J Comb Optim (2013) 25:536–542
DOI 10.1007/s10878-012-9455-1

The decycling number of outerplanar graphs

Huilan Chang · Hung-Lin Fu · Min-Yun Lien

Published online: 25 February 2012

Abstract For a graph G, let $\tau(G)$ be the decycling number of G and $c(G)$ be the number of vertex-disjoint cycles of G. It has been proved that $c(G) \leq \tau(G) \leq 2c(G)$ for an outerplanar graph G. An outerplanar graph G is called *lower-extremal* if $\tau(G) = c(G)$ and *upper-extremal* if $\tau(G) = 2c(G)$. In this paper, we provide a necessary and sufficient condition for an outerplanar graph being upper-extremal. On the other hand, we find a class $\mathcal{S}$ of outerplanar graphs none of which is lower-extremal and show that if G has no subdivision of S for all $S \in \mathcal{S}$, then G is lower-extremal.

Keywords Decycling number · Feedback vertex number · Cycle packing number · Outerplanar graph

1 Introduction

The problem of destroying all cycles in a graph by deleting a set of vertices originated from applications in combinatorial circuit design (Johnson 1974). Also, it has found applications in deadlock prevention in operating systems (Wang et al. 1985; Silberschatz et al. 2003), the constraint satisfaction problem and Bayesian inference in artificial intelligence (Bar-Yehuda et al. 1998), monopolies in synchronous distributed systems (Peleg 1998, 2002), the converters' placement problem in optical networks (Kleinberg and Kumar 1999), and VLSI chip design (Festa et al. 2000).

This research is partially supported by NSC 99-2811-M-009-056 and NSC 100-2115-M-390-004-MY2.

H. Chang (✉) · H.-L. Fu · M.-Y. Lien
Department of Applied Mathematics, National Chiao Tung University, Hsinchu, 30010, Taiwan, ROC
e-mail: huilan0102@gmail.com

H. Chang
Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan, ROC

In the literature, a set of vertices of a graph whose removal leaves an acyclic graph is referred to as a *decycling set* (Beineke and Vandell 1997), or a *feedback vertex set* (Wang et al. 1985), of the graph. The minimum cardinality of a decycling set of G, denoted by $\tau(G)$, is referred to as the *decycling number* of G. Determining the decycling number is equivalent to finding the greatest order of an induced forest of G proposed first by Erdös et al. (1986). The problem of determining the decycling number has been proved to be *NP*-complete for general graphs (Karp et al. 1975), which also shows that even for planar graphs, bipartite graphs and perfect graphs, the computation complexity of finding their decycling numbers is not reduced.

Besides searching for the value (or an upper bound) of the decycling number in the order of a graph, another parameter that is closely related to the decycling number is the *cycle packing number*, which is the maximum number of vertex-disjoint cycles. We denote this parameter by $c(G)$. Determining the cycle packing number of a graph is also known to be *NP*-complete (Bodlaender 1994). A trivial relation between the decycling number and the cycle packing number is $c(G) \le \tau(G)$.

A graph is said to be *outerplanar* provided that all its vertices lie on the boundary of a face (after embedding the graph in a sphere). Even for an outerplanar graph G, not much is known about $\tau(G)$. Bau et al. (1998) found formulas of decycling numbers for subclasses of outerplanar graphs. For maximal outerplanar graph of order n, they provided a sharp upper bound $\lfloor \frac{n}{3} \rfloor$, which can be derived by the acyclic coloring argument (Fertin et al. 2002). Kloks et al. (2002) proved that $\tau(G) \le 2c(G)$ by a greedy algorithm.

An outerplanar graph G is called *lower-extremal* if $\tau(G) = c(G)$ and *upper-extremal* if $\tau(G) = 2c(G)$. In this paper, we provide a necessary and sufficient condition for an outerplanar graph being upper-extremal. On the other hand, we provide a sufficient condition for an outerplanar graph being lower-extremal. We find a class $\mathcal{S}$ of outerplanar graphs none of which is lower-extremal and show that if G has no subdivision of S (or S-subdivision) for all $S \in \mathcal{S}$, then G is lower-extremal.

For graphs notations and terminologies here, we refer to West (2001).

