On the Decycling Numbers of Generalized Kautz Digraphs

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Abstract

A set of vertices of a graph whose removal leaves an acyclic graph is called a decycling set of the graph. The minimum size of a decycling set of a graph G is referred to as the decycling number of G. Let $f(d,n)$ be the decycling number of the generalized Kautz digraph $G_K(d,n)$. In this paper, we obtain the upper bound of $f(d,n)$ for all $n \geq d \geq 2$.

Keywords: generalized Kautz digraph, Interconnection networks, feedback vertex number, decycling number.

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1 Introduction

A set of vertices of a graph whose removal leaves an acyclic graph is referred as a *decycling set*, or a *feedback vertex set*, of the graph. The minimum cardinality of a decycling set of a graph G is referres to as the decycling number of G. The problem was originally formulated in the area of combinatorial circuit design. Other applications of the problem are connected with resource allocation mechanisms in operating systems that prevent deadlocks, to the constraint satisfaction problem and Bayesian inference in artificial intelligence, to the study of monopolies in synchronous distributed systems and to converter placement problems in optical networks.

The Kautz digraphs $K(d, D)$ have many attractive features superior to the hypercube, and been thought of as a good candidate for the next generation of parallel system architectures, after the hypercube network. In [7], Panchapakesan et al. analyzed the behavior of a Kautz digraph as the logical topology of a multihop optical network. Such networks have been shown to have average distance very close to the lower bound and have queuing delay better than the logical topology based on de Bruijn digraph. However, one of the disadvantages of Kautz digraphs is the restriction on the number of vertices. The Kautz digraph was also generalized by Imase and Itoh [4, 5] in 1981. The generalized Kautz digraphs $G_K(d, n)$ retain all of the properties of the Kautz digraphs, but have no restrictions on the number of vertices. Formally, for $n > d \geq 2$, the generalized Kautz digraph $G_K(d, n)$ is defined by congruence equations as follows:

$$\begin{cases} V(G_K(d,n)) = \{0, 1, 2, \cdots, n-1\}; \\ A(G_K(d,n)) = \{(x,y) | y \equiv -dx - i \pmod{n}, 1 \leq i \leq d\}. \end{cases}$$

See [1, 2, 3, 6] for references.

Xu et al. [8] have determined the exact values of the decycling number of the Kautz digraph $K(d, n)$ for $1 \leq n \leq 7$ and the asymptotical vaules for

$n \geq 8$. Recently, Xu et al. [10] provided the bounds of the decycling number of the undirected Kautz graph.

In this paper, we consider the decycling number of generalized Kautz digraph $G_K(d,n)$ for $n \geq d \geq 2$. Let $f(d,n)$ be the decycling number of the generalized Kautz digraph $G_K(d,n)$. We obtain the upper bound of $f(d,n)$ as follows

$$f(d,n) \leq \begin{cases} 8m+3t+1, m=\lfloor \frac{n}{36} \rfloor, n \equiv t \pmod{36}, \text{ for } d=2. \\ 12m+t+\lceil \frac{t}{2} \rceil + \lfloor \frac{3t}{4} \rfloor + 5, m=\lfloor \frac{n}{36} \rfloor, n \equiv t \pmod{36}, \text{ for } d=3. \\ (\frac{1}{2}-\frac{d-1}{2d^2})n+\frac{d}{2}(d-t+5)-2, m=\lfloor \frac{n}{(d+1)} \rfloor, \ n \equiv t \pmod{(d+1)}, \text{ for } d \geq 4. \end{cases}$$

2 Preliminaries

In this paper, we deal with generalized Kautz digraph $G_K(d,n)$, where $n \geq d \geq 2$.

For $\alpha \in V(G_K(d,n))$, $R(\alpha) = \{-d\alpha - i \pmod{n}, 1 \leq i \leq d\}$. For a set $S \subset V(G_K(d,n))$, $R(S) = \bigcup_{v \in S} R(v)$

For convenience, we use the notation as follows:

$$[a,b] = \begin{cases} \{a, a+1, \cdots, b\} & \text{if } a \leq b, \\ \{a, a+1, \cdots, n+b-1, n+b\} & \text{otherwise.} \end{cases}$$

We can see that $R([a,b]) = [-db-d, -da-1] \pmod{n}$.

