The Decycling Number of $P_m \square P_n$*

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Abstract

A set of vertices of a graph whose removal leaves an acyclic graph is called a decycling set of the graph. The minimum size of a decycling set of a graph G is referred to as the decycling number of G, denoted by $\nabla(G)$. In this paper, we study the decycling number of the Cartesian product of two paths, $\nabla(P_m \square P_n)$, and obtain several new results. Mainly, we prove that $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil \leq \nabla(P_m \square P_n) \leq \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil + 1$. Moreover, we obtain the exact value of $\nabla(P_m \square P_n)$ for some classes (modulo 6) of pairs (m, n).

Key words and phrases: Decycling number, Feedback vertex number, Grids.

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1 Introduction

A set of vertices of a graph whose removal leaves an acyclic graph is referred to as a *decycling set*, or a *feedback vertex set*, of the graph. The minimum cardinality of a decycling set of G denoted by $\nabla(G)$, is referred to as the *decycling number* of G. In [11], Pike and Zou proved that the decycling number of Cartesian product of two cycles $C_m \square C_n$ is $\lceil \frac{mn+2}{3} \rceil$. But, finding $\nabla(P_m \square P_n)$ remains unsettled. For more results

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on the research of this parameter, see [1, 2, 5, 6, 7, 10, 12, 13, 14, 16, 17, 18, 19]. As to the notations and terminologies of graphs, we refer to [15].

It is well-known that $P_m \Box P_n$ has vertex set $V(P_m \Box P_n) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ and edge set $E(P_m \Box P_n) = \{(v_{i,j}, v_{i+1,j}) : 1 \le i \le m-1, 1 \le j \le n\} \cup \{(v_{i,j}, v_{i,j+1}) : 1 \le i \le m, 1 \le j \le n-1\}$. It was shown by Luccio in [8] that the decycling number of the grid $P_m \Box P_n$ is at most $\left\lfloor \frac{mn}{3} + \frac{m+n}{6} + o(m,n) \right\rfloor$ and at least $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$. Subsequently, in [4], Caragiannis, Kaklamanis and Kanellopoulos improved the upper bound. They showed that the decycling number of the grid $P_m \Box P_n$ is at most $\left\lfloor \frac{mn}{3} - \frac{m+n-5}{6} \right\rfloor$. Finally, Madelaine and Stewart [9] construct new decycling sets in grids so that for certain number of pairs (m, n), the size of decycling set in the grid $P_m \Box P_n$ matches the best lower bound $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$, and for all other pairs the size of decycling set is at most this lower bound plus 2.

In this paper, we further improve both the lower and upper bounds of $\nabla(P_m \Box P_n)$ for several classes of (m, n) such that for more (m, n) the decycling number of $P_m \Box P_n$ matches the lower bound and for all others it differs from the known lower bound by at most 1, i.e., $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil \le \nabla(P_m \Box P_n) \le \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil + 1$.

2 Decycling Sets of $P_m \Box P_n$

A decycling set S of a graph G gives the graph $G \backslash S$ a forest. Therefore, to obtain a minimum decycling set, we try to find an S for which $G \backslash S$ is a forest and $c(G \backslash S) + |E(G[S])|$ is as small as possible, where $c(G)$ is the number of components of G. Clearly, if $G \backslash S$ is a tree and S induces no edges, then we have a minimum decycling set S. But, for general graphs G, we may end up with more components in $G \backslash S$ or S induces at least one edge, and S is of minimum size. Therefore, we can use the number of components in $G \backslash S$ and the number of edges in $G[S]$ to characterize whether S is indeed a minimum decycling set.

The following result was obtained by F. L. Luccio in 1998.

Theorem 2.1. [8] *If $m, n \ge 2$, then* $\nabla(P_m \Box P_n) \ge \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

For convenience, we use $F_{m,n}$ and $f_{m,n}$ to denote $\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$ and $\frac{(m-1)(n-1)+1}{3}$ respectively. The following proposition is implicit in the proof of the above theorem.

Proposition 2.2. *If $m \geq 5$ and $f_{m,n}$ is an integer, then each decycling set S of size $f_{m,n}$ satisfies the following two properties:*
(1) S contains exactly one vertex of degree 3 and contains no vertex of degree 2; and
(2) S induces a subgraph of $P_m \Box P_n$ with no edges.