2 Upper-extremal graphs

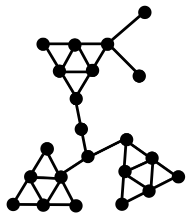
For simplicity, we use ij to denote an edge $\{i, j\}$. We start by presenting an upper-extremal graph with simplest structure.

Definition 1 S_k is a graph with vertex set $V = \{0, 1, \dots, 2k-1\}$ and edge set $E = \{i(i+1) : 0 \le i \le 2k-1\} \cup \{i(i+2) : i \text{ is even}\}$ (the indices are under modulo $2k$).

Then $\tau(S_k) = \lceil \frac{k}{2} \rceil$ and $c(S_k) = \lfloor \frac{k}{2} \rfloor$. S_3 is clearly an upper-extremal graph; indeed, its subdivisions are the only 2-edge-connected outerplanar graphs that are upper-extremal and have cycle packing number one. We define the *simplified graph* of a graph G to be the graph obtained from G by continuously deleting isolated vertices or degree one vertices until there is no more such vertex and denote it by $\lfloor G \rfloor$.

Throughout the paper, let $F(G)$ denote the outer face of an outerplanar G. An edge uv is called a *basic edge* of G if uv and some u,v-path on the boundary of $F(G)$ form the boundary of a face of G. Then, we have

Fig. 1 An S_3-tree G of order 3, where $\tau(G) = 6 = 2c(G)$



Lemma 1 *For an outerplanar graph G with $c(G) = 1$, G is upper-extremal if and only if $\lfloor G \rfloor$ is an S_3-subdivision.*

Proof It suffices to prove the necessity. If $\lfloor G \rfloor$ has a cut-vertex v, then v belongs to two blocks of $\lfloor G \rfloor$, say G_1 and G_2, and $\lfloor G \rfloor - v$ has a cycle which is vertex-disjoint with G_1 or G_2. Then $\lfloor G \rfloor$ has two vertex-disjoint cycles, a contradiction. Thus $\lfloor G \rfloor$ is 2-connected. Any two basic edges of $\lfloor G \rfloor$ have a common vertex; otherwise, we can find two vertex-disjoint cycles. This implies that $\lfloor G \rfloor$ has at most three basic edges. Then $\lfloor G \rfloor$ has exactly three basic edges; otherwise we can decycle it by deleting one vertex. Hence it is an S_3-subdivision. □

To characterize the upper-extremal graphs, we first define a class of special upper-extremal graphs—S_3-trees. A graph is an *S_3-tree of order t* if it has exactly t vertex-disjoint S_3-subdivisions and every edge not on these S_3-subdivisions belongs to no cycle (see Fig. 1 for an example). It is easy to verify that any S_3-tree of order t has exactly t vertex-disjoint cycles, and to decycle an S_3-tree, we have to delete two vertices from each S_3-subdivision. Hence, all S_3-trees are upper-extremal. We will show that there is no other upper-extremal outerplanar graph.

For $X, Y \subseteq V(G)$, an *X, Y-path* is a path having one endpoint in X, the other one in Y, and no other vertex in $X \cup Y$, and a $\{v\}, Y$-path is simply written as a v, Y-path. Then,

Lemma 2 *An outerplanar graph G comprised of a connected S_3-tree H of order t and two internally disjoint $v, V(H)$-paths has $t + 1$ vertex-disjoint cycles for $v \notin V(H)$.*

Proof Suppose that $v_1, v_2 \in V(H)$ are the endpoints of these two $v, V(H)$-paths. Let C be the cycle comprised of these two $v, V(H)$-paths and the v_1, v_2-path in H such that C is the boundary of some face of G. Then the intersection (vertex and edge) of C and any S_3-subdivision S in H is either an edge on the boundary of the outer face of S or a vertex of S; otherwise, there would be a subdivision of $K_{2,3}$ or K_4, a contradiction. Hence, we can easily find a cycle in every S_3-subdivision that is vertex-disjoint with C. □

Theorem 3 *An outerplanar graph G is upper-extremal if and only if G is an S_3-tree.*

Proof It suffices to consider the necessity. We prove it by induction on $c(G)$. The statement is clearly true for G if $c(G) = 0$. Let G be an upper-extremal graph. Then we can find a maximal induced path P with some endpoints u and v such that uv