Let V_d be the subset of $V(G_K(d,n))$ which contains the vertices deleted. The remaining vertex set of $V(G_K(d,n))$ is denoted by V_r, where $V_r = V(G_K(d,n)) - V_d$. Let $G[V_r]$ be the subgraph of $V(G_K(d,n))$ induced by the vertices set V_r.

Let V_c be the subset of V_r which contains the vertices lying on a cycle of induced subgraph $G[V_r]$. Let $V_{nc} = V(G_K(d,n)) - V_c$. It implies that V_{nc} contains all the vertices without lying on any cycle of $G[V_r]$. If we want to prove V_d is a decycling set, then it is equal to prove $G[V_r]$ is acyclic. It suffices to prove that $V_c = \emptyset$ or $V_{nc} = V(G_K(d,n))$. We will use the following Lemma throughout the paper.

Lemma 1. *For any arbitrary subset $S \subset V_r$, if $R(S) \subset V_{nc}$, then $S \subset V_{nc}$.*

Lemma 2. $V_d \subseteq V_{nc}$.

Proof. For all $x \in V_d$. Since V_d contains the vertices deleted from $V(G_K(d,n))$, $x \notin V_r$. This implies $x \notin V_c$. That is $x \in V_{nc}$. Thus, $V_d \subseteq V_{nc}$.∎

3 Decycling Set of $G_K(2,n)$

Let $m = \lfloor \frac{n}{36} \rfloor$ and $n \equiv t \pmod{36}$, then $n = 36m + t$. Define

$$A_1 = [28m, n-1],$$

$$A_2 = [24m, 24m + t - 1],$$

$$A_3 = [12m, 12m + t].$$

We will show that $A_1 \cup A_2 \cup A_3$ is a decycling set of $G_K(2,n)$.

(1)Let $S = [0, 4m-1]$.

$$R(S) = [-2(4m-1) - 2 + 36m + t, -1 + n] = [28m + t, n-1] \subset V_{nc}.$$

Therefore, $[0, 4m-1] \cup A_1 \cup A_2 \cup A_3 \subset V_{nc}$.

(2)Let $S = [16m + t, 22m - 1]$.

$$R(S) = [28m + 2t, n-1] \cup [0, 4m - t - 1] \subset V_{nc}.$$

Therefore, $[0, 4m-1] \cup [16m + t, 22m - 1] \cup A_1 \cup A_2 \cup A_3 \subset V_{nc}$.

(3)Let $S = [7m + t, 10m - 1]$.

$$R(S) = [16m + t, 22m - t - 1] \subset V_{nc}.$$

Therefore, $[0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 22m-1] \cup A_1 \cup A_2 \cup A_3 \subset V_{nc}$.

(4)Let $S = [25m + t, 28m - 1]$.

$$R(S) = [16m + 2t, 22m - 1] \subset V_{nc}.$$

Therefore, $[0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 22m-1] \cup [25m+t, n-1] \cup A_2 \cup A_3 \subset V_{nc}$.

(5)Let $k = \lceil \log_2 m \rceil$, then $k \geq \log_2 m$ and $2^k \geq m$.

(5-1)Let $S_e = [22m, 24m - 2^{k-1} - 1]$.

$$R(S_e) = [24m+2^k+2t, 28m+2t-1] \subset V_{nc}, \text{ since } 24m+2^k+2t \geq 25m+t.$$

Therefore, $[0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 24m-2^{k-1}-1] \cup [25m+t, n-1] \cup A_2 \cup A_3 \subset V_{nc}$.

Let $S_r = [24m + 2^{k-1} + t, 25m + t - 1]$.

$$R(S_r) = [22m, 24m-2^k-1] \subset V_{nc}, \text{ since } 24m-2^k-1 \leq 24m-2^{k-1}-1.$$

Therefore, $[0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 24m-2^{k-1}-1] \cup [24m+2^{k-1}+t, n-1] \cup A_2 \cup A_3 \subset V_{nc}$.

(5-2)Let $S_e = [24m - 2^{k-1}, 24m - 2^{k-2} - 1]$.

$$R(S_e) = [24m + 2^{k-1} + 2t, 24m + 2^k + 2t - 1] \subset V_{nc}.$$

Therefore, $[0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 24m-2^{k-2}-1] \cup [24m+2^{k-1}+t, n-1] \cup A_2 \cup A_3 \subset V_{nc}$.

