The Absorbant Number of Generalized de Bruijn Digraphs

Jyhmin Kuo*and Hung-Lin Fu†

Abstract

Let $D = (V, A)$ be a digraph with the vertex set V and the arc set A. An absorbant of D is a set $S \subseteq V$ such that for each $v \in V \setminus S$, $O(v) \cap S \neq \emptyset$ where $O(v)$ is the out-neighborhood of v. The absorbant number of D, denoted by $\gamma_a(D)$, is defined as the minimum cardinality of an absorbant of D. The generalized de Bruijn digraph $G_B(n, d)$ is a digraph with the vertex set $V(G_B(n, d)) = \{0, 1, 2, \cdots, n-1\}$ and the arc set $A(G_B(n, d)) = \{(x, y) | y \equiv dx + i \pmod{n}, 0 \leq i < d\}$. In this paper, we determine $\gamma_a(G_B(n, d))$ for all $d \leq n \leq 4d$.

Keywords: generalized de Bruijn digraph; absorbant number; resource location problem

1 Introduction and preliminaries

Let $D = (V, A)$ be a digraph with the vertex set V and the arc set A. If $(x, y) \in A$, then the vertex x is called a predecessor of y and y is called a successor of x. For a vertex $v \in V$, the out-neighborhood of v is $O(v) = \{w | (v, w) \in A\}$ and the in-neighborhood of v is $I(v) = \{u | (u, v) \in A\}$. The closed out-neighborhood of v is the set $O[v] = O(v) \cup \{v\}$ and the closed in-neighborhood is the set $I[v] = I(v) \cup \{v\}$. For a set $S \subseteq V$, the out-neighborhood of S is the set $O(S) = \bigcup_{s \in S} O(s)$. $O[S]$, $I(S)$ and $I[S]$ are defined accordingly.

An *absorbant* of $D = (V, A)$ is a set $S \subseteq V$ such that for each $v \in V \setminus S$, $O(v) \cap S \neq \emptyset$, i.e., $I[S] = V$. The absorbant number of D, denoted by $\gamma_a(D)$, is defined as the minimum cardinality of an absorbant of D.

Note here that a *dominating set* of $D = (V, A)$ is a set $T \subseteq V$ such that for all $v \in V \setminus T$, $I(v) \cap T \neq \emptyset$, i.e., $O[T] = V$. So, from the definitions, it

*Chen-De Senior High School, Hsin Chu, Taiwan 30047; E-mail: jyhminkuo@gmail.com

†Department of Applied Mathematics, National Chiao Tung University, Hsin Chu, Taiwan 30050; E-mail: hlfu@math.nctu.edu.tw

is not difficult to realize that the absorbant and also the dominating set of D play an important role in resource location problem and facility location problem [4]. The general facility location problem is: given a set of facility locations and a set of customers who are served from the facilities then:

- which facilities should be used
- which customers should be served from which facilities so as to minimize the total cost of serving all the customers.

Imase and Itoh [3] were the first to generalize the well-known de Bruijn network $B(d, D)$ [1], independently followed by Reddy, Pradhan and Kuhl [5]. The generalized de Bruijn digraph $G_B(n, d)$ is defined by congruence equations,

$$\begin{cases} V(G_B(n,d)) = \{0, 1, 2, \cdots, n-1\} = Z_n \\ A(G_B(n,d)) = \{(x,y) | y \equiv dx + i \pmod{n}, 0 \leq i < d\}. \end{cases}$$

Note that if $n = d^D$, $G_B(n, d)$ is the de Bruijn digraph $B(d, D)$. It is well-known that the de Bruijn graph is a highly reliable and efficient network which was proposed as suitable processor interconnection network for VLSI implementation [7]. It was verified later by Xu et al. [8] that directed de Bruijn networks are suitable model for interconnection networks in parallel and distributed processing systems. However, one of the disadvantages of de Bruijn digraphs $B(d, D)$ is the restriction on the number of vertices d^D. This phenomenon can be overcome by $G_B(n, d)$. As a matter of fact, $G_B(n, d)$ retains all the properties of the de Bruijn digraphs and has no restrictions on the number of vertices [2]. So determining the connectivity, diameter and the absorbant number of $G_B(n, d)$ is of relevant interest and important. Throughout this paper, we assume that $n > d \geq 2$ in $G_B(n, d)$. In [6], the authors Shan et al. studied the absorbant and several interesting results were obtained. The following theorem gives the upper and lower bound of $\gamma_a(G_B(n, d))$.