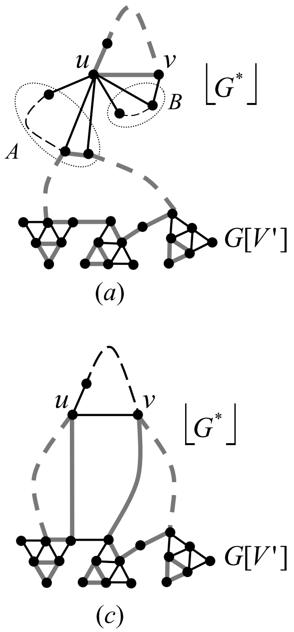


Fig. 2 Gray edges form some vertex-disjoint cycles

is an edge of G ($u \neq v$ since G is upper-extremal). Then $G \setminus \{u, v\}$ must be upper-extremal and $c(G \setminus \{u, v\}) \leq c(G) - 1$. Thus we can assume that $G \setminus \{u, v\}$ is an S_3-tree of order t. Then $c(G) \geq t + 1$. Since $\tau(G) \leq 2t + 2$ and G is upper-extremal, $c(G) = t + 1$ and thus $\tau(G) = 2t + 2$.

Define $G^* := \lfloor G \setminus \{x : x \text{ is on some cycle of } G \setminus \{u, v\}\ \}\rfloor$. Then $c(G^*) = 1$. If $\tau(G^*) = 2$, then by Lemma 1 G^* is an S_3-tree of order one. This implies that G contains $t + 1$ vertex-disjoint S_3-subdivisions. By Lemma 2, there exists at most one path between any two S_3-subdivisions and thus G is an S_3-tree. Now, we consider w.l.o.g. that $G^* - u$ is acyclic. Let $V^* := V(G^*)$. Then G is a graph comprised of G^*, $\lfloor G \setminus V^* \rfloor$, and some internally disjoint V^*, $V(\lfloor G \setminus V^* \rfloor)$-paths. Notice that there is at most one w, V^*-path if $w \in V(\lfloor G \setminus V^* \rfloor)$ is not on any S_3-subdivision. We classify the vertices in $V^* \setminus V(P)$ into two disjoint sets A and B where A is the union of the vertex sets of components of $G^* - u$ except the one containing v. Let V' be the vertex set of a component of $\lfloor G \setminus V^* \rfloor$. Then each component of $G[A]$ has at most one path to V' and there is at most one B, V'-path; otherwise, by Lemma 2 $c(G) \geq t + 2$ (see Fig. 2(a)), a contradiction. We consider the following cases.

Case 1: *G^* has a cycle containing u but not v*. Then there is at most one v, V'-path; otherwise, $c(G) \geq t + 2$. Then the remaining case we have to deal with is that there is exactly one B, V'-path and one u, V'-path. Let x, y be the endpoints of these two paths in V'. Then at least one of x and y is on an S_3-subdivision in $G[V']$ and thus we can decycle G by deleting u and a minimum decycling set of $G \setminus \{u, v\}$ including it, contradicting the fact that $\tau(G) = 2t + 2$.

Case 2: *Every cycle of* G^* *contains both* u *and* v. Then $G^* - v$ is also acyclic. Suppose that $V_u \subseteq V'$ is the set of vertices as the endpoints of some u,V'-paths and $V_v \subseteq V'$ is the set of vertices as the endpoints of some $B \cup \{v\}$,V'-paths. If $\min(|V_u|, |V_v|) \geq 2$ and $\max(|V_u|, |V_v|) \geq 3$, then by Lemma 2 $c(G) \geq t + 2$ (see Fig. 2(b) for an example), a contradiction. Thus $|V_u| = 2 = |V_v|$ or $|V_u| = 1$ or $|V_v| = 1$. If $|V_u| = 1$ (or $|V_v| = 1$), then G can be decycled by deleting v (or u) and a minimum decycling set of $G \setminus \{u, v\}$, contradicting that $\tau(G) = 2t + 2$. It remains to consider that $|V_u| = 2 = |V_v|$. If $V_u \cap V_v = \emptyset$, then $c(G) \geq t + 2$ (see Fig. 2(c) for an example), a contradiction. Suppose that $V_u \cap V_v = \{w\}$. Then w must be on some S_3-subdivision. Therefore, we can decycle G by deleting u and a minimum decycling set of $G \setminus \{u, v\}$ with w included (see Fig. 2(d) for an example), again a contradiction. □

3 Lower-extremal graphs

To prove that a property is sufficient for a graph being lower-extremal, we will use induction. In order to facilitate the proof of the induction step, we need a hereditary graph property. A graph property is called *monotone* if it is closed under removal of vertices. We provide the following general result that is applicable to all graphs.