Now, we have a result on the lower bound of $\nabla(P_m \Box P_n)$.

Theorem 2.3. *If $m \geq 5$, mn is even and $f_{m,n}$ is an integer, then $\nabla(P_m \Box P_n) \geq f_{m,n} + 1 = F_{m,n} + 1$.*

Proof. Suppose not. Assume that $\nabla(P_m \Box P_n) = f_{m,n} = F_{m,n}$ and S is a decycling set with size $f_{m,n}$. By Proposition 2.2, we may let $v_{i,1}$ be the vertex of S with degree 3 where $2 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$. Since S is a decycling set and induces no edges in $P_m \Box P_n$, $v_{m-1,2} \in S$ and $v_{m-1,3} \notin S$. For otherwise, we have a 4-cycle $(v_{m-1,1}, v_{m-1,2}, v_{m,2}, v_{m,1})$ or $v_{m-1,2}, v_{m-1,3}$ is an edge in $(P_m \Box P_n)[S]$. Following this observation, we conclude that S contains $v_{m-1,2}, v_{m-1,4}, \cdots, v_{m-1,n-1}$ since S has no other vertices on the boundary of $P_m \Box P_n$. Hence, $n-1$ is even and n is odd. Similarly, $v_{m-3,n-1}, v_{m-5,n-1}, \cdots, v_{2,n-1}$ are contained in S and therefore, m is also odd. This contradicts to the assumption and we have the proof. ∎

Corollary 2.4. *For $m \geq 5$, if $m \equiv 0$ (mod 6) and $n \equiv 2$ (mod 3) or $(m,n) \equiv (3,2)$ (mod 6), $\nabla(P_m \Box P_n) \geq F_{m,n} + 1$.*

Proof. By direct checking, $f_{m,n}$ is an integer and $m \cdot n$ is even. ∎

Using this fact, we can estimate $\nabla(P_m \Box P_n)$ for more pairs (m,n) by using the following theorem which was obtained by Madelaine and Stewart. For clearness, we use Table 1 to depict these results (which take symmetry into consideration).

Theorem 2.5. [9]

Table 1:

m \ n	0	1	2	3	4	5
0	B	A	B	B	A	B
1	A	A	A	A	A	A
2	B	A	B	B	A	B
3	B	A	B	B	A	C
4	A	A	A	A	A	A
5	B	A	B	C	A	C

◯ : increasing the lower bound in this paper

◌ : decreasing the upper bound in this paper

In Table 1, A: $\nabla(P_m \Box P_n) = F_{m,n}$*, B:* $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$*, C:* $\nabla(P_m \Box P_n) \leq F_{m,n} + 2$*.*

Now, combining Theorem 2.5 with Corollary 2.4, we have

Theorem 2.6. *For* $m \geq 5$*, if* $(m, n) \equiv (0, 2), (0, 5), (3, 2), (2, 0), (5, 0), (2, 3)$ $(mod\ 6)$*, then* $\nabla(P_m \Box P_n) = F_{m,n} + 1$*.*

In what follows, we prove that for cases in class "C" mentioned above $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$ for $m \geq 6$. Before we go any further, we need to introduce a couple of new notations. We shall use $P_m \Box P_r \mid P_m \Box P_k$ to represent that $P_m \Box P_{r+k-1}$ can be separated into $P_m \Box P_r$ and $P_m \Box P_k$ with a common vertical path P_m (see Figure 1(a)). Similarly, we use $\frac{P_r \Box P_n}{P_k \Box P_n}$ to represent that $P_{r+k-1} \Box P_n$ can be separated into $P_r \Box P_n$ and $P_k \Box P_n$ and they overlap a horizontal path P_n (see Figure 1(b) for an example). For $x, y \in V(G)$, an x, y-path is a path beginning at x and ending at y.

In order to prove the main theorem, we need the following three smaller cases.

Lemma 2.7. *For* $(m, n) = \{(6, 6), (6, 8), (8, 8)\}$*,* $\nabla(P_m \Box P_n) \leq F_{m,n} + 1$*.*

Proof. Beineke and Vandell [3] have already proved the first two cases. By direct checking, the third one is also true. For clearness, we include a decycling set of $P_8 \Box P_8$ in Figure 2.