Theorem 1. *[6]* $\lceil \frac{n}{d+1} \rceil \leq \gamma_a(G_B(n, d)) \leq \lceil \frac{n}{d} \rceil$.

Also, in the paper, they proposed five open problems, three of them are on the absorbant number of $G_B(n, d)$.

1. Find sufficient conditions for the absorbant number of $G_B(n, d)$ to be the lower bound $\lceil n/(d + 1) \rceil$.
2. Is it true that $\gamma_a(G_B(8k - 4, 4k - 3)) = 3$ for $k \geq 2$?
3. Is it true that $\gamma_a(G_B(6k, 2k - 1)) = 4$ for $k \geq 2$?

In this paper, we solve the second and third problems. In fact, we obtain a more general result by showing

$$(a)\ \gamma_a(G_B(2d + 1, d))) = \begin{cases} 2 & \text{if } d \not\equiv 1 \pmod{3}, \\ 3 & \text{otherwise,} \end{cases}$$

(*b*) $\gamma_a(G_B(2d+2,d))) = \begin{cases} 2 & \text{if } d \not\equiv 1 \pmod 4, \\ 3 & \text{otherwise,} \end{cases}$

(*c*) $\gamma_a(G_B(3d+1,d))) = \begin{cases} 3 & \text{if } d \not\equiv 1 \pmod 4, \\ 4 & \text{otherwise,} \end{cases}$

(*d*) $\gamma_a(G_B(3d+2,d))) = \begin{cases} 3 & \text{if } d = 2,4,5, \\ 4 & \text{otherwise,} \end{cases}$

(*e*) $\gamma_a(G_B(3d+3,d))) = \begin{cases} 3 & \text{if } d \equiv 0 \pmod 2, \\ 4 & \text{otherwise.} \end{cases}$

Therefore, combine with Theorem 1, we have determined $\gamma_a(G_B(n,d))$ for each $d \leq n \leq 4d$.

2 The main results

For convenience, we shall use $[a, b]$ to denote the set of non-negative integers $\{a, a+1, \cdots, b-1, b\}$ where $a < b$ are non-negative integers and the integers are taking modulo n. For example, in Z_8, $[7, 9] = [7, 1]$ denotes $\{7, 0, 1\}$ and $[4, 6]$ denotes $\{4, 5, 6\}$.

Lemma 2. *Let n and d be positive integers with $2 \leq d < n$, and let $gcd(n, d) = 1$. For a subset S of Z_n, if*

$$Z_n \setminus (\bigcup_{s\in S}[s-d+1, s]) \subseteq \{ds | s \in S\} \tag{2.1}$$

holds, then S is an absorbant of $G_B(n, d)$.

Proof. Since $gcd(n, d) = 1$, we have $Z_n = \{ds|s \in S\} \cup \{ds'|s' \in Z_n \setminus S\}$. Therefore, if S is a subset of Z_n such that

$$Z_n \setminus (\bigcup_{s\in S}[s-d+1, s]) \subseteq \{ds | s \in S\}$$

then for each $x \in Z_n \setminus S$, $dx \in [s-d+1, s]$ for some $s \in S$. This implies that (x, s) is an arc of $G_B(n, d)$ and thus S is an absorbant of $G_B(n, d)$. ∎

Example 1. Let $S = \{1, 6\}$ be a subset of $V(G_B(8, 3)) = Z_8$. Since $Z_8 \setminus (\{7, 0, 1\} \cup \{4, 5, 6\}) = \{2, 3\} \subseteq \{ds|s \in S\} = \{3, 2\}$, S is an absorbant of $G_B(8, 3)$ by Lemma 2. ∎

If fact, the reverse statement of Lemma 2 is also true.