Let $S_r = [24m + 2^{k-2} + t, 24m + 2^{k-1} + t - 1]$.

$$R(S_r) = [24m - 2^k, 24m - 2^{k-1} - 1] \subset V_{nc}.$$

Therefore, $[0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, 24m-2^{k-2}-1] \cup [24m+2^{k-2}+t, n-1] \cup A_2 \cup A_3 \subset V_{nc}$.

Continuing in this way, we have $[24m - 2^{L+1}, 24m - 2^L - 1] \subset V_{nc}$, and $[24m + 2^L + t, 24m + 2^{L+1} + t - 1] \subset V_{nc}$ for $L = k-3, k-4, ..., 1, 0$. Since

$R(\{24m-1\}) = [24m+2t, 24m+2t+1] \subset V_{nc}$, and $R(\{24m+t\}) = [24m-2, 24m-1] \subset V_{nc}$. We have $[0, 4m-1] \cup [7m+t, 10m-1] \cup [16m+t, n-1] \cup A_3 \subset V_{nc}$.

(6)Let $S = [4m, 7m+t-1]$.

$$R(S) = [22m-t, 28m+t-1] \subset V_{nc}.$$

Therefore, $[0, 10m-1] \cup [16m+t, n-1] \cup A_3 \subset V_{nc}$.

(7)Let $S = [13m+t, 16m+t-1]$.

$$R(S) = [4m-t, 10m-t-1] \subset V_{nc}.$$

Therefore, $[0, 10m-1] \cup [13m+t, n-1] \cup A_3 \subset V_{nc}$.

(8-1)Let $k = \lfloor \log_2 m \rfloor$ and $S_e = [10m, 12m-2^k-1]$.

$$R(S_e) = [12m+2^{k+1}+t, 16m+t-1] \subset V_{nc}, \text{ since } 12m+2^{k+1}+t \geq 13m+t.$$

Therefore, $[0, 12m-2^k-1] \cup [13m+t, n-1] \cup A_3 \subset V_{nc}$.

Let $S_r = [12m+2^k+t, 13m+t-1]$.

$$R(S_r) = [10m-t, 12m-2^{k+1}-t-1] \subset V_{nc}, \text{ since } 12m-2^{k+1}-t-1 \leq 12m-2^k-1.$$

Therefore, $[0, 12m-2^k-1] \cup [12m+2^k+t, n-1] \cup A_3 \subset V_{nc}$.

(8-2)Let $S_e = [12m-2^k, 12m-2^{k-1}-1]$.

$$R(S_e) = [12m+2^k+t, 12m+2^{k+1}+t-1] \subset V_{nc}.$$

Therefore, $[0, 12m-2^{k-1}-1] \cup [12m+2^k+t, n-1] \cup A_3 \subset V_{nc}$.

Let $S_r = [12m+2^{k-1}+t, 12m+2^k+t-1]$.

$$R(S_r) = [12m-2^{k+1}-t, 12m-2^k-t-1] \subset V_{nc}.$$

Therefore, $[0, 12m-2^{k-1}-1] \cup [12m+2^{k-1}+t, n-1] \cup A_3 \subset V_{nc}$.

Continuing in this way, we have $[12m-2^{L+1}, 12m-2^L-1] \subset V_{nc}$, $[12m+2^L+t, 12m+2^{L+1}+t-1] \subset V_{nc}$ for $L = k-2, k-3, ..., 1, 0$ and $R(\{12m-1\}) = [12m+t, 12m+t+1]$. Therefore, $[0, n-1] \subset V_{nc}$. Thus, $V(G_K(d,n)) = V_{nc}$.

Lemma 3. *For $G_K(2,n)$, let $m = \lfloor \frac{n}{36} \rfloor$ and $n \equiv t \pmod{36}$, i.e. $n = 36m+t$, then $f(2,n) \le 8m+3t+1$.*

Proof. $A_1 \cup A_2 \cup A_3$ is a decycling set of $G_K(2,n)$.

4 Decycling Set of $G_K(3,n)$

Let $m = \lfloor \frac{n}{36} \rfloor$ and $n \equiv t \pmod{36}$, then $n = 36m+t$. Define

$$A_1 = [0, 6m+t],$$
$$A_2 = [12m + \lfloor \frac{t}{3} \rfloor, 18m + \lceil \frac{t}{2} \rceil],$$
$$A_3 = [9m-1, 9m + \lfloor \frac{t}{3} \rfloor],$$
$$A_4 = [27m, 27m + \lfloor \frac{3t}{4} \rfloor].$$

We will show that $A_1 \cup A_2 \cup A_3 \cup A_4$ is a decycling set of $G_K(3,n)$.