Lemma 4 *Suppose that a* 2*-connected graph is lower-extremal provided that it satisfies a monotone property* $\mathcal{P}$. *Then* G *is lower-extremal if* G *satisfies* $\mathcal{P}$.

Proof We prove the statement by induction on $|G|$. The statement is true for graphs with $c(G) = 0$ or $|V(G)| = 1$. For a graph G of connectivity one, let G_1 be a leaf block of G and v be the cut-vertex of G in $V(G_1)$. Let $G_2 = G \setminus V(G_1 - v)$. Then $c(G)$ is either $c(G_1) + c(G_2)$ or $c(G_1) + c(G_2) - 1$, and $\tau(G) \leq \tau(G_1) + \tau(G_2)$. Thus suppose to the contrary that $\tau(G) > c(G)$. Then $c(G) = c(G_1) + c(G_2) - 1$ and $\tau(G) = \tau(G_1) + \tau(G_2)$. The first equality shows that every maximum set of vertex-disjoint cycles of G_i must contain a cycle with v for $i = 1, 2$, and thus $c(G_i - v) < c(G_i)$ for $i = 1, 2$. The second equality shows that v does not belong to any minimum decycling set of G^* where $G^* = G_1$ or G_2 and thus $\tau(G^* - v) = \tau(G^*)$. Thus by the monotonicity of $\mathcal{P}$ and the induction hypothesis, $c(G^* - v) = \tau(G^* - v) = \tau(G^*) = c(G^*)$, a contradiction. □

To introduce a sufficient condition for a graph being lower-extremal, we first classify all edges of an outerplanar graph. For a 2-connected outerplanar graph G, let $E_0(G)$ and $E_1(G)$ be the set of edges on the boundary of $F(G)$ and the set of basic edges of G, respectively. For $k \geq 2$, define $E_k(G)$ to be the set of basic edges of $G \setminus \bigcup_{i=1}^{k-1} E_i(G)$. For an edge $uv \in E_k(G)$, we use $C(uv)$ to denote a cycle generated by uv and a u, v-path on the boundary of $F(G)$ such that the cycle is the boundary of a face of $G \setminus \bigcup_{i=1}^{k-1} E_i(G)$. We also call it a *basic cycle* of the graph $G \setminus \bigcup_{i=1}^{k-1} E_i(G)$ generated from edge uv.

Lemma 5 *If* G *is a* 2*-connected outerplanar graph with no* S_k*-subdivision for all odd number* k, *then* G *is lower-extremal.*

Proof We prove the statement by induction on $|E(G)|$. It is easy to verify that the statement is true for graphs with at most three edges. It suffices to prove that there exists a 2-connected subgraph G' of G that has fewer number of edges and no S_k-subdivision for all odd number k and satisfies $\tau(G) \le \tau(G')$ (then $\tau(G) \le \tau(G') = c(G') \le c(G)$).

The statement is clearly true for G with $|E_2(G)| = 0$. Suppose $|E_2(G)| \ge 1$ (and thus $|E_1(G)| \ge 1$). Take an edge $e = xy \in E_2(G)$ and a basic cycle $C(e)$ of $G \setminus E_1(G)$. Let $E \subseteq E_1(G)$ be the set of edges with both endpoints on $C(e)$. We consider the following cases.

Case 1: *E induces an x, y-path of G, say $xv_1v_2 \cdots v_t y$.* Here, t must be even since G contains an S_{t+2}-subdivision. Let D be a minimum decycling set of $G - e$. If D contains x or y, then D is also a decycling set of G and thus $\tau(G) \le \tau(G - e)$. Suppose $x, y \notin S$. W.l.o.g., we can assume that $D \cap C(e)$ contains only vertices of degree larger than two. Then $|D \cap C(e)| \ge (t + 2)/2$. Let $D' = (D \setminus C(e)) \cup \{x, v_2, v_4, \ldots, v_t\}$. Then D' is a decycling set of G of size at most $\tau(G - e)$. Thus, $\tau(G) \le \tau(G - e)$.