Lemma 3. *Let n and d be positive integers with $2 \le d < n$, and let $gcd(n,d) = 1$. If a subset S of Z_n is an absorbant of $G_B(n,d)$ then*

$$Z_n \setminus (\bigcup_{s \in S} [s-d+1, s]) \subseteq \{ds | s \in S\}. \tag{2.2}$$

Proof. Since $gcd(n,d) = 1$, we have $Z_n = \{ds|s \in S\} \cup \{ds'|s' \in Z_n \setminus S\}$. Since S is an absorbant, for each $x \in Z_n \setminus S$, $dx \in [s-d+1, s]$ for some $s \in S$. Hence $\{dx|x \in Z_n \setminus S\} \subseteq \bigcup_{s \in S}[s-d+1, s]$. Therefore, $Z_n \setminus (\bigcup_{s \in S}[s-d+1, s]) \subseteq \{ds|s \in S\}$. ∎

In what follows, we consider $G_B(n,d)$ and let $T_S = Z_n \setminus (\bigcup_{s \in S}[s-d+1, s])$ for brevity. Now, we consider the case $gcd(n,d) > 1$. For convenience, let $gcd(n,d) = \lambda$.

Lemma 4. *Let n and d be positive integers with $2 \le d < n$ and let $\lambda > 1$. For a subset S of Z_n, define $T'_S = \{t : t \in T_S$ and $\lambda | t\}$. If*

$$T'_S \subseteq \{ds|s \in S\} \tag{2.3}$$

and

$$\bigcup_{t \in T'_S} \{x : x \in Z_n, dx \equiv t \pmod{n}\} \subseteq S \tag{2.4}$$

hold, then S is an absorbant of $G_B(n,d)$. Furthermore,

$$\lambda |T'_S| \le |S|. \tag{2.5}$$

Proof. Let S be a subset of Z_n satisfying (2.3) and (2.4). Since $\lambda > 1$, $\{dx|x \in Z_n\} = \{ds|s \in S\} \cup \{ds'|s' \in Z_n \setminus S\} = \{t : t \in Z_n$ and $\lambda|t\} = T'_S \cup \{t : t \in (\bigcup_{s \in S}[s-d+1, s]), \lambda|t\}$. Since (2.4) holds, $T'_S \cap \{ds' : s' \in Z_n \setminus S\} = \emptyset$. Since (2.3) holds, $\{ds'|s' \in Z_n \setminus S\} \subseteq \{t : t \in (\bigcup_{s \in S}[s-d+1, s]), \lambda|t\}$. This implies that for each $s' \in Z_n \setminus S$, $ds' \in [s''-d+1, s'']$ for some $s'' \in S$. Hence $(s', s'') \in A(G_B(n,d))$ and S is an absorbant.

For every $t \in T'_S$, there exists a set $X = \{x', x' + \frac{n}{\lambda}, x' + 2(\frac{n}{\lambda}), \cdots, x' + (\lambda-1)\frac{n}{\lambda}\}$ where $dx' \equiv t \pmod{n}$, such that $\forall x \in X$, $dx \equiv t \pmod{n}$. Since (2.4) holds, $\lambda|T'_S| \le |S|$. ∎

Example 2. Let $S = \{3, 10, 17\}$ be a subset of $V(G_B(21,6))$. Since $T'_S = \{18\} \subseteq \{ds|s \in S\} = \{18\}$, (2.3) holds. Since $\bigcup_{t \in T'_S}\{x : x \in Z_n, dx \equiv t \pmod{n}\} \subseteq S$, (2.4) holds. Therefore, S is an absorbant by Lemma 4. ∎

Lemma 5. *Let n and d be positive integers with $2 \le d < n$ and let $\lambda > 1$. If a subset S of Z_n is an absorbant of $G_B(n,d)$, then*

$$T'_S \subseteq \{ds|s \in S\} \tag{2.6}$$

and

$$\bigcup_{t\in T'_S}\{x : x \in Z_n, dx \equiv t \pmod{n}\} \subseteq S, \tag{2.7}$$

where $T'_S = \{t : t \in T_S$ *and* $\lambda | t\}$.

Proof. Let S be an absorbant of $G_B(n,d)$. Suppose (2.7) is false. Hence there exist $t_0 \in T'_S$ and $x_0 \in Z_n \setminus S$ such that $dx_0 \equiv t_0 \pmod{n}$. This implies that $dx_0 \in Z_n \setminus \bigcup_{s\in S}[s-d+1, s]$. Therefore S is not an absorbant, a contradiction. Furthermore, we have $T'_S \cap \{ds' : s' \in Z_n \setminus S\} = \emptyset$.