(1)Let $S = [10m, 12m + \lfloor \frac{t}{3} \rfloor - 1]$,

$$R(S) = [t - 3\lfloor \frac{t}{3} \rfloor, 6m+t-1] \subset A_1 \subset V_{nc}.$$

Now, $A_1 \cup [10m, 18m + \lceil \frac{t}{2} \rceil] \cup A_3 \cup A_4 \subset V_{nc}$.

(2)Let $S = [6m+t+1, 8m-1]$,

$$R(S) = [12m+t, 18m-2t-4] \subset V_{nc}.$$

We have $[0, 8m-1] \cup [10m, 18m + \lceil \frac{t}{2} \rceil] \cup A_3 \cup A_4 \subset V_{nc}$.

(3)Let $S_r = [9m + \lfloor \frac{m}{3} \rfloor + \lfloor \frac{t}{3} \rfloor + 1, 10m-1]$,

$$R(S_r) = [6m+t, 9m - 3\lfloor \frac{m}{3} \rfloor + t - \lfloor \frac{t}{3} \rfloor - 4] \subset V_{nc}.$$

Let $S_e = [8m, 8m + \lfloor \frac{2m}{3} \rfloor - 1]$,

$$R(S_e) = [12m - 3\lfloor \frac{2m}{3} \rfloor + t, 12m+t-1] \subset V_{nc}$$

. Hence, $[0, 8m+\lfloor\frac{2m}{3}\rfloor-1] \cup [9m+\lfloor\frac{m}{3}\rfloor+\lfloor\frac{t}{3}\rfloor+1, 18m+\lceil\frac{t}{2}\rceil] \cup A_3 \cup A_4 \subset V_{nc}$.

(3-1)Let $k = \lceil \log_3(m+t) \rceil$, then $9m + 3^{k-1} \geq 9m + \lfloor\frac{m}{3}\rfloor + \lfloor\frac{t}{3}\rfloor$.
Let $S_r = [9m + 3^{k-2} + \lfloor\frac{t}{3}\rfloor + 1, 9m + 3^{k-1}]$,

$$R(S_r) = [9m - 3^k + t - 3, 9m - 3^{k-1} + t - 3\lfloor\frac{t}{3}\rfloor - 4] \subset V_{nc},$$

since $9m - 3^{k-1} + t - 3\lfloor\frac{t}{3}\rfloor - 4 \leq 8m + \lfloor\frac{2m}{3}\rfloor - 1$.
We have $[0, 8m+\lfloor\frac{2m}{3}\rfloor-1] \cup [9m+3^{k-2}+\lfloor\frac{t}{3}\rfloor+1, 18m+\lceil\frac{t}{2}\rceil] \cup A_3 \cup A_4 \subset V_{nc}$.

Let $S_e = [9m - 3^{k-1} - 1, 9m - 3^{k-2} - 1]$.

$$R(S_e) = [9m + 3^{k-1} + t, 9m + 3^k + t + 2] \subset V_{nc}.$$

We have $[0, 9m-3^{k-2}-1] \cup [9m+3^{k-2}+\lfloor\frac{t}{3}\rfloor+1, 18m+\lceil\frac{t}{2}\rceil] \cup A_3 \cup A_4 \subset V_{nc}$.

(3-2)Let $S_r = [9m + 3^{k-3} + \lfloor\frac{t}{3}\rfloor + 1, 9m + 3^{k-2} + \lfloor\frac{t}{3}\rfloor]$ Then

$$R(S_r) = [9m - 3^{k-1} + t - 3\lfloor\frac{t}{3}\rfloor - 3, 9m - 3^{k-2} + t - 3\lfloor\frac{t}{3}\rfloor - 4] \subset V_{nc}.$$

We have $[0, 9m-3^{k-2}-1] \cup [9m+3^{k-3}+\lfloor\frac{t}{3}\rfloor+1, 18m+\lceil\frac{t}{2}\rceil] \cup A_3 \cup A_4 \subset V_{nc}$.

Let $S_e = [9m - 3^{k-2}, 9m - 3^{k-3} - 1]$.

$$R(S_e) = [9m + 3^{k-2} + t, 9m + 3^{k-1} + t - 1] \subset V_{nc}.$$

We have $[0, 9m-3^{k-3}-1] \cup [9m+3^{k-3}+\lfloor\frac{t}{3}\rfloor+1, 18m+\lceil\frac{t}{2}\rceil] \cup A_3 \cup A_4 \subset V_{nc}$.

