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On the Intersections of Latin Squares with Holes

Chin-Mei Fu and Hung-Lin Fu

Electronics Research & Service Organization
195-4-E000, Sec. 4, Chunc-Hsing Road
Chu-Tung, Hsin-Chu 31015
Taiwan, Republic of China

Department of Applied Mathematics
National Chiao Tung University
Hsin-Chu, Taiwan, Republic of China

1. Introduction

Let $P = \{S_1, S_2, \ldots, S_n\}$ be a partition of a set S $(n \geq 2)$. A partitioned incomplete latin square, (or PILS), having partition P, is an $|S|$ by $|S|$ array L, indexed by S, satisfying the following properties:

(1) a cell of L either contains an element of S or is empty,
(2) the subarray indexed by $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (we will refer to these subarrays as holes),
(3) The elements occuring in row (or column) s of L are precisely those in $S \backslash S_i$, where $s \in S_i$.

If the PILS is commutative, we denote it by CPILS.

We will say that the type of L is multiset $\{|S_1|, |S_2|, \ldots, |S_n|\}$, and we use the notation $t_i^{u_1} \cdot t_2^{u_2} \cdot \cdots \cdot t_k^{u_k}$ to describe the type of a PILS of CPILS, where there are precisely $u_i S_j's$ of cardinality t_i, for $1 \leq i \leq k$.

Denote by $J[2n]$ the set of all positve integers k such that there exists a pair of PILS of order $2n$ with type 2^n which have exactly k entries (not in holes) in common, and set $I[2n] = \{0, 1, \ldots, (2n)^2 - 4n - 6, (2n)^2 - 4n - 4, (2n)^2 - 4n\}$. Also, we let $\overline{J}[2n]$ be the set of all positive integers h such that exists two CPILS's of order $2n$ with type 2^n which have exactly h entries (in upper triangular portion but not in holes) in common, and set $\overline{I}[2n] = \{0, 1, \ldots, 2n^2 - 2n - 6, 2n^2 - 2n - 4, 2n^2 - 2n\}$.

In this paper, we first use a $2n$ construction and the results in [1] to show that $J[2n] = I[2n]$ for every $n \geq 5$, and then with a v to $2v + 4$ construction prove that $\overline{J}[2n] = \overline{I}[2n]$ for every $n \geq 5$. Moreover, we solve the intersection problem of half-idempotent latin squares.

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2. The main theorems

Let A_1, A_2 be two independent latin squares based on the set $\{1, 2, \ldots, n\}$, and A'_1, A'_2 be the partial latin squares obtained by deleting the diagonal entries of A_1, A_2, respectively. Similarly, let B_1, B_2 be two idempotent latin squares based on the set $\{n+1, n+2, \ldots, 2n\}$, and B'_1, B'_2 be the partial latin squares obtained by taking away the diagonal entries of B_1, B_2, respectively. Hence we can construct a PILS L of order $2n$ (Figure 2.1) with type 2^n where $S_i = \{i, n+i\}, 1 \leq i \leq n$.

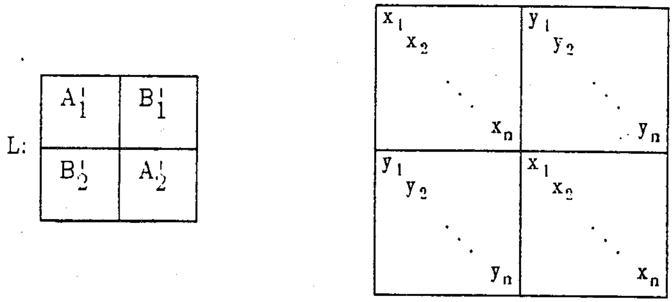


Figure 2.1

Lemma 2.1. [1] *For each* $k \in K[n] = \{0, 1, \ldots, n^2 - n - 6, n^2 - n - 4, n^2 - n\}$, *and* $n \geq 6$, *there exists a pair of idempotent latin squares of order* n *which have exactly* k *entries (outside the diagonal) in common.*

Before we prove the following theorem, we define two sets. Let $A + B = \{a + b | a \in A \text{ and } b \in B\}$ and $t \cdot A$ be the sum of t copies of A.

Lemma 2.2. $J[2n] \subseteq I[2n]$.

Proof: It suffices to show that there does not exist a pair of PILS($2n$) which have $4n^2 - 4n - 1, 4n^2 - 4n - 2, 4n^2 - 4n - 3$, or $4n^2 - 4n - 5$ entries not in holes in common. Since it is impossible to find a pair of PILS($2n$) which have $2n - 1$ entries in common that contains in the same row or column. This completes the proof.