Since S is an absorbant, $(x, s') \in A(G_B(n,d))$, $\forall x \in Z_n \setminus S$ and for some $s' \in S$. Therefore, $dx \in \bigcup_{s\in S}[s-d+1, s]$, $\forall x \in Z_n \setminus S$. Since $\lambda > 1$, $\{dx | x \in Z_n\} = \{ds | s \in S\} \cup \{ds' | s' \in Z_n \setminus S\} = \{t | t \in Z_n$ and $\lambda | t\} = T'_S \cup \{t : t \in (\bigcup_{s\in S}[s-d+1, s]), \lambda | t\}\}$. Since $\{ds' | s' \in Z_n \setminus S\} \subseteq \{t : t \in (\bigcup_{s\in S}[s-d+1, s]), \lambda | t\}$ and $T'_S \cap \{ds' : s' \in Z_n \setminus S\} = \emptyset$, we have $T'_S \subseteq \{ds | s \in S\}$ and (2.6) holds. ∎

Now, we are ready to prove our main results. First, we consider the second problem. Instead of finding $\gamma_a(G_B(8k-4, 4k-3))$ only, we prove a more general theorem.

Theorem 6. $\gamma_a(G_B(2d+2, d)) = \begin{cases} 2 & \textit{if } d \not\equiv 1 \pmod{4}, \\ 3 & \textit{otherwise.} \end{cases}$

Proof. To prove the first part, it suffices to construct an absorbant S with cardinality 2. Let $S = \{\lfloor \frac{d}{2} \rfloor, n - \lfloor \frac{d}{2} \rfloor - 1\}$. We claim that S is an absorbant of $G_B(2d+2, d)$ for $d \not\equiv 1 \pmod{4}$. According to the congruent classes modulo 4, we split the proof into 3 cases.

Case 1. $d \equiv 0 \pmod{4}$. Let $d = 4p$. Hence $n = 8p+2$ and $S = \{2p, 6p+1\}$. We have $\lambda = 2$ and $T'_S = \{6p+2\} \subseteq \{ds | s \in S\} = \{6p+2\}$. By Lemma 4, we have the proof.

Case 2. $d \equiv 2 \pmod{4}$. Let $d = 4p+2$. Hence $n = 8p+6$ and $S = \{2p+1, 6p+4\}$. Therefore, $\lambda = 2$ and $T'_S = \{2p+2\} \subseteq \{ds | s \in S\} = \{2p+2\}$. By Lemma 4, we have the proof.

Case 3. $d \equiv 3 \pmod{4}$. Let $d = 4p+3$. Hence $n = 8p+8$ and $S = \{2p+1, 6p+6\}$. We get $T_S = \{2p+2, 2p+3\} \subseteq \{ds | s \in S\} = \{2p+3, 2p+2\}$. By Lemma 2, we have the proof.

Now, we are left with the case $d \equiv 1 \pmod{4}$. Let $d = 4p+1$ where $p \geq 1$ then $n = 8p+4$ and $\lambda = 1$. Suppose there exists an absorbant S with cardinality 2. We claim that S does not exist. By Lemma 3, we may let $S = \{x, x+d\}$ or $S = \{x, x+d+1\}$. Otherwise, $|T_S| > |S|$, a contradiction.

Case 1. $S = \{x, x+d\}$. Hence we have $T_S = \{x+d+1, x+d+2\} \subseteq \{ds | s \in S\} = \{dx, dx+1\}$. Therefore,

$$dx \equiv x + d + 1 \pmod{n},$$

$$x(4p+1) \equiv x + (4p+1) + 1 \pmod{n},$$

$$4xp \equiv 4p+2 \pmod{8p+4}, \text{ a contradiction.}$$

Case 2. $S = \{x, x+d+1\}$. This implies that $T_S = \{x+1, x+d+2\} \subseteq \{ds|s \in S\} = \{dx, dx+d+1\}$. If $dx \equiv x+1$, then $4px \equiv 1 \pmod{8p+4}$, a contradiction. If $dx \equiv x+d+2$, then $4px \equiv 4p+3 \pmod{8p+4}$, a contradiction.

Since (2.2) fails to hold, S is not an absorbant by Lemma 3. ∎

Theorem 7. $\gamma_a(G_B(3d+3,d))) = \begin{cases} 3 & \textit{for even } d, \\ 4 & \textit{otherwise.} \end{cases}$

Proof. By Theorem 1, we have $3 \leq \gamma_a(G_B(3d+3,d)) \leq 4$. If there exists an absorbant S with cardinality 3, then we complete the proof of the first part where d is even. Let $S = \{d/2, (3d/2)+1, (5d/2)+2\}$. According to d modulo 6, we have 3 cases to consider.

