The Hamilton-Waterloo problem for triangle-factors and heptagon-factors

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Abstract

Given 2-factors R and S of order n, let r and s be nonnegative integers with $r+s=\lfloor\frac{n-1}{2}\rfloor$, the Hamilton-Waterloo problem asks for a 2-factorization of K_n if n is odd, or of K_n-I if n is even, in which r of its 2-factors are isomorphic to R and the other s 2-factors are isomorphic to S. In this paper, we solve the problem for the case of triangle-factors and heptagon-factors for odd n with 3 possible exceptions when $n=21$.

Keywords: cycle decomposition; triangle-factor; heptagon-factor; 2-factorization

1 Introduction

A subgraph F of a graph G is a factor if F contains all the vertices of G. If each component of F is isomorphic to a graph H, then F is called an H-factor of G, while if F is a d-regular graph, then we call F a d-factor. For example, a C_k-factor is a 2-factor consists of cycles of length k. A factorization of a graph G is a collection of factors whose edges partition the edges of G, if the factors are all 2-factors then called a 2-factorization. An $\{H_1^{m_1}, H_2^{m_2}, \ldots, H_t^{m_t}\}$-factorization of a graph G is a factorization of G in which there are precisely m_i H_i-factors. If there exists such a factorization we say that $(G; H_1^{m_1}, H_2^{m_2}, \ldots, H_t^{m_t})$ exists.

Given 2-factors R and S of oder n, let r and s be nonnegative integers with $r+s=\lfloor\frac{n-1}{2}\rfloor$, the Hamilton-Waterloo problem asks for a 2-factorization of the complete graph K_n if n is odd, or K_n-I when n is even, in which r of its 2-factors are isomorphic to R and the other s 2-factors are isomorphic to S, where I is a 1-factor. The goal of the problem is to determine the spectrum of r (or s) for all possible n. If R is a C_m-factor and S is a C_k-factor, i.e. each 2-factor is uniform, then such a 2-factorization is denoted by $HW(n; r, s; m, k)$.

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The uniform cases of the Hamilton-Waterloo problem have attracted much attention in the last decade. The existence of $HW(n;r,s;m,k)$ has been settled when $r=0$ or $s=0$ in [2, 3, 10]. So we only discuss the case $rs\neq 0$ in this paper.

Theorem 1.1. *[2, 3, 10] Let $n\geq 3$ and $m\geq 3$. Let $G=K_n$ if n is odd, $G=K_n-I$ if n is even. Then $(G;C_m^{\lfloor\frac{n-1}{2}\rfloor})$ exists if and only if $n\equiv 0 \pmod m$ and $(n,m)\notin\{(6,3),(12,3)\}$.*

Adams et al. [1] dealt with the cases $(m,k)\in\{(4,6),\ (4,8),\ (4,16),\ (8,16),\ (3,5),\ (3,15),\ (5,15)\}$ and completely solved some of them, they also introduced some methods. Danziger et al.[6] completely solved the case $(m,k)=(3,4)$ with only 9 possible exceptions. The case $(m,k)=(n,3)$, i.e. R is a Hamilton cycle and S is a triangle-factor, was studied in [7, 8, 11, 13], and is still open. In the recent years, remarkable progress has been made on the Hamilton-Waterloo problem when both R and S are consists of even cycles, see [4, 5, 9, 12].

The next two lemmas are useful for our constructions, and have been used in many papers, for example see [9].

