. Let $C$ be the column containing the elements $a_i - 1$ and $b_i - 1$, for $1 \le i \le 2k+1$, and $4k+2$ such that
  (a) $2k$ is in the $(2k+2)$-th position,
  (b) $4k+2$ is in the $(2k+3)$-th position, and,
  (c) for $1 \le i \le 2k+1$, elements $a_i - 1$ and $b_i - 1$ are in adjacent positions.

$4^0$. Construct an $n \times n$ array $W$ such that the last column of $W$ is $C$ and (except for the last column) the $i$th row ($i = 1,2,\dots,n$) of $W$ is the $(j+1)$-th row of $R'$ whenever the $i$th entry of $C$ is $j$.

$5^0$. By pairing the entries (cells) of $W$, we obtain an $n \times n$ array which is covered by dominoes as in Figure 2.2.

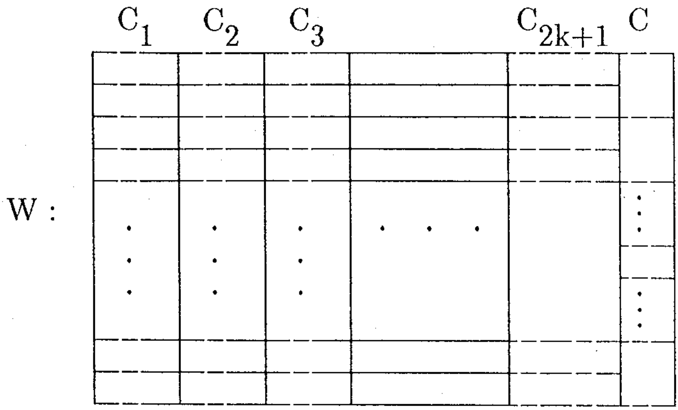


Figure 2.2

$6^0$. By permuting the columns of $W$, and possibly flipping all the dominoes in one column, we obtain an array $W'$, where $C$ is in the $(2k+3)$-th column and the column to the left of $C$, $C'$, satisfies the following property: the $(2k+3)$-th cell of $C'$ is filled with $2k$.

$7^0$. By changing the dominoes in the middle of $W'$ (Figure 2.3) we obtain a whim domino square of side $4k+3$ with $k$ an even number.

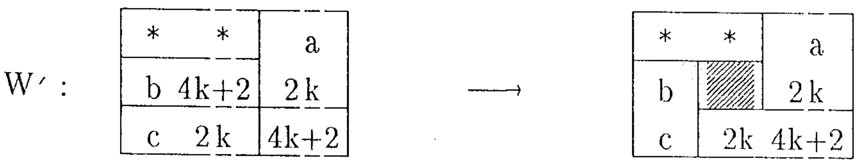


Figure 2.3

**Lemma 2.2.** *There exists a whim domino square of side $n$ for each $n = 4k+3$ where $k$ is an even number.*

Proof: (To follow the proof, it is advisable to look in the appendix at the example of the construction of a whim domino square of side 11). It is clear that no symbol is repeated in any row or column. None of the differences $b_i - a_i$ is congruent to $2k+2$ module $n+1 = 4k+4$. It follows that none of the dominoes

$$\boxed{0,\ 2k+2},\quad \boxed{1,\ 2k+3},\ldots,\ \boxed{2k+1,\ 4k+3}$$

occurs in the squares $W$ or $W'$. In $W'$ row $2k+3$ can be obtained from row $2k+2$ by adding $2k+2$ and reducing module $4k+4$. Therefore, the two dominoes which are introduced in the last stage, $\boxed{2k,\ 4k+2}$ and $\boxed{b, c}$, are both in the set of dominoes which do not occur in $W$ or $W'$. It follows that we obtain finally a whim domino square of side $n$.

**Theorem 2.3.** *A whim domino square of side $n$ exists for each* $n \equiv 3 \pmod 4$.

Proof: Let $n = 4k + 3$, $k$ is an integer. By Lemma 2.2 we have the proof for the case $k$ is an even integer. For the case $k$ is an odd integer, we simply replace A-system by B-system in $2^0$, $2k$ by $2k-1$ in $3^0$, $4k+2$ by $4k+1$ in $3^0$, the last column of $A$ by $[n, 1, 2, \ldots, n-2]^T$, and the other steps are similar; we omit the details.