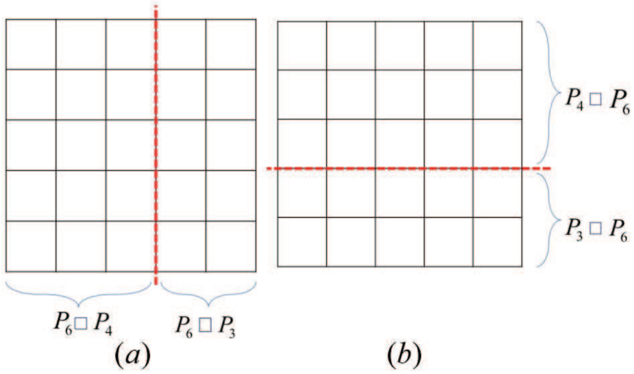

Figure 1: (a)$P_6 \square P_6 = P_6 \square P_4 \mid P_6 \square P_3$; (b)$P_6 \square P_6 = \frac{P_4 \square P_6}{P_3 \square P_6}$

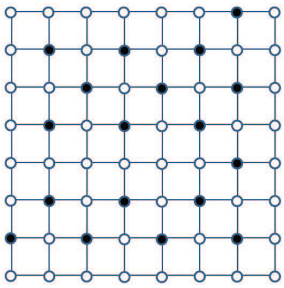

Figure 2: A decycling set of $P_8 \square P_8$.

Lemma 2.8. [3] *If G and H are homeomorphic graphs, then* $\nabla(G) = \nabla(H)$.

Theorem 2.9. *For $m, n \geq 6$, $\nabla(P_m \square P_n) \leq F_{m,n} + 1$.*

Proof. By Theorem 2.5, Lemma 2.7 and the symmetry of the graph, it suffices to consider the following 2 cases.

Case 1. $m \equiv 5 \pmod 6$ and $n \equiv 5 \pmod 6$.

Let $X_{6k+5,6r+5} = \{v_{i,j} : i \text{ and } j \text{ are even}, 1 \leq i \leq 6k+5, 1 \leq j \leq 6r+5\}$. Then $P_{6k+5} \square P_{6r+5} \backslash X_{6k+5,6r+5}$ is homeomorphic to the graph $P_{3k+3} \square P_{3r+3}$. By Lemma 2.8, for $k, r \geq 0, \nabla(P_{6k+5} \square P_{6r+5}) \leq (3k+2)(3r+2) + \lceil \frac{(3k+2)(3r+2)+1}{3} \rceil + 1 = F_{6k+5,6r+5} + 1$.

Case 2. $m \equiv 3 \pmod 6$ and $n \equiv 5 \pmod 6$.

First, we can find a decycling set of $P_9 \Box P_{11}$ directly. (See Figure 3, $\nabla(P_9 \Box P_{11}) \leq 28 = F_{9,11} + 1$.) Then, we partition this case into 3 subcases and apply the case $m \equiv 1 \pmod 3$ in [9] to solve the following.

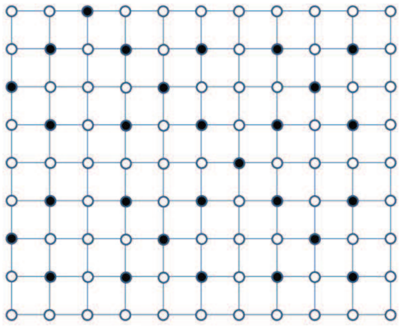

Figure 3: Decycling set (black vertices) of $P_9 \Box P_{11}$.

Subcase 2.1. $m = 9$ and $n \equiv 5 \pmod 6$.

Separate $P_9 \Box P_{6k+5}$ into $P_9 \Box P_{6(k-1)+1} \mid P_9 \Box P_{11}$. We can find a set of vertices $X_{9,6(k-1)+1}$ in $P_9 \Box P_{6(k-1)+1}$ by using Madelaine and Stewart's method [9]. Define $X_{9,6(k-1)+1}$

$$
\begin{aligned}
&= \{v_{i,j} : 5 \leq i \leq 7, i \text{ is odd}, 3 \leq j \leq 6(k-1)+1, j \equiv 3, 5 \pmod 6\} \\
&\bigcup \{v_{i,j} : 5 \leq i \leq 8, i \text{ is even}, 2 \leq j \leq 6(k-1), j \equiv 0, 2 \pmod 6\} \\
&\bigcup \{v_{5,j} : 2 \leq j \leq 6(k-1)+1, j \equiv 1 \pmod 6\} \\
&\bigcup \{v_{8,j} : 2 \leq j \leq 6(k-1)+1, j \equiv 4 \pmod 6\} \\
&\bigcup \{v_{2,j} : 2 \leq j \leq 6(k-1), j \text{ is even}\} \\
&\bigcup \{v_{3,j} : 3 \leq j \leq 6(k-1)+1, j \text{ is odd}\} \\
&\bigcup \{v_{4,2}\}.
\end{aligned}
$$