Case 1. $d \equiv 2 \pmod 6$. Let $d = 6p+2$. Hence $n = 18p+9$ and $S = \{3p+1, 9p+4, 15p+7\}$. We have $T_S = \{3p+2, 9p+5, 15p+8\} \subseteq \{ds|s \in S\} = \{3p+2, 15p+8, 9p+5\}$. By Lemma 2, S is an absorbant.

Case 2. $d \equiv 4 \pmod 6$. Let $d = 6p+4$. Hence $n = 18p+15$ and $S = \{3p+2, 9p+7, 15p+12\}$. We have $T_S = \{3p+3, 9p+8, 15p+13\} \subseteq \{ds|s \in S\} = \{9p+8, 15p+13, 3p+3\}$. By Lemma 2, S is an absorbant.

Case 3. $d \equiv 0 \pmod 6$. Let $d = 6p$. Hence $n = 18p+3$ and $S = \{3p, 9p+1, 15p+2\}$. Therefore, $\lambda = 3$ and $T'_S = \{15p+3\} \subseteq \{ds|s \in S\} = \{15p+3\}$. By Lemma 4, S is an absorbant.

For the second part, according to the congruent classes modulo 6, there are also 3 cases to consider.

Case 1. $d \equiv 3 \pmod 6$. Let $d = 6p+3$. Hence $n = 18p+12$ and $\lambda = 3$. By Lemma 5, we may let $S = \{x, x+d+1, x+2d+2\}$. Hence $T_S = \{x+1, x+d+2, x+2d+3\}$.

Case 1-1. $x \equiv 0 \pmod 3$. Hence $T'_S = \{x+2d+3\}$. Therefore, $dx \equiv x+2d+3 \pmod n$, $6px+2x-12p-6 \equiv 3 \pmod{18p+12}$, a contradiction.

Case 1-2. $x \equiv 1 \pmod 3$. Hence $T'_S = \{x+d+2\}$. Therefore, $dx \equiv x+d+2 \pmod n$, $6px-6p \equiv 5-2x \pmod{18p+12}$, a contradiction.

Case 1-3. $x \equiv 2 \pmod 3$. Hence $T'_S = \{x+1\}$. Therefore, $dx \equiv x+1 \pmod n$, $6px+2p \equiv 1 \pmod{18p+12}$, a contradiction.

Since (2.6) fails to hold, S is not an absorbant by Lemma 5.

Case 2. $d \equiv 1 \pmod 6$. Let $d = 6p+1$, then $n = 18p+6$ and $\lambda = 1$. By Lemma 3, we have 4 subcases to consider.

Case 2-1. $S = \{x, x+d, x+2d\}$. Hence we have $T_S = \{x+2d+1, x+2d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+1, dx+2\}$. But $dx \equiv x+2d+1 \pmod n$, $6px \equiv 12p+3 \pmod{n = 12p+6}$, a contradiction.

Case 2-2. $S = \{x, x+d, x+2d+1\}$. Hence we have $T_S = \{x+d+1, x+2d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+1, dx+d+2\}$. Since $dx \equiv x+2d+2 \pmod{n}$ has no solutions, it is a contradiction.

Case 2-3. $S = \{x, x+d, x+2d+2\}$. Hence we have $T_S = \{x+d+1, x+d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+1, dx+2d+2\}$. Since $dx \equiv x+d+1 \pmod{n}$, we have $6px \equiv 6p+2 \pmod{n = 12p+6}$, a contradiction.

Case 2-4. $S = \{x, x+d+1, x+2d+2\}$. We have $T_S = \{x+1, x+d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+d+1, dx+2d+2\} \pmod{n}$. By all of the three congruences $dx \equiv x+1$, $dx \equiv x+d+1$ and $dx \equiv dx+2d+2 \pmod{n}$ fail to hold, it is a contradiction.

Since (2.2) fails to hold, S is not an absorbant by Lemma 3.

For the last case $d \equiv 5 \pmod{6}$, let $d = 6p+5$, and we have $n = 18p+18$ and $\lambda = 1$. By Lemma 3, there are 4 subcases to consider.