Case 2: *E generates a maximal path that contains none of x and y, say $v_1v_2 \cdots v_t$.* We let G' to denote $G \setminus V(C(e) - x - y)$ if $E = \{v_iv_{i+1} : i = 1, \ldots, t - 1\}$ and $G \setminus \{v_iv_{i+1} : i = 1, \ldots, t - 1\}$ otherwise. Then G' is clearly 2-connected. Thus we have $\tau(G) \le \tau(G') + \lfloor \frac{t}{2} \rfloor = c(G') + \lfloor \frac{t}{2} \rfloor = c(G)$.

Case 3: *E induces at most two components which are paths as $xv_1v_2 \cdots v_t$ and $yu_1u_2 \cdots u_{t'}$.* Suppose t (or t') is odd. Let D be a minimum decycling set of $G - e$. Similar to the argument in Case 1, suppose that $x, y \notin D$. Then $|D \cap \{v_1, v_2, \ldots, v_t\}| \ge (t + 1)/2$ and thus $(D \setminus \{v_1, v_2, \ldots, v_t\}) \cup \{x, v_2, v_4, \ldots, v_{t-1}\}$ is a decycling set of G. Hence $\tau(G) \le \tau(G - e)$. It remains to consider that t and t' are even. Let $G' = G \setminus V(C(e) - x - y)$ and D be a minimum decycling set of G'. Then $D \cup \{v_1, v_3, \ldots, v_{t-1}\} \cup \{u_1, u_2, \ldots, u_{t'-1}\}$ is a decycling set of G of size $\tau(G') + (t + t')/2$. Since $G[V(C(e))]$ has $(t + t')/2$ vertex-disjoint cycles that do not contain x and y, $\tau(G) \le \tau(G') + (t + t')/2 = c(G') + (t + t')/2 \le c(G)$. This concludes the proof. □

The property of being without S_k-subdivision is monotone. Therefore, by Lemma 4 and Lemma 5, we have

Theorem 6 *For an outerplanar graph G, if G has no S_k-subdivision for all odd number k, then G is lower-extremal.*

References

Bar-Yehuda R, Geiger D, Naor J, Roth RM (1998) Approximation algorithms for the feedback vertex set problem with applications to constraint satisfaction and Bayesian inference. SIAM J Comput 27:942–959

Bau S, Beineke LW, Vandell RC (1998) Decycling snakes. Congr Numer 134:79–87

Beineke LW, Vandell RC (1997) Decycling graphs. J Graph Theory 25:59–77

Bodlaender HL (1994) On disjoint cycles. Int J Found Comput Sci 5:59–68

Erdös P, Saks M, Sós VT (1986) Maximum induced trees in graphs. J Comb Theory, Ser B 41:61–79

Fertin G, Godard E, Raspaud A (2002) Minimum feedback vertex set and acyclic coloring. Inf Process Lett 84:131–139

Festa P, Pardalos PM, Resende MGC (2000) Feedback set problems. In: Du D-Z, Pardalos PM (eds) Handbook of combinatorial optimization, supplement A. Kluwer Academic, Dordrecht, pp 209–259

Johnson DS (1974) Approximation algorithms for combinatorial problems. J Comput Syst Sci 9:256–278

Karp RM, Miller RE, Thatcher JW (1975) Reducibility among combinatorial problems. J Symb Log 40:618–619

Kleinberg J, Kumar A (1999) Wavelength conversion in optical networks. In: Proc 10th annual ACM-SIAM symposium on discrete algorithms (SODA 99), pp 566–575

Kloks T, Lee C-M, Liu J (2002) New algorithms for k-face cover, k-feedback vertex set, and k-disjoint cycles on plane and planar graphs. In: Proceedings of the 28th international workshop on graph-theoretic concepts in computer science (WG 2002), vol 2573. Springer, Berlin, pp 282–295

Peleg D (1998) Size bounds for dynamic monopolies. Discrete Appl Math 86:263–273

Peleg D (2002) Local majority voting, small coalitions and controlling monopolies in graphs: a review. Theor Comput Sci 282:231–257

Silberschatz A, Galvin PB, Gagne G (2003) Operating systems concepts, 6th edn. Wiley, New York

Wang C, Lloyd EL, Soffa ML (1985) Feedback vertex sets and cyclically reducible graphs. J ACM 32:296–313

West DB (2001) Introduction to graph theory, 2nd edn. Prentice-Hall, New York