And we find $X_{9,11}$ in $P_9 \Box P_{11}$ by letting $X_{9,11}$

$$
\begin{aligned}
&= \{v_{i,j} : 2 \leq i \leq 8, i \text{ is even}, 6(k-1)+1 \leq j \leq 6k+5, j \text{ is even}\} \\
&\bigcup \{v_{3,j}, v_{7,j} : j = 6(k-1)+1, 6(k-1)+5, 6k+3\} \\
&\bigcup \{v_{1,6(k-1)+3}, v_{5,6k+1}\}.
\end{aligned}
$$

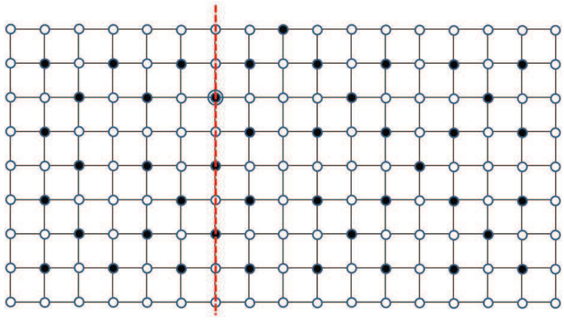

Figure 4: Decycling set of $P_9 \square P_{17}$.

Define $X_{9,6k+5} = X_{9,6(k-1)+1} \cup X_{9,11}$. The set $X_{9,17}$ is shown in Figure 4. We claim that $X_{9,6k+5}$ is a decycling set. Observe that if there is a cycle in $P_9 \square P_{6k+5} \setminus X_{9,6k+5}$, then the cycle must use the perimeter vertices of $P_9 \square P_{6(k-1)+1}$ excluding $\{v_{i,6k-5} : 3 \leq 7\}$ and a $(v_{2,6k-5}, v_{8,6k-5})$ -path in $P_9 \square P_{11} \setminus X_{9,11}$. However, there is no $(v_{2,6k-5}, v_{8,6k-5})$ -path in $P_9 \square P_{11} \setminus X_{9,11}$. Hence, $X_{9,6k+5}$ is a decycling set of $P_9 \square P_{6k+5}$. Since $v_{3,6(k-1)+1}$ belongs to both $X_{9,6(k-1)+1}$ and $X_{9,11}$, the size of $X_{9,6k+5}$ is

$$\left\lceil \frac{8 \cdot 6(k-1)+1}{3} \right\rceil + 28 - 1 = \left\lceil \frac{8(6k+4)+1}{3} \right\rceil + 1.$$

Subcase 2.2. $m \equiv 3 \pmod 6$ and $n = 11$.

Similar to **Subcase 3.1**, we let $P_{6k+3} \square P_{11} = \frac{P_{6(k-1)+1} \square P_{11}}{P_9 \square P_{11}}$ and let $X_{6(k-1)+1,11}$

$$\begin{aligned}
&= \{v_{i,j} : 1 \leq i \leq 6(k-1)+1, i \equiv 0, 2 \pmod 6, 2 \leq j \leq 7, j \text{ is even}\} \\
&\bigcup \{v_{i,j} : 1 \leq i \leq 6(k-1)+1, i \equiv 3, 5 \pmod 6, 2 \leq j \leq 7, j \text{ is odd}\} \\
&\bigcup \{v_{i,7} : 2 \leq i \leq 6(k-1)+1, i \equiv 1 \pmod 6\} \\
&\bigcup \{v_{i,2} : 2 \leq i \leq 6(k-1)+1, i \equiv 4 \pmod 6\} \\
&\bigcup \{v_{i,10} : 1 \leq i \leq 6(k-1), i \text{ is even}\} \\
&\bigcup \{v_{i,9} : 3 \leq i \leq 6(k-1)+1, i \text{ is odd}\} \\
&\bigcup \{v_{2,8}\}.
\end{aligned}$$