Lemma 2.3. $J[2n] = I[2n]$ *for every* $n \geq 6$.

Proof: We start with two PILS of order $2n$ with type 2^n which are obtained by the $2n$ construction. (Figure 2.1.) Since A'_1, A'_2, B'_1, and B'_2 can be replaced by any idempotent latin square (without diagonal and based on the same set) independently, hence we have $K[n] + K[n] + K[n] + K[n] \subseteq J[2n]$ which is $I[2n] \subseteq J[2n]$. By Lemma 2.2, we conclude the proof.

For the case $n \leq 5$, we have the following results. Since a part of the conclusion is from tedious direct construction, we omit the details here.

Theorem 2.4. $J[6] = \{0,4,8,12,16,20,24\}$, $J[8] = I[8]\backslash\{35,41\}$, *and* $J[10] = I[10]$.

A latin square $L = [l_{i,j}]$ of order $2n$ is said to be half-idempotent if $l_{i,i} = i$ whenever $1 \leq i \leq n$, and $l_{i,i} = i - n$ whenever $n+1 \leq i \leq 2n$. It is not difficult to see that the latin square M in Figure 2.2 is half-idempotent latin square of order $2n$ where A_1, A_2 are idempotent latin squares of order n based on the set $\{1,2,\ldots,n\}$ and C_1, C_2 are latin squares of order n based on the set $\{n+1, n+2, \ldots, 2n\}$.

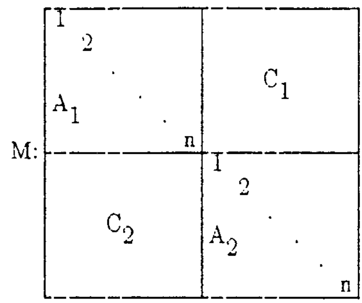


Figure 2.2

Lemma 2.5. [1] *For each* $k \in \{0,1,\ldots,n^2-6, n^2-4, n^2\}$ *and* $n \geq 5$, *there exists a pair of latin squares of order* n *which have exactly* k *entries in common.*

Since the latin squares C_1, C_2 can be replaced by any latin squares based upon the same set, and A_1, A_2 can be replaced by any other idempotent latin squares based on the same set independently, we have the following theorem.

Theorem 2.6. *For each* $k \in \{0,1,\ldots,4n^2-2n-6, 4n^2-2n-4, 4n^2-2n\}$ *and* $n \geq 5$ *there exists a pair of half-idempotent latin squares which have exactly* k *entries (outside the diagonal) in common.*

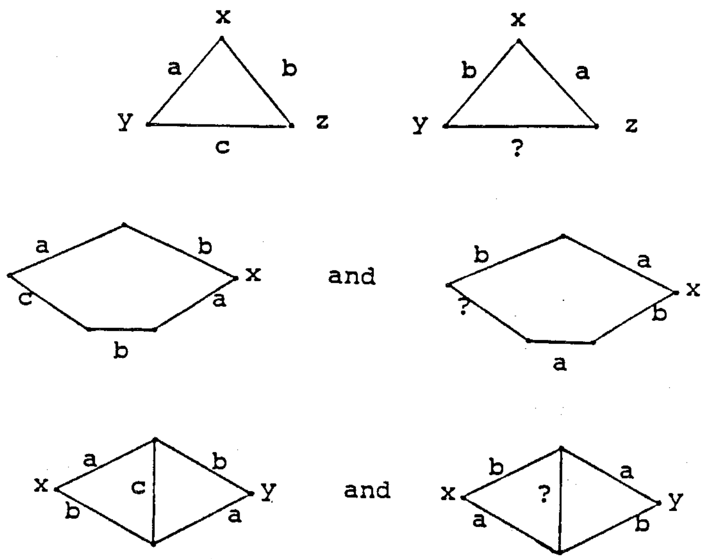
Proof: Similar to the proof of Theorem 2.3.

By direct construction we can also obtain some results on the cases $1 \leq n \leq 4$. Again we omit the details.

Now we are ready for the commutative case. If n is odd, we can replace A_1', A_2' in Figure 2.1 by commutative idempotent latin squares without diagonals, and replace B_1', B_2' by any idempotent latin square and its transpose (without diagonals) respectively, then we have a CPILS of order $2n$, denoted by CPILS($2n$). By the same ideas of the proof in Theorem 2.3, Lemma 2.7, and Theorem 2.8, we have $\overline{J}[2n] = \overline{I}[2n]$ whenever n is odd and $n \geq 7$.