Lemma 1.2. *Suppose G_1 and G_2 are two vertex-disjoint graphs. If $(G_1;C_m^r,C_k^s)$ and $(G_2;C_m^r,C_k^s)$ both exist, then $(G_1\cup G_2;C_m^r,C_k^s)$ exists.*

Lemma 1.3. *Suppose G_1 and G_2 are two edge-disjoint graphs with the same vertex set. If $(G_1;C_m^{r_1},C_k^{s_1})$ and $(G_2;C_m^{r_2},C_k^{s_2})$ both exist, then $(G_1\cup G_2;C_m^{r_1+r_2},C_k^{s_1+s_2})$ exists.*

In this paper, we deal with the case $(m,k)=(3,7)$ with n odd. Lemma 3.3 in [1] shows that if $HW(21;r,s;3,7)$ exists for all nonnegative integers r and s with $r+s=10$ then the problem is settled. Unfortunately we can't construct all possible 2-factorizations of this kind for $n=21$. Instead, with 2-factorizations of $K_{7,7,7}$ we have the following result.

Theorem 1.4. *Let $n\equiv 1 \pmod 2$ and $rs\neq 0$ with $r+s=\frac{n-1}{2}$, then there exists an $HW(n;r,s;3,7)$ if and only if $n\equiv 21 \pmod{42}$ except possibly when $n=21$ and $r=2,4,6$.*

In Section 2, we decompose $K_{7,7,7}$ into C_3-factors and C_7-factors. In Section 3, we deal with K_{21}. In Section 4, we show how to decompose K_n into $K_{7,7,7}$-factors and K_{21}-factors, then prove Theorem 1.4.

2 Decompositions of $K_{7,7,7}$

Let $V(K_{7,7,7})=\{j_i\mid j\in Z_7, i\in Z_3\}$, and let $V_i=\{j_i\mid j\in Z_7\}$ for $i\in Z_3$ be the three partite sets of $K_{7,7,7}$. Denote the complete graph on V_i by K_{V_i}, the complete bipartite graph on V_i and V_j by K_{V_i,V_j}, and the complete tripartite graph $K_{7,7,7}$ on V_0, V_1 and V_2 by K_{V_0,V_1,V_2}. Then

$$E(K_{V_0,V_1,V_2})=E(K_{V_0,V_1})\cup E(K_{V_1,V_2})\cup E(K_{V_2,V_0}).$$

For $i, j \in Z_3$ and $d \in Z_7$, let $E_{ij}(d) = \{\{l_i, (l+d)_j\} \mid l \in Z_7\}$. It is easy to verify that

$$E(K_{V_i}) = \bigcup_{d=1}^{3} E_{ii}(d),$$

$$E(K_{V_i,V_j}) = \bigcup_{d=0}^{6} E_{ij}(d) \text{ for } i \neq j.$$

Some of the techniques used in the following lemmas are widely used in combinatorial designs, see [14] for example. In the beginning we give a few basic constructions. The first two lemmas are easy to see, so we omit the proofs.

Lemma 2.1. *Let $d_0, d_1, d_2 \in Z_7$. If $d_0 + d_1 + d_2 \equiv 0 \pmod 7$, then the edges of $E_{01}(d_0) \cup E_{12}(d_1) \cup E_{20}(d_2)$ form a C_3-factor of $K_{7,7,7}$.*

Lemma 2.2. *If $(d, 7) = 1$, then the edges of $E_{ii}(d)$ form a Hamilton cycle, i.e. a C_7-factor of K_{V_i}.*

Lemma 2.3. *The edges of $\bigcup_{d\in\{1,6\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d))$ can be decomposed into 2 C_7-factors of $K_{7,7,7}$.*

Proof. Let

$$F_1 = \{(0_i, 1_{i+1}, 2_{i+2}, 3_i, 4_{i+1}, 5_i, 6_{i+1}) \mid i \in Z_3\},$$
$$F_2 = \{(0_i, 1_{i+2}, 2_{i+1}, 3_i, 4_{i+2}, 5_i, 6_{i+2}) \mid i \in Z_3\},$$

then both F_1 and F_2 are C_7-factors of $K_{7,7,7}$. It is straightforward to verify that

$$E(F_1) \cup E(F_2) = \bigcup_{d\in\{1,6\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d)).$$