We use a different construction to find $X_{9,11}$ in $P_9 \Box P_{11}$, where $X_{9,11} = \{v_{i,j} : 6(k-1)+1 \leq i \leq 6k+3, i \text{ is even}, 1 \leq j \leq 11, j \text{ is even}\} \bigcup \{v_{6k-5,9}, v_{6k-3,3}, v_{6k-3,5}, v_{6k-1,1}, v_{6k-1,9}, v_{6k+1,3}, v_{6k+1,7}, v_{6k+3,9}\}$.

Define $X_{6k+3,11} = X_{6(k-1)+1,11} \cup X_{9,11}$. The construction of $X_{15,11}$ can be visualized as in Figure 5. The argument is similar to Subcase 3.1 which yields that $X_{6k+3,11}$ is a decycling set of $P_{6k+3} \Box P_{11}$. Since $v_{6(k-1)+1,9}$ belongs to both $X_{6(k-1)+1,11}$ and $X_{9,11}$, the size of $X_{6k+3,11}$ is

$$\left\lceil \frac{6(k-1)10+1}{3} \right\rceil + 28 - 1 = \left\lceil \frac{(6k+2)10+1}{3} \right\rceil + 1.$$

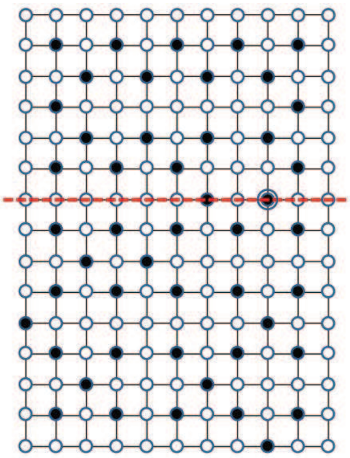

Figure 5: Decycling set of $P_{15} \Box P_{11}$.

Subcase 2.3. $m \equiv 3 \pmod 6$ and $n \equiv 5 \pmod 6$ and $m > 9, n > 11$.

Let $P_{6k+3} \Box P_{6r+5}$ be $\frac{P_{6(k-1)+1} \Box P_{6r+5}}{P_9 \Box P_{6(r-1)+1} | P_9 \Box P_{11}}$. We note that the labeling of each vertex in the following is the same as the labeling used in the original grid.

Now, define $X_{6(k-1)+1,6r+5}$ in $P_{6(k-1)+1}\Box P_{6r+5}$ as

$$\begin{aligned}
& \{v_{i,j} : 1 \le i \le 6(k-1)+1, i \equiv 0, 2 \pmod 6, 2 \le j \le 6r+1, j \text{ even}\} \\
\bigcup\ & \{v_{i,j} : 1 \le i \le 6(k-1)+1, i \equiv 3, 5 \pmod 6, 2 \le j \le 6r+1, j \text{ odd}\} \\
\bigcup\ & \{v_{i,6r+1} : 2 \le i \le 6(k-1)+1, i \equiv 1 \pmod 6\} \\
\bigcup\ & \{v_{i,2} : 2 \le i \le 6(k-1)+1, i \equiv 4 \pmod 6\} \\
\bigcup\ & \{v_{i,6r+4} : 1 \le i \le 6(k-1), i \text{ even}\} \\
\bigcup\ & \{v_{i,6r+3} : 3 \le i \le 6(k-1)+1, i \text{ odd} \\
\bigcup\ & \{v_{2,6r+2}\}.
\end{aligned}$$

Define $X_{9,6(r-1)+1}$ in $P_9\Box P_{6(r-1)+1}$ as following. $X_{9,6(r-1)+1}$

$$\begin{aligned}
= \ & \{v_{i,j} : 6k-1 \le i \le 6k+1, i \text{ odd}, 3 \le j \le 6r-5, j \equiv 3, 5 \pmod 6\} \\
\bigcup\ & \{v_{i,j} : 6k-1 \le i \le 6k+2, i \text{ even}, 2 \le j \le 6r-6, j \equiv 0, 2 \pmod 6\} \\
\bigcup\ & \{v_{6(k-1)+5,j} : 2 \le j \le 6(r-1)+1, j \equiv 1 \pmod 6\} \\
\bigcup\ & \{v_{6k+2,j} : 2 \le j \le 6(r-1)+1, j \equiv 4 \pmod 6\} \\
\bigcup\ & \{v_{6(k-1)+2,j} : 2 \le j \le 6(r-1), j \text{ even}\} \\
\bigcup\ & \{v_{6(k-1)+3,j} : 3 \le j \le 6(r-1)+1, j \text{ odd}\} \\
\bigcup\ & \{v_{6(k-1)+4,2}\}.
\end{aligned}$$