Case 3-1. $S = \{x, x+d, x+2d\}$. Hence $T_S = \{x+2d+1, x+2d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+6p+7, dx+12p+14\}$, a contradiction.

Case 3-2. $S = \{x, x+d, x+2d+1\}$. Hence $T_S = \{x+d+1, x+2d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+6p+7, dx+1\}$, a contradiction.

Case 3-3. $S = \{x, x+d, x+2d+2\}$. Hence $T_S = \{x+d+1, x+d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+6p+7, dx+6p-30\}$, a contradiction.

Case 3-4. $S = \{x, x+d+1, x+2d+2\}$. Hence $T_S = \{x+1, x+d+2, x+2d+3\} \subseteq \{ds|s \in S\} = \{dx, dx+12p+12, dx+6p-30\}$, a contradiction.

Since (2.2) fails to hold, S is not an absorbant by Lemma 3 and we complete the proof. ∎

Theorem 8. $\gamma_a(G_B(2d+1,d))) = \begin{cases} 2 & \text{for } d \not\equiv 1 \pmod{3}, \\ 3 & \text{otherwise.} \end{cases}$

Proof. Clearly $\lambda = 1$. By Theorem 1, we have $2 \le \gamma_a(G_B(2d+1,d)) \le 3$. For the first part, it suffices to construct an absorbant S with cardinality 2.

Case 1. $d \equiv 0 \pmod{6}$. Clearly if $d = 6p$ then $n = 12p+1$. Let $S = \{4p, 10p\}$. We have $T_S = \{10p+1\} \subseteq \{dx|x \in S\} = \{10p+1, 7p+1\}$.

Case 2. $d \equiv 2 \pmod{6}$. Clearly if $d = 6p+2$ then $n = 12p+5$. Let $S = \{2p, 8p+3\}$. We have $T_S = \{2p+1\} \subseteq \{dx|x \in S\} = \{11p+5, 2p+1\}$.

Case 3. $d \equiv 3 \pmod{6}$. Clearly if $d = 6p+3$ then $n = 12p+7$. Let $S = \{2p+1, 8p+4\}$. We have $T_S = \{8p+5\} \subseteq \{dx|x \in S\} = \{5p+3, 8p+5\}$.

Case 4. $d \equiv 5 \pmod{6}$. Let $d = 6p+5$ then $n = 12p+11$ and $S = \{4p+3, 10p+9\}$. We have $T_S = \{4p+4\} \subseteq \{dx|x \in S\} = \{4p+4, p+1\}$.

For the second part, if S is an absorbant then $|T_S| \le |S|$ by Lemma 3. Hence we have two cases to consider.

Case 1. $S = \{x, x+d-1\}$. Hence $T_S = \{x+d, x+d+1\}$.

Case 1-1. $d = 6p+1$. Hence $n = 12p+3$. We have $T_S = \{x+d, x+d+1\} \subseteq \{ds|s \in S\} = \{dx, dx-3p\}$, a contradiction.

Case 1-2. $d = 6p+4$. Hence $n = 12p+9$. We have $T_S = \{x+d, x+d+1\} \subseteq \{ds|s \in S\} = \{dx, dx+3p+3\}$, a contradiction.

Case 2. $S = \{x, x+d\}$. Hence $T_S = \{x+d+1\} = \{x+3p+2\}$. Let $d = 3p+1$, then $n = 6p+3$. One get $dx \equiv (3p+1)x \equiv (x+3p+2)+(3px-3p-2) \not\equiv (x+3p+2) \pmod{n}$ and $(x+d)d \equiv (x+3p+2)+(9p^2+3px+3p-1) \not\equiv (x+3p+2) \pmod{n}$. Hence (2.2) fails to hold. By Lemma 3, S is not an absorbant. ∎

Theorem 9. $\gamma_a(G_B(3d+1, d))) = \begin{cases} 3 & \textit{for } d \not\equiv 1 \pmod{4}, \\ 4 & \textit{otherwise.} \end{cases}$

Proof. Clearly $\lambda = 1$. By Theorem 1, it suffices to construct an absorbant S with cardinality 3 to prove the first part. By taking the congruent classes modulo 4, we have 3 cases to consider.

Case 1. $d \equiv 0 \pmod{4}$. Let $d = 4p$ and $S = \{2p, 6p, 10p\}$. Hence $n = 12p+1$ and $T_S = \{10p+1\}$. Since $(6p)(4p) \equiv 10p+1 \pmod{n}$, $T_S \subseteq \{(6p)d\}$.