Lemma 2.7. $\overline{J}[2n] \subseteq \overline{I}[2n]$.

Proof: It suffices to show that there does not exist a pair of CPILS($2n$) which have $2n^2-2n-1, 2n^2-2n-2, 2n^2-2n-3$, or $2n^2-2n-5$ entries in common. Define two edge-colored graphs G_1 and G_2 corresponding to two CPILS($2n$) L_1 and L_2 respectively. The vertices of G_i represent the rows in which L_1 and L_2 are not identical, $i = 1, 2$, and if the (x, y) entry of L_1 is not equal to the (x, y) entry of L_2, i.e., if $L_1(x, y) \neq L_2(x, y)$, then xy is an edge of both graphs. We color the edge xy with the color $L_i(x, y)$ in G_i, $i = 1, 2$, respectively. Then the minimum degree of each vertex is ≥ 2. Therefore $2n^2-2n-1, 2n^2-2n-2 \notin \overline{J}[2n]$. By the following figures we obtain that $2n^2-2n-3, 2n^2-2n-5 \notin \overline{J}[2n]$.



a,b,c represent the colors

? represents that it's impossible to color that edge with any color a,b,c

Theorem 2.8. [3] *For each* $k \in \{0, 1, 2, \ldots, \frac{1}{2}(n^2-n)-6, \frac{1}{2}(n^2-n)-4, \frac{1}{2}(n^2-n)\}$ *there exists a pair of commutative idempotent latin squares which have exactly k entries in common (outside the diagonal and in upper right trianglular portion) whenever n is odd and not less than 7.*

Theorem 2.9. $\overline{J}[2n] = \overline{I}[2n]$, *n is odd and* $n > 7$.

For the case n is even, we need the following lemmas.

Therem 2.10. *A CPILS(v) can be embedded in CPILS(*$2v+4$*) provided there exists a CPILS(*$v+2$*).*

Proof: Let A be a CPILS(v) of type $2^{v/2}$ based on $\{1, 2, \ldots, v\}$, and $B = [b_{ij}]$ be a CPILS($v + 2$) of type $2^{(v+2)/2}$ with its 2×2 holes filled with suitable 2×2 latin squares. (Figure 2.3.) We construct a $(v + 4) \times (v + 4)$ array B' as in Figure 2.4, where $b'_{i,j} = b_{i-2,j-2}$ (note that B' does not have the latin property). By deleting each occurance of $v + 1$ and $v + 2$ from B' that is not in a hole, we get the partial latin rectangle B'' as in Figure 2.5 where if the (i, j) entry is not in the holes and $b'_{i,j}$ is not equal to $v + 1$ or $v + 2$ then $b''_{i,j} = b'_{i,j}$, and otherwise $b''_{i,j}$ is undefined. Without loss of generality, we suppose in B'' the empty cells occupy the cells $(i_1, j_1), (i_2, j_2), \ldots, (i_{v+4}, j_{v+4})$ above the diagonal. Let $S = \{v + 1, v + 2, \ldots, 2v + 4\}$ and S_t be the set of elements which are in S but not in the i_t^{th} row and the j_t^{th} column of B''. Then $|S_t| = v$ and each element of S occurs in exactly v different sets of $S_1, S_2, \ldots, S_{v+4}$. By P. Hall's theorem [2], the sets $S_1, S_2, \ldots, S_{v+4}$ have an SDR (system of distinct representatives). Replacing the empty cells with this SDR gives a $(v + 4) \times (v + 4)$ array B'''. Using the same technique recursively, we can complete C to a latin rectangle which completes L as in Figure 2.6 to a CPILS($2v + 4$). We conclude the proof.

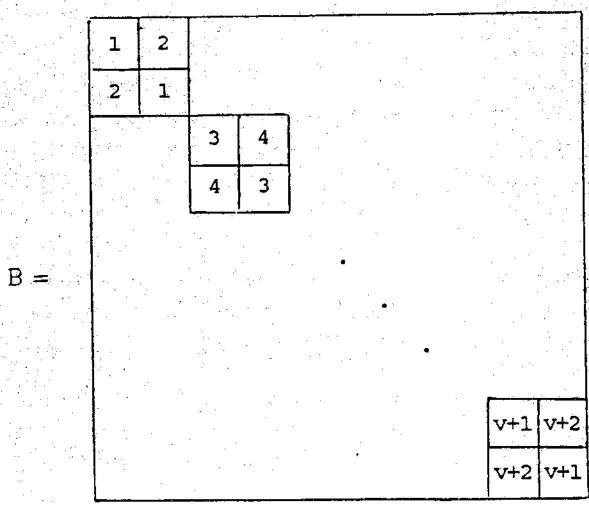


Figure 2.3

Lemma 2.11. *If* $\overline{J}[v] = \overline{I}[v]$ *where* v *is even and* $v \geq 8$, *then* $\overline{J}[2v + 4] = \overline{I}[2v + 4]$.