Continuing in this way, we have $[9m+3^L+\lfloor\frac{t}{3}\rfloor+1, 9m+3^{L+1}+\lfloor\frac{t}{3}\rfloor] \subset V_{nc}$ and $[9m-3^{L+1}, 9m-3^L-1] \subset V_{nc}$ for $L = k-4, k-5, ..., 1, 0$. Now we have $[0, 18m+\lceil\frac{t}{2}\rceil] \cup A_4 \subset V_{nc}$.

(4)Let $S = [18m + \lceil\frac{t}{2}\rceil + 1, 20m]$,

$$R(S) = [12m + 2t - 3, 18m + 2t - 3\lceil\frac{t}{2}\rceil - 4] \subset V_{nc}.$$

Then, $[0, 20m] \cup A_4 \subset V_{nc}$.

(5)Let $S = [n - \lfloor\frac{20m}{3}\rfloor, n-1]$,

$$R(S) = [0, 3\lfloor\frac{20m}{3}\rfloor - 1] \subset V_{nc}.$$

Then, $[0, 20m] \cup A_4 \cup [n - \lfloor \frac{20m}{3} \rfloor, n-1] \subset V_{nc}$.

(6)Let $S = [20m+1, 26m-1]$,

$$R(S) = [30m + 3t, 12m + 2t - 4] \subset V_{nc}.$$

Then, $[0, 26m-1] \cup A_4 \cup [n - \lfloor \frac{20m}{3} \rfloor, n-1] \subset V_{nc}$.

(7)Let $S = [28m + t, n - \lfloor \frac{20m}{3} \rfloor - 1]$,

$$R(S) = [3\lfloor \frac{20m}{3} \rfloor, 24m - 1] \subset V_{nc}.$$

Then, $[0, 26m-1] \cup A_4 \cup [28m + t, n-1] \subset V_{nc}$.

(8)Let $S_r = [27m + 3^k + t, 27m + 3^{k+1}]$,where $k = \lceil \log_3(m+t) \rceil$. Then

$$R(S_r) = [27m - 3^{k+2} + 3t - 3, 27m - 3^{k+1} - 1] \subset V_{nc}, \text{ since } 27m - 3^{k+1} - 1 \leq 26m - 1.$$

We have $[0, 26m-1] \cup [27m + 3^k + t, n-1] \cup A_4 \subset V_{nc}$.

Let $S_e = [27m - 3^{k+1} - 1, 27m - 3^k - 1]$.

$$R(S_e) = [27m + 3^{k+1} + 3t, 27m + 3^{k+2} + 3t + 2] \subset V_{nc}.$$

We have $[0, 27m - 3^k - 1] \cup [27m + 3^k + t, n-1] \cup A_4 \subset V_{nc}$.

Continuing in this way, we have $[27m + 3^L + t, 27m + 3^{L+1} + t - 1] \subset V_{nc}$ and $[27m - 3^{L+1} - 1, 27m - 3^L - 1] \subset V_{nc}$ for $L = k-1, k-2, ..., 1, 0$. It follows $[0, 27m-2] \cup [27m + t + 1, n-1] \cup A_4 \subset V_{nc}$.

Since $R(\{27m-1\}) = [27m + 3t, 27m + 3t + 2] \subset V_{nc}$, we have $[0, 27m + \lfloor \frac{3t}{4} \rfloor] \cup [27m + t + 1, n-1] \subset V_{nc}$.

(9) Let $S = [27m + \lfloor \frac{3t}{4} \rfloor + 1, 27m + t]$.

$$R(S) = [27m - 3, 27m + 3t - 3\lfloor \frac{3t}{4} \rfloor - 4] \subset V_{nc}, \text{ since } 27m + 3t - 3\lfloor \frac{3t}{4} \rfloor - 4 \leq 27m + \lfloor \frac{3t}{4} \rfloor.$$

Now, $V(G_K(3,n)) = V_{nc}$. Therefore, $A_1 \cup A_2 \cup A_3 \cup A_4$ is a decycling set of $G_K(3,n)$.