Case 2. $d \equiv 2 \pmod{4}$. Let $d = 4p+2$ and $S = \{2p+1, 6p+3, 10p+5\}$. Hence $n = 12p+7$ and $T_S = \{10p+6\}$. Since $(6p+3)(4p+2) \equiv 24p^2 + 24p+12 \equiv 2p(12p+7)+(10p+6) \equiv 10p+6 \pmod{n}$, $T_S \subseteq \{(6p+3)d\}$.

Case 3. $d \equiv 3 \pmod{4}$. Let $d = 4p+3$ and $S = \{p, 5p+4, 9p+7\}$. Hence $n = 12p+10$ and $T_S = \{p+1\}$. Since $(9p+7)(4p+3) \equiv (3p+2)(12p+10)+(p+1) \equiv p+1 \pmod{n}$, $T_S \subseteq \{(9p+7)d\}$.

In all the 3 cases, (2.1) hold, S is an absorbant by Lemma 2.

Now we show the second part. Let $d = 4p+1$ where $p \geq 2$ then $n = 12p+4$ and $\lambda = 1$. Suppose S is an absorbant with cardinality 3. We will claim S doesn't exist. By Lemma 3, we have $|T_S| \leq |S| = 3$. Withourt loss of generality, we may let S be one of the following sets $\{x, x+d, x+2d\}$, $\{x, x+d-1, x+2d-1\}$, $\{x, x+d+1, x+2d\}$, $\{x, x+d-1, x+2d-2\}$, $\{x, x+d+1, x+2d-1\}$. Thus T_S corresponds to $\{x+2d+1\}$, $\{x+2d, x+2d+1\}$, $\{x+1, x+2d+1\}$, $\{x+2d-1, x+2d, x+2d+1\}$ and $\{x+1, x+2d, x+2d+1\}$, respectively. For each case, $x+2d+1 \in T_S$ and $x+2d+1 \equiv x+3 \pmod{4}$. On the other hand, $\{ds \pmod{4} | s \in \{x, x+d-1, x+d, x+d+1, x+2d-2, x+2d-1, x+2d\}\} = \{x, x+1, x+2\} \pmod{4}$. Hence, $x+2d+1 \in T_S \nsubseteq \{ds|s \in S\}$. Therefore equation (2.2) fails to hold and S is not an absorbant by Lemma 3. ∎

Theorem 10. $\gamma_a(G_B(3d+2, d)) = \begin{cases} 3, & \textit{if } d = 2, 4, 5, \\ 4, & \textit{otherwise.} \end{cases}$

Proof. If $d = 2$, let $S = \{0, 2, 6\}$. If $d = 4$, let $S = \{3, 7, 10\}$. If $d = 5$, let $S = \{3, 8, 13\}$. These 3 cases can be checked directly.

For the second part, we split d into two parts: odd and even. First, d is odd. Hence $\lambda = 1$. Suppose S is an absorbant with cardinality 3. By Lemma 3 we have 4 cases to consider. They are $S_1 = \{x, x+d, x+2d\}$, $S_2 = \{x, x+d+1, x+2d+2\}$, $S_3 = \{x, x+d-1, x+2d\}$ and $S_4 = \{x, x+d, x+2d-1\}$. Therefore, the corresponding T_S are as follows: $T_{S_1} = \{x+2d+1, x+2d+2\}$, $T_{S_2} = \{x+1, x+d+2\}$, $T_{S_3} = \{x+d, x+2d+1, x+2d+2\}$ and $T_{S_4} = \{x+2d, x+2d+1, x+2d+2\}$. According to d modulo 6, we have 3 cases to consider. We claim all of them are all impossible by (2.2) fails to hold.