Proof: Since the entries $v + 1, v + 2, \ldots, 2v + 4$ outside A can be permuted pairwisely $(v + 1, v + 2; v + 3, v + 4; \ldots)$, the entries in B''' which are $1, 2, \ldots, v$ can be permuted, and the CPILS(v) A can be replaced by any other CPILS(v), hence we can construct two CPILS($2v+4$) which have exactly k entries in common where $k \in \{0,(2v+2),2(2v+2),\ldots,(v+4)(2v+2)/2\} + \{0,(v+4)/2,\ldots,(v-2)(v+4)/2, v(v+4)/2\} + \overline{J}[v]$, i.e., $\overline{I}[2v+4] \subseteq \overline{J}[2v+4]$ (direct calculation). By Lemma 2.7, $\overline{J}[2v+4] \subseteq \overline{I}[2v+4]$, we conclude the proof.

B' =

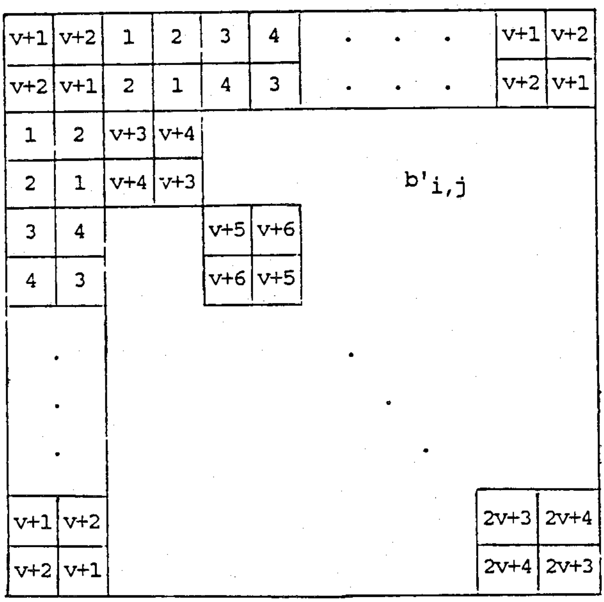


Figure 2.4

Lemma 2.12. $\overline{J}[6] = \{0,4,8,12\}$, $\overline{J}[8] = \overline{I}[8]\backslash\{3,7,11,15,16,17,20\}$, *and* $\overline{J}[v] = \overline{I}[v]$ *when* $v = 10,12,16,20$.

Proof: By direct construction in [5].

Theorem 2.13. $\overline{J}[2n] = \overline{I}[2n]$ *for every* $n \geq 5$.

Proof: By Lemma 2.9,2.11, and 2.12.

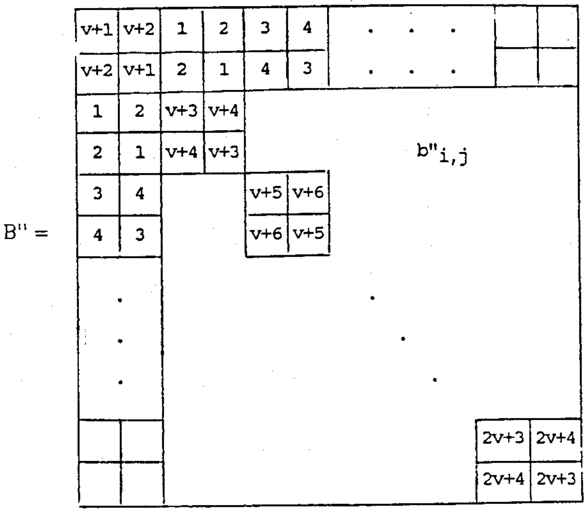


Figure 2.5

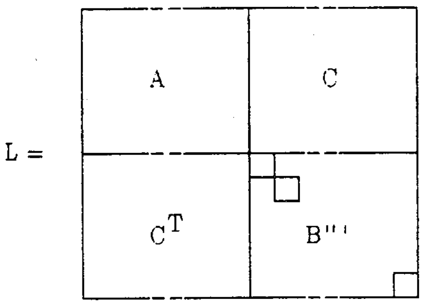


Figure 2.6

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