Lemma 4. *For $G_K(3,n)$, let $m = \lfloor \frac{n}{36} \rfloor$ and $n \equiv t \pmod{36}$, i.e. $n = 36m + t$, then $f(3,n) \leq 12m + t + \lceil \frac{t}{2} \rceil + \lfloor \frac{3t}{4} \rfloor + 5$.*

Proof. Since $A_1 \cup A_2 \cup A_3 \cup A_4$ is a decycling set of $G_K(3,n)$.

5 Decycling Set of $G_K(d,n)$

Now we consider the decycling set of $G_K(d,n)$ for $d \geq 4$.

Let $m = \lfloor \frac{n}{d+1} \rfloor$ and $n \equiv t \pmod{d+1}$, then $n = (d+1)m + t$, where $0 \leq t \leq d$. Define

$$A_1 = [0, m],$$
$$A_2 = [\lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^3} \rfloor, 2m+1],$$
$$A_i = [\lfloor \frac{(i-1)n}{d} \rfloor + \lfloor \frac{(i-1)n}{d^3} \rfloor, im + (i-1)], \text{ for } i = 3, 4, \cdots d.$$

$$\text{We will show that } \bigcup_{i=1}^{d} A_i \text{ is a decycling set of } G_K(d,n)$$

.

Process 1.

Let $S_1 = [m+1, \lfloor \frac{n}{d} \rfloor - 1]$.

$$R(S_1) = [n - d\lfloor \frac{n}{d} \rfloor, m - (d-t) - 1] \subset V_{nc}.$$

Let $X_1 = [n - \lfloor \frac{n}{d^2} \rfloor, n-1]$.

$$R(X_1) = [0, d\lfloor \frac{n}{d^2} \rfloor - 1] \subset V_{nc}.$$

Let $Y_1 = [\lfloor \frac{n}{d} \rfloor, \lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^3} \rfloor - 1]$.

$$R(Y_1) = [2n - d\lfloor \frac{n}{d} \rfloor - d\lfloor \frac{n}{d^3} \rfloor, 2n - d\lfloor \frac{n}{d} \rfloor - 1] \subset V_{nc}.$$

$$\text{Therefore, } [0, 2m+1] \cup [n - \lfloor \frac{n}{d^2} \rfloor, n-1] \cup \bigcup_{i=3}^{d} A_i \subset V_{nc}.$$

Process 2.

Let $S_2 = [2m+2, \lfloor \frac{2n}{d} \rfloor - 1]$.

$$R(S_2) = [2n - d\lfloor \frac{2n}{d} \rfloor, 2m - 2(d-t) - 1] \subset V_{nc}.$$

Let $X_2 = [n - \lfloor \frac{2n}{d^2} \rfloor, n - \lfloor \frac{n}{d^2} \rfloor - 1]$.

$$R(X_2) = [d\lfloor \frac{n}{d^2} \rfloor, d\lfloor \frac{2n}{d^2} \rfloor - 1] \subset V_{nc}.$$

Let $Y_2 = [\lfloor \frac{2n}{d} \rfloor, \lfloor \frac{2n}{d} \rfloor + \lfloor \frac{2n}{d^3} \rfloor - 1]$.

$$R(Y_2) = [3n - d\lfloor \frac{2n}{d} \rfloor - d\lfloor \frac{2n}{d^3} \rfloor, 3n - d\lfloor \frac{2n}{d} \rfloor - 1] \subset V_{nc}.$$

$$\text{Therefore, } [0, 3m+2] \cup [n - \lfloor \frac{2n}{d^2} \rfloor, n-1] \cup \bigcup_{i=4}^{d} A_i \subset V_{nc}.$$

Using the same method, we have

Process k.

$$\text{After } k-1 \text{ steps, we have } [0, km+(k-1)] \cup [n - \lfloor \frac{(k-1)n}{d^2} \rfloor, n-1] \cup \bigcup_{i=k+1}^{d} A_i \subset V_{nc}.$$

Let $S_k = [km + k, \lfloor \frac{kn}{d} \rfloor - 1]$.

$$R(S_k) = [kn - d\lfloor \frac{kn}{d} \rfloor, km - k(d-t) - 1] \subset V_{nc}.$$

Let $X_k = [n - \lfloor \frac{kn}{d^2} \rfloor, n - \lfloor \frac{(k-1)n}{d^2} \rfloor - 1]$.

$$R(X_k) = [d\lfloor \frac{(k-1)n}{d^2} \rfloor, d\lfloor \frac{kn}{d^2} \rfloor - 1] \subset V_{nc}.$$