□

Lemma 2.4. *The edges of $\bigcup_{d\in\{2,5\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d))$ can be decomposed into 2 C_7-factors of $K_{7,7,7}$.*

Proof. The proof is similar to Lemma 2.3, let the 2 C_7-factors be

$$F_1 = \{(0_i, 2_{i+1}, 4_{i+2}, 6_i, 1_{i+1}, 3_i, 5_{i+1}) \mid i \in Z_3\},$$
$$F_2 = \{(0_i, 2_{i+2}, 4_{i+1}, 6_i, 1_{i+2}, 3_i, 5_{i+2}) \mid i \in Z_3\}.$$

□

Lemma 2.5. *The edges of $\bigcup_{d\in\{3,4\}} (E_{01}(d) \cup E_{12}(d) \cup E_{20}(d))$ can be decomposed into 2 C_7-factors of $K_{7,7,7}$.*

Proof. Let the 2 C_7-factors be

$$F_1 = \{(0_i, 3_{i+1}, 6_{i+2}, 2_i, 5_{i+1}, 1_i, 4_{i+1}) \mid i \in Z_3\},$$
$$F_2 = \{(0_i, 3_{i+2}, 6_{i+1}, 2_i, 5_{i+2}, 1_i, 4_{i+2}) \mid i \in Z_3\}.$$

□

Lemma 2.6. *[3] Let $K_{d(m)}$ be the complete multipartite graph with d parts of size m, if d and m are both odd integers, then there is a 2-factorization of $K_{d(m)}$, in which each 2-factor is a C_m-factor.*

Now we decompose $K_{7,7,7}$ into C_3-factors and C_7-factors.

Lemma 2.7. *$(K_{7,7,7}; C_3^\alpha, C_7^\beta)$ exists for $\alpha \in \{0, 1, 3, 5, 7\}$ with $\alpha + \beta = 7$.*

Proof. For $\alpha = 0$, $(K_{7,7,7}; C_7^7)$ exists by Lemma 2.6.

For $\alpha = 1$, decompose $\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d = 1, 2, \ldots, 6\}$ into 6 C_7-factors by Lemma 2.3-2.5, the remaining edges $E_{01}(0) \cup E_{12}(0) \cup E_{20}(0)$ form a C_3-factor by Lemma 2.1.

For $\alpha = 3$, decompose $\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d = 2, 3, 4, 5\}$ into 4 C_7-factors by Lemma 2.4 and Lemma 2.5. The 3 C_3-factors are $E_{i(i+1)}(0) \cup E_{(i+1)(i+2)}(1) \cup E_{(i+2)i}(6), i \in Z_3$ by Lemma 2.1.

For $\alpha = 5$, decompose $\{E_{01}(d) \cup E_{12}(d) \cup E_{20}(d) \mid d = 3, 4\}$ into 2 C_7-factors by Lemma 2.5. By Lemma 2.1 the 5 C_3-factors are

$$E_{01}(0) \cup E_{12}(1) \cup E_{23}(6),\ E_{01}(2) \cup E_{12}(0) \cup E_{23}(5),$$
$$E_{01}(5) \cup E_{12}(2) \cup E_{23}(0),\ E_{01}(6) \cup E_{12}(6) \cup E_{23}(2),$$
$$E_{01}(1) \cup E_{12}(5) \cup E_{23}(1).$$

For $\alpha = 7$, by Lemma 2.1 the 7 C_3-factors are

$$E_{i(i+1)}(1) \cup E_{(i+1)(i+2)}(2) \cup E_{(i+2)i}(4),\ i \in Z_3;$$
$$E_{j(j+1)}(3) \cup E_{(j+1)(j+2)}(5) \cup E_{(j+2)j}(6),\ j \in Z_3;$$
$$E_{01}(0) \cup E_{12}(0) \cup E_{23}(0).$$

□

3 Decompositions of K_{21}

In this section, $V_i, K_{V_i}, K_{V_i,V_j}, K_{V_0,V_1,V_2}$, and $E_{ij}(d)$ have the same meanings as given in Section 2. Let $V(K_{21}) = V_0 \cup V_1 \cup V_2$, then the edge set is

$$E(K_{21}) = \bigcup_{i \in Z_3} E(K_{V_i}) \cup E(K_{V_0,V_1,V_2}).$$

Now we decompose K_{21} into γ C_3-factors and δ C_7-factors with $\gamma + \delta = 10$.