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## References

1. J.A. Edwards, G.M. Hamilton, A.J.W. Hilton, and Bill Jackson, *Domino squares,*, Annals of Discrete Mathematics **12** (1982), 95–111.
2. A.J.W. Hilton, *On Steiner and similar triple systems*, Math. Scand. **24** (1969), 208–216.

# Appendix

$(1^0)$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 |
| 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 |
| 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 |
| 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 |
| 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

$(2^0)$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 9 | 1 | 3 | 4 | 7 | 2 | 6 | 5 | 10 |
| 9 | 10 | 2 | 4 | 5 | 8 | 3 | 7 | 6 | 11 |
| 10 | 11 | 3 | 5 | 6 | 9 | 4 | 8 | 7 | 0 |
| 11 | 0 | 4 | 6 | 7 | 10 | 5 | 9 | 8 | 1 |
| 0 | 1 | 5 | 7 | 8 | 11 | 6 | 10 | 9 | 2 |
| 1 | 2 | 6 | 8 | 9 | 0 | 7 | 11 | 10 | 3 |
| 2 | 3 | 7 | 9 | 10 | 1 | 8 | 0 | 11 | 4 |
| 3 | 4 | 8 | 10 | 11 | 2 | 9 | 1 | 0 | 5 |
| 4 | 5 | 9 | 11 | 0 | 3 | 10 | 2 | 1 | 6 |
| 5 | 6 | 10 | 0 | 1 | 4 | 11 | 3 | 2 | 7 |
| 6 | 7 | 11 | 1 | 2 | 5 | 0 | 4 | 3 | 8 |

$(3^0)$

| |
|---|
| 7 |
| 8 |
| 0 |
| 2 |
| 9 |
| 4 |
| 10 |
| 3 |
| 6 |
| 1 |
| 5 |

$(4^0, 5^0)$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 8 | 10 | 11 | 2 | 9 | 1 | 0 | 5 | 7 |
| 4 | 5 | 9 | 11 | 0 | 3 | 10 | 2 | 1 | 6 | 8 |
| 8 | 9 | 1 | 3 | 4 | 7 | 2 | 6 | 5 | 10 | 0 |
| 10 | 11 | 3 | 5 | 6 | 9 | 4 | 8 | 7 | 0 | 2 |
| 5 | 6 | 10 | 0 | 1 | 4 | 11 | 3 | 2 | 7 | 9 |
| 0 | 1 | 5 | 7 | 8 | 11 | 6 | 10 | 9 | 2 | 4 |
| 6 | 7 | 11 | 1 | 2 | 5 | 0 | 4 | 3 | 8 | 10 |
| 11 | 0 | 4 | 6 | 7 | 10 | 5 | 9 | 8 | 1 | 3 |
| 2 | 3 | 7 | 9 | 10 | 1 | 8 | 0 | 11 | 4 | 6 |
| 9 | 10 | 2 | 4 | 5 | 8 | 3 | 7 | 6 | 11 | 1 |
| 1 | 2 | 6 | 8 | 9 | 0 | 7 | 11 | 10 | 3 | 5 |

$(6^0)$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 8 | 10 | 9 | 1 | 7 | 11 | 2 | 0 | 5 |
| 4 | 5 | 9 | 11 | 10 | 2 | 8 | 0 | 3 | 1 | 6 |
| 8 | 9 | 1 | 3 | 2 | 6 | 0 | 4 | 7 | 5 | 10 |
| 10 | 11 | 3 | 5 | 4 | 8 | 2 | 6 | 9 | 7 | 0 |
| 5 | 6 | 10 | 0 | 11 | 3 | 9 | 1 | 4 | 2 | 7 |
| 0 | 1 | 5 | 7 | 6 | 10 | 4 | 8 | 11 | 9 | 2 |
| 6 | 7 | 11 | 1 | 0 | 4 | 10 | 2 | 5 | 3 | 8 |
| 11 | 0 | 4 | 6 | 5 | 9 | 3 | 7 | 10 | 8 | 1 |
| 2 | 3 | 7 | 9 | 8 | 0 | 6 | 10 | 1 | 11 | 4 |
| 9 | 10 | 2 | 4 | 3 | 7 | 1 | 5 | 8 | 6 | 11 |
| 1 | 2 | 6 | 8 | 7 | 11 | 5 | 9 | 0 | 10 | 3 |

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$(7^0)$