Case 1. $d \equiv 1 \pmod 6$. Let $d = 6p+1$. Hence

$$T_{S_1} \not\subseteq \{ds | s \in S_1\} = \{dx, dx+2p+1, dx+4p+2\},$$

$$T_{S_2} \not\subseteq \{ds | s \in S_2\} = \{dx, dx+d+2p+1, dx+2d+4p+2\},$$

$$T_{S_3} \not\subseteq \{ds | s \in S_3\} = \{dx, dx+2p-2, dx+4p+2\},$$

$$T_{S_4} \not\subseteq \{ds | s \in S_4\} = \{dx, dx+2p+1, dx-2p+1\}.$$

Case 2. $d \equiv 3 \pmod 6$. Let $d = 6p+3$. Hence

$$T_{S_1} \not\subseteq \{ds | s \in S_1\} = \{dx, dx+14p+9, dx+10p+7\},$$

$$T_{S_2} \not\subseteq \{ds | s \in S_2\} = \{dx, dx+d+14p+9, dx+2d+10p+7\},$$

$$T_{S_3} \not\subseteq \{ds | s \in S_3\} = \{dx, dx+8p+6, dx+10p+7\},$$

$$T_{S_4} \not\subseteq \{ds | s \in S_4\} = \{dx, dx+14p+9, dx+8p-7\}.$$

Case 3. $d \equiv 5 \pmod 6$. Let $d = 6p+5$ where $p > 0$. Hence

$$T_{S_1} \not\subseteq \{ds | s \in S_1\} = \{dx, dx+8p+8, dx+16p+16\},$$

$$T_{S_2} \not\subseteq \{ds | s \in S_2\} = \{dx, dx+d+8p+8, dx+2d+16p+16\},$$

$$T_{S_3} \not\subseteq \{ds | s \in S_3\} = \{dx, dx+8p+6, dx+10p+7\},$$

$$T_{S_4} \not\subseteq \{ds | s \in S_4\} = \{dx, dx+14p+9, dx+8p-7\}.$$

Since (2.2) does not hold, we have $\gamma_a(G_B(3d+2, d)) \geq |S| + 1 = 4$.

Now, it is left to consider the case: d is even. Let $d \geq 6$, then $n = 3d+2$ and $\lambda = 2$. Suppose $S = \{x, y, z\}$ is an absorbant of $G_B(3d+2, d)$. We claim S does not exist. According to the number of elements in T'_S, we have 3 cases to consider.

Case 1. $|T'_S| = 0$. By definition, $T_S = Z_n \backslash ([x-d+1, x] \cup [y-d+1, y] \cup [z-d+1, z])$. T_S can not contain two consecutive integers, otherwise T_S contains at least one even integer and this implies $|T'_S| \geq 1$, a contradiction. Therefore S has two possibilities: $\{x, x+d+1, x+2d+1\}$ and $\{x, x+d, x+2d+1\}$. The corresponding T_S are $\{x+1, x+2d+2\}$ and

$\{x+d+1, x+2d+2\}$, respectively. In both of the two cases, T_S contains exactly one even integer by d is even. Hence $|T'_S| \geq 1$, a contradiction.

Case 2. $|T'_S| = 1$. Suppose $T'_S = \{t_0\}$. Since $\lambda = 2$, there exist two integers, x and $x + \frac{n}{2}$, equivalence to $t_0 \pmod{n}$. Hence $\{x, x+\frac{n}{2}\} \subseteq S$. Without loss of generality we may let $[x+1, x+\frac{n}{2}-d] \subseteq O(y)$ such that $S = \{x, x+\frac{n}{2}, y\}$. By (2.1)

$$T_S = Z_n \setminus (\bigcup_{s \in S}[s-d+1, s]) = [x + \frac{n}{2} + 1, x-d], |T_S| = \frac{d}{2} + 1.$$

Since T'_S is the subset of even integer in T_S, we have $|T_S| = \frac{d}{2} + 1 \leq 3$ and $d \leq 4$, a contradiction.

Case 3. $|T'_S| \geq 2$. This implies that $|S| \geq 4$ by Lemma 5, a contradiction. ∎

3 Conclusion

Observe that if $\lceil \frac{n}{d+1} \rceil = \lceil \frac{n}{d} \rceil$, then $\gamma_a(G_B(n,d)) = \lceil \frac{n}{d} \rceil$. Therefore, we have

$$\gamma_a(G_B(n,d))) = \begin{cases} 2 & \text{if } d+2 \leq n \leq 2d, d \geq 2; \\ 3 & \text{if } 2d+3 \leq n \leq 3d, d \geq 3; \text{ and} \\ 4 & \text{if } 3d+4 \leq n \leq 4d, d \geq 4. \end{cases}$$

So, combine with Theorem 2 - 5, we have determined $\gamma_a(G_B(n,d))$ for each $d \leq n \leq 4d$.

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