Lemma 3.1. *$(K_{21}; C_3^{\gamma}, C_7^{\delta})$ exists for $\gamma \in \{0, 1, 3, 5, 7, 10\}$ with $\gamma + \delta = 10$.*

Proof. Since $E(K_{V_i}) = \bigcup_{d=1}^{3} E_{ii}(d)$ for $i \in Z_3$, by Lemma 2.2, $(K_{V_i}; C_7^3)$ exists. Then by Lemma 1.2, $(\bigcup_{i\in Z_3} K_{V_i}; C_7^3)$ exists. Hence, it is easy to observe that if $(K_{V_0,V_1,V_2}; C_3^{\alpha}, C_7^{\beta})$ exists, then $(K_{V_1\cup V_2\cup V_3}; C_3^{\alpha}, C_7^{\beta+3})$ exists by Lemma 1.3. Thus by Lemma 2.7, $(K_{21}; C_3^{\gamma}, C_7^{\delta})$ exists for $\gamma \in \{0, 1, 3, 5, 7\}$ with $\gamma + \delta = 10$.

For $(\gamma, \delta) = (10, 0)$, $(K_{21}; C_3^{\gamma}, C_7^{\delta})$ exists by Theorem 1.1. □

Lemma 3.2. *$(K_{21}; C_3^{\gamma}, C_7^{\delta})$ exists for $(\gamma, \delta) = (8, 2)$.*

Proof. Let

$$\begin{aligned} F_0 =\{&(0_0, 1_0, 2_1), (1_1, 4_1, 5_2), (1_2, 6_2, 3_0), (2_0, 6_1, 4_2), \\ &(4_0, 0_1, 3_2), (5_0, 3_1, 2_2), (6_0, 5_1, 0_2)\}, \end{aligned}$$

then F_0 is a C_3-factor of K_{21}. Six additional C_3-factors, denoted by $F_1, F_2, \ldots, F_6$, are formed by developing F_0 mod$(7, -)$. Let $F_7 = E_{01}(0) \cup E_{12}(0) \cup E_{20}(0)$, then by Lemma 2.1 F_7 is a C_3-factor. Let $F_8 = E_{00}(2) \cup E_{11}(1) \cup E_{22}(1)$, $F_9 = E_{00}(3) \cup E_{11}(2) \cup E_{22}(3)$, then F_8 and F_9 are both C_7-factors of K_{21} by Lemma 1.2 and Lemma 2.2. Finally, one can check that each edge of K_{21} is used exactly once. □

Lemma 3.3. *$(K_{21}; C_3^{\gamma}, C_7^{\delta})$ exists for $(\gamma, \delta) = (9, 1)$.*

Proof. Let

$$\begin{aligned} F_0 =\{&(0_0, 1_0, 6_1), (0_1, 1_1, 4_2), (0_2, 2_2, 3_0), (2_0, 4_0, 4_1), \\ &(3_1, 5_1, 3_2), (5_2, 6_2, 5_0), (6_0, 2_1, 1_2)\}, \end{aligned}$$

then F_0 is a C_3-factor of K_{21}. Six additional C_3-factors, denoted by $F_1, F_2, \ldots, F_6$, are formed by developing F_0 mod$(7, -)$. Let $F_7 = E_{01}(1) \cup E_{12}(2) \cup E_{20}(4)$, $F_8 = E_{01}(4) \cup E_{12}(1) \cup E_{20}(2)$, then by Lemma 2.1 F_7 and F_8 are both C_3-factors. Let $F_9 = E_{00}(3) \cup E_{11}(3) \cup E_{22}(3)$, then F_9 is a C_7-factor of K_{21} by Lemma 1.2 and Lemma 2.2. Again, one can check that each edge of K_{21} is used exactly once. □

Combining Lemma 3.1-3.3, we have the following result.