Define $X_{9,11}$ in $P_9\Box P_{11}$ as the following Figure 6, the size of $X_{9,11}$ is 30.

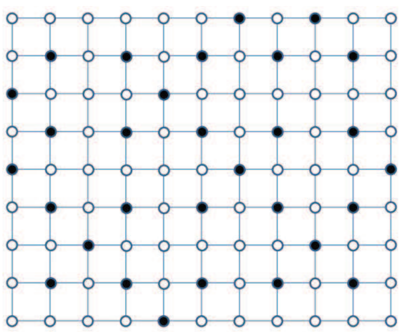

Figure 6: Decycling set of $P_9\Box P_{11}$ (Different from Figure 3).

Define $X_{6k+3,6r+5} = X_{6(k-1)+1,6r+5} \cup X_{9,6(r-1)+1} \cup X_{9,11}$. The construction is illustrated for $P_{15}\Box P_{17}$ in Figure 7.

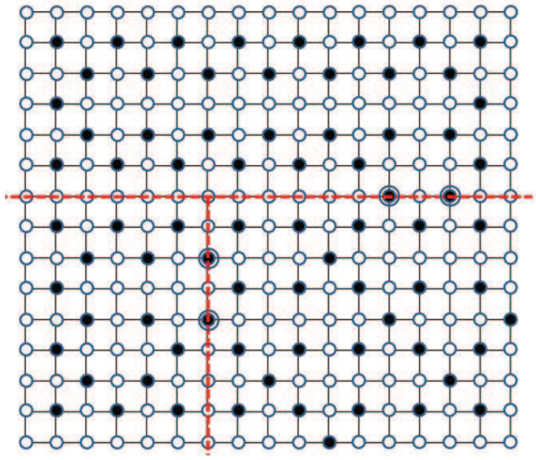

Figure 7: Decycling set of $P_{15}\Box P_{17}$.

We claim that $X_{6k+3,6r+5}$ is a decycling set. Observe that if there is a cycle in $P_{6k+3}\Box P_{6r+5} \setminus X_{6k+3,6r+5}$ then the cycle must use the perimeter vertices of $P_{6(k-1)+1}\Box P_{6r+5}$ excluding $\{v_{6(k-1)+1,6r+j} : j = 1,2,3\}$ and a $(v_{6(k-1)+1,6r}, v_{6(k-1)+1,6r+4})$ -path in $(P_9\Box P_{6(r-1)+1} \mid P_9\Box P_{11}) \setminus (X_{9,6(r-1)+1} \cup X_{9,11})$. By directly checking, there is no path from the right boundary of $P_9\Box P_{11}$ to the left boundary of $P_9\Box P_{11}$. There is no $(v_{6(k-1)+1,6r}, v_{6(k-1)+1,6r+4})$ -path in $(P_9\Box P_{6(r-1)+1} \mid P_9\Box P_{11}) \setminus (X_{9,6(r-1)+1} \cup X_{9,11})$. Hence $X_{6k+3,6r+5}$ is a decycling set of $P_{6k+3}\Box P_{6r+5}$. Since $v_{6(k-1)+1,6r+1}, v_{6(k-1)+1,6r+3} \in X_{9,11} \cap X_{6(k-1)+1,6r+5}$ and $v_{6(k-1)+3,6(r-1)+1}, v_{6(k-1)+5,6(r-1)+1} \in X_{9,11} \cap X_{9,6(r-1)+1}$, the size of $X_{6k+3,6r+5}$ is $\left\lceil \frac{6(k-1)(6r+4)+1}{3} \right\rceil + \left\lceil \frac{8\cdot 6(r-1)+1}{3} \right\rceil + 30 - 4 = \left\lceil \frac{(6k+2)(6r+4)+1}{3} \right\rceil + 1$. We complete the proof. ∎

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