Let $Y_k = [\lfloor \frac{kn}{d} \rfloor, \lfloor \frac{kn}{d} \rfloor + \lfloor \frac{kn}{d^3} \rfloor - 1]$.

$$R(Y_k) = [(k+1)n - d\lfloor \frac{kn}{d} \rfloor - d\lfloor \frac{kn}{d^3} \rfloor, (k+1)n - d\lfloor \frac{kn}{d} \rfloor - 1] \subset V_{nc}.$$

$$\text{Therefore, } [0, (k+1)m + k] \cup [n - \lfloor \frac{kn}{d^2} \rfloor, n-1] \cup \bigcup_{i=k+2}^{d} A_i \subset V_{nc}.$$

$$\text{After } (d-1) \text{ processes, we have } [0, dm+(d-1)] \cup [n - \lfloor \frac{(d-1)n}{d^2} \rfloor, n-1] \subset V_{nc}.$$

Process d.

Let $S_d = [dm + d, n - 1]$.

$$R(S_d) = [0, dm - d(d - t) - 1] \subset V_{nc}.$$

Now, $V_{nc} = G_K(d, n)$. We complete the proof.

Lemma 5. *For $G_K(d, n), d \geq 4$, let $m = \lfloor \frac{n}{d+1} \rfloor$ and $n \equiv t \pmod{d+1}$. Then $f(d, n) \leq (\frac{1}{2} - \frac{d-1}{2d^2})n + \frac{d}{2}(d - t + 5) - 2$.*

Proof.

By above, we have $\bigcup_{i=1}^{d} A_i$ is a decycling set of $G_K(d, n)$.

$$\begin{aligned} f(d,n) \leq \sum_{i=1}^{d} |A_i| &= (m+1) + [(2m+1) - (\lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^3} \rfloor) + 1] + \cdots \\ &+ [(km + (k-1)) - (\lfloor \frac{(k-1)n}{d} \rfloor + \lfloor \frac{(k-1)n}{d^3} \rfloor) + 1] + \cdots \\ &+ [(dm + (d-1)) - (\lfloor \frac{(d-1)n}{d} \rfloor + \lfloor \frac{(d-1)n}{d^3} \rfloor) + 1] \\ &\leq (\frac{1}{2} - \frac{d-1}{2d^2})n + \frac{d}{2}(d - t + 5) - 2, \end{aligned}$$

since $\frac{kn}{d} - 1 \leq \lfloor \frac{kn}{d} \rfloor \leq \frac{kn}{d}$ and $\frac{kn}{d^3} - 1 \leq \lfloor \frac{kn}{d^3} \rfloor \leq \frac{kn}{d^3}$. ∎

6 Conclusions

References

[1] D. Barth and M.C. Heydemann, A new digraphs composition with applications to de Bruijn and generalized de Bruijn digraphs, Discrete Applied Math., 77 (1996), 99-118.

[2] D.Z. Du, D.F. Hsu, H.Q. Ngo, and G.W. Peck, On connectivity of consecutive-d digraphs, Discrete Math., 257 (2002), 371-384.

[3] T. Hasunuma, Y. Kikuchi, T. Mori and Y. Shibata, On the number of cycles in generalized Kautz digraphs, Discrete math., 285 (2004), 127-140.

[4] M. Imase and M. Itoh, Design to minimize diameter on building-block network, IEEE Trans. on Computers, C-30 (1981), 439-442.

[5] M. Imase and M. Itoh, A design for directed graphs with minimum diameter, IEEE Trans. on Computers C-32 (1981), 782-784.

[6] Y. Kikuchi and Y. Shibata, On the domination numbers of generalized de Bruijn digraphs and generalized Kautz digraphs, Information Processing Letters, 86 (2003), 79-85.

[7] G. Panchapakesan and A. Sengupta, On a lightwave network topology using Kautz digraphs, IEEE Transactions on Computers, Vol. 48, No. 10, 1999.

[8] J. M. Xu, Y. Z. Wu, J. Huang, C. Yang, Feedback numbers of Kautz digraphs, Discrete Math., 307(13) (2007), 1589-1599.

[9] X. Xu, J.-M. Xu and Y. Cao, Bounds on feedback numbers of de Bruijn graphs, Taiwaness J. Math., 15(3) (2011), 1101-1113.

[10] X. Xu, J. Wang, J. M. Xu, Y. Cao, Feedback numbers of Kautz undirected graphs, Australasian Journal of Combinatorics, 52(2012), 3-9.