Lemma 3.4. *$(K_{21}; C_3^{\gamma}, C_7^{\delta})$ exists for $\gamma \in \{0, 1, 3, 5, 7, 8, 9, 10\}$ with $\gamma + \delta = 10$.*

4 Main Results

Let n be an odd integer. Let r and s be positive intergers with $r + s = \frac{n-1}{2}$. It is easy to see that a necessary condition for the existence of an $HW(n; r, s; 3, 7)$ is $n \equiv 21 \pmod{42}$, then let $n = 42t + 21$, $t \geq 0$. Let the vertex set of K_n be $V(K_n) = \{j_i \mid j \in Z_7, i \in Z_{6t+3}\}$, denote $V_i = Z_7 \times \{i\}$ for $i \in Z_{6t+3}$. The next lemma is based on a construction given in the paper [1].

Lemma 4.1. *For $n = 42t + 21$ and $t \geq 0$, $(K_n; K_{7,7,7}^{3t}, K_{21})$ exists.*

Proof. By Theorem 1.1, $(K_{6t+3}; C_3^{3t+1})$ exits for $t \geq 0$, it is actually the well known Kirkman triple system of order $6t+3$. Let the vertex set of K_{6t+3} be $\{V_i \mid i \in Z_{6t+3}\}$, replace each 3-cycle (V_i, V_j, V_k) with the complete tripartite graph $K_{7,7,7}$ on vertex sets V_i, V_j and V_k, then each C_3-factor of K_{6t+3} corresponds to a $K_{7,7,7}$-factor of K_n, also these $K_{7,7,7}$-factors form the complete multipartite graph $K_{(6t+3)(7)}$ on vertex sets $V_0, V_1, \ldots, V_{6t+2}$, i.e. $(K_{(6t+3)(7)}; K_{7,7,7}^{3t+1})$ exists. Hence $(K_n; K_{7,7,7}^{3t+1}, K_7)$ exists. Since the union of any $K_{7,7,7}$-factor and the K_7-factor of K_n is actually a K_{21}-factor, $(K_n; K_{7,7,7}^{3t}, K_{21})$ exists. $\square$

Lemma 4.2. *Let $\alpha_i \in \{0,1,3,5,7\}$ with $\alpha_i + \beta_i = 7$ for $i = 1,2,\ldots,3t$, and $\gamma \in \{0,1,3,5,7,8,9,10\}$ with $\gamma + \delta = 10$, then there exists an $HW(n; \sum_{i=1}^{3t} \alpha_i + \gamma, \sum_{i=1}^{3t} \beta_i + \delta; 3, 7)$.*

Proof. By Lemma 4.1, we decompose K_n into $3t$ $K_{7,7,7}$-factors and a K_{21}-factor.

For the ith $K_{7,7,7}$-factor, let $\alpha_i \in \{0,1,3,5,7\}$ and $\alpha_i + \beta_i = 7$. Then decompose each $K_{7,7,7}$ of this $K_{7,7,7}$-factor into α_i C_3-factors and β_i C_7-factors by Lemma 2.7, then by Lemma 1.2 these 2-factors of $K_{7,7,7}$ form α_i C_3-factors and β_i C_7-factors of K_n.

Similarly, the K_{21}-factor of K_n can be decomposed into γ C_3-factors and δ C_7-factors for $\gamma \in \{0,1,3,5,7,8,9,10\}$ with $\gamma + \delta = 10$ by Lemma 1.2 and 3.4.

Then by Lemma 1.4, $(K_n; C_3^{\sum_{i=1}^{3t} \alpha_i + \gamma}, C_7^{\sum_{i=1}^{3t} \beta_i + \delta})$ exists, i.e. there exists an $HW(n; \sum_{i=1}^{3t} \alpha_i + \gamma, \sum_{i=1}^{3t} \beta_i + \delta; 3, 7)$. $\square$

We are now ready to prove the following theorem.

Theorem 4.3. *Let n be an odd integer. Let r and s be positive intergers with $r+s = \frac{n-1}{2}$. Then there exists an $HW(n; r, s; 3, 7)$ if and only if $n \equiv 21 \pmod{42}$ except possibly when $n = 21$ and $r = 2, 4, 6$.*

Proof. The case $t = 0$ (i.e. $n = 21$) is solved by Lemma 3.4.

For the case $t > 0$, let $r = 7a + b$, where $0 \leq b < 7$, we only need to assign a proper value to each of $\{\gamma, \alpha_i \mid i = 1,2,\ldots,3t\}$ in Lemma 4.2.

If $b = 0$ and $a < 3t+1$, let $\gamma = 0$ and $\alpha_i = \begin{cases} 7, & \text{for } 1 \leq i \leq a, \\ 0, & \text{for } a < i \leq 3t. \end{cases}$

If $b = 0$ and $a = 3t+1$, let $\gamma = 7$ and $\alpha_i = 7$ for $i = 1,2,\ldots,3t$.

If $b = 1$ and $a < 3t+1$, let $\gamma = 1$ and $\alpha_i = \begin{cases} 7, & \text{for } 1 \leq i \leq a, \\ 0, & \text{for } a < i \leq 3t. \end{cases}$

If $b = 1$ and $a = 3t+1$, let $\gamma = 8$ and $\alpha_i = 7$ for $i = 1,2,\ldots,3t$.

If $b = 2$ and $a < 3t$, let $\gamma = 1$ and $\alpha_i = \begin{cases} 1, & \text{for } i = 1, \\ 7, & \text{for } 2 \leq i \leq a+1, \\ 0, & \text{for } a+1 < i \leq 3t. \end{cases}$

If $b = 2$ and $a = 3t$, let $\gamma = 9$ and $\alpha_i = \begin{cases} 1, & \text{for } i = 1, \\ 7, & \text{for } 2 \leq i \leq 3t. \end{cases}$

If $b = 2$ and $a = 3t+1$, let $\gamma = 9$ and $\alpha_i = 7$ for $i = 1,2,\ldots,3t$.

If $b = 3$, let $\gamma = 3$ and $\alpha_i = \begin{cases} 7, & \text{for } 1 \leq i \leq a, \\ 0, & \text{for } a < i \leq 3t. \end{cases}$

If $b=4$ and $a<3t$, let $\gamma=3$ and $\alpha_i=\begin{cases}1, & \text{for } i=1,\\ 7, & \text{for } 2\le i\le a+1,\\ 0, & \text{for } a+1<i\le 3t.\end{cases}$

If $b=4$ and $a=3t+1$, let $\gamma=8$ and $\alpha_i=\begin{cases}3, & \text{for } i=1,\\ 7, & \text{for } 1<i\le 3t.\end{cases}$

If $b=5$, let $\gamma=5$ and $\alpha_i=\begin{cases}7, & \text{for } 1\le i\le a,\\ 0, & \text{for } a<i\le 3t.\end{cases}$

If $b=6$ and $a<3t$, let $\gamma=1$ and $\alpha_i=\begin{cases}5, & \text{for } i=1,\\ 7, & \text{for } 2\le i\le a+1,\\ 0, & \text{for } a+1<i\le 3t.\end{cases}$

If $b=6$ and $a=3t$, let $\gamma=8$ and $\alpha_i=\begin{cases}5, & \text{for } i=1,\\ 7, & \text{for } 1<i\le 3t.\end{cases}$ □

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