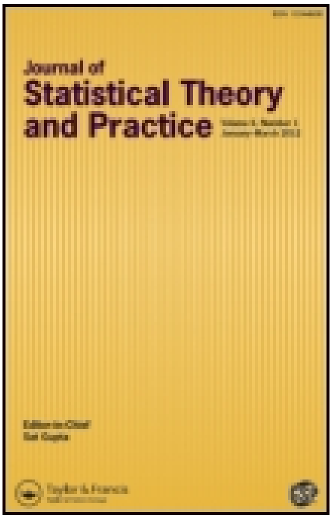

Journal of Statistical Theory and Practice
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/ujsp20

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Published online: 08 Mar 2012.

To cite this article: Hung-Lin Fu & Chang-Chun Lee (2012) Transversals in m × n Arrays, Journal of Statistical Theory and Practice, 6:1, 139-146, DOI: 10.1080/15598608.2012.647554

To link to this article: http://dx.doi.org/10.1080/15598608.2012.647554

Journal of Statistical Theory and Practice, 6:139–146, 2012

ISSN: 1559-8608 print / 1559-8616 online
DOI: 10.1080/15598608.2012.647554

Transversals in $m \times n$ Arrays

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An m by n array consists of mn cells in m rows and n columns, where $2 \leq m \leq n$. A partial transversal in an m by n array is a set of m cells, one from each row and no two from the same column. A transversal in an m by n array is a partial transversal in which m symbols are distinct. Define $L(m, n)$ as the largest integer such that if each symbol in an m by n array appears at most $L(m, n)$ times, then the array must have a transversal. In this article, we first obtain a better lower bound of $L(m, n)$ by using a probabilistic method and then find $L(m, n)$ for certain positive integers m and n.

AMS Subject Classification: 05B15.

Keywords: Transversal; $m \times n$ arrays.

1. Introduction

A *Latin square M* of order n based on an n-set S is an $n \times n$ array such that each symbol of S occurs in each row and each column exactly once. For convenience, we may use $S = \{1, 2, 3, ..., n\}$, and the symbol appearing in the i-th row and j-th column is called the (i, j)-entry of the Latin square, denoted by $M(i, j)$. Then the following figures are examples of a Latin square of order 4 and a Latin square of order 5, respectively.

$$M_1 = \begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{matrix} \qquad M_2 = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{matrix}$$

A *transversal T* of a Latin square is a set of n cells such that no two are in the same row and the same column and the symbols that occur in T are distinct. It is not difficult to see that the squares just given have transversals, respectively: for examples, $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$ and $\{(1, 1), (2, 3), (3, 5), (4, 2), (5, 4)\}$. These two sets are the transversals of M_1 and M_2, respectively. But not every Latin square has a transversal—for example,

Received March 30, 2010; accepted October 21, 2010.
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$$M_3 = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 4 & 2 & 3 & 1 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{matrix}$$

It is easy to check that M_3 has no transversals. Therefore, to determine whether a Latin square has a transversal or not is an interesting problem. More than 250 years ago, Euler conjectured that there do not exist two orthogonal Latin squares of order $4k+2$ for each positive integer k. It is believed that the idea is mainly originated from the fact that there exists a Latin square of order $4k+2$ that does not have a transversal. This is easy to see from M_3.

Now, we know that a pair of orthogonal Latin squares of order $4k+2$, $k \geq 2$, does exist (Stinson 2004). But for a given Latin square, to determine whether a transversal exists is still an open problem. Toward solving this problem, in 1967, Ryser (1967) conjectured that every Latin square of odd order has a transversal, and the number of transversals of a Latin square has the same parity as the order of the square. But Parker pointed out in 1989 that many Latin square of order 7 have an even number of transversals. Balasubramanian (1990) proved that a Latin square of even order has an even number of transversals.

Unfortunately, the results just described do not provide any assistance in determining whether there exists a transversal in a given Latin square or not. An intuitive approach is to find as many distinct elements from distinct rows and columns as possible. A *partial transversal* of a Latin square of order n is a set of n cells from distinct rows and columns. The *size* of a partial transversal is the number of distinct symbols that appears in the partial transversal. For example, $P_1 = \{(1,1),(2,3),(3,2),(4,4)\}$ is a partial transversal of M_1 of size 2. $P_2 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$ is a partial transversal of M_2 of size 1. It is easy to see that we can always find a partial transversal of size at least $n/2$ in a Latin square of order n. But for larger size, it takes a while to get to the best known result today. First, Koksma (1969) showed that the size of a partial transversal in a Latin square is at least $n - (1/3)n$. Later, Drake (1977) showed that the lower bound is $n-(1/4)n$. Then, by using the idea of matchings in the bipartite graph $K_{n,n}$, Woolbright (1978) improved this lower bound to $n-\sqrt{n}$ in 1978. Four years later, Shor (1982) gave a better bound $n-(5.53)(\ln n)^2$. Finally, by using a careful calculation in Shor's technique, Fu et al. (2002) improved this the lower bound to $n-(5.518)(\ln n)^2$. Unfortunately, the result in Shor (1982) is not correct. A revised paper by Hatami and Shor appeared (2008), in which mainly it is proved that the lower bound should be $n-(11.053)(\ln n)^2$.

Recently, the notion of *transversals in a Latin square* has been generalized to that of arrays where we allow common symbols, in both rows and columns. For $2 \leq m \leq n$, an m by n array A consists of mn cells and each cell contains one symbol, and for $1 \leq i \leq m$ and $1 \leq j \leq n$, we use $A(i,j)$ to denote the symbol that appears in the row i and column j. A *partial transversal* in an m by n array is a set of m cells such that no two symbols are in the same row and the same column. A partial transversal of *size* k contains exactly k distinct symbols that appears in the partial transversal. A *transversal* is a partial transversal of size m. Let $L(m,n)$ be the largest integer such that if each symbol in an m by n array appears at most $L(m,n)$ times, then the array must have a transversal—For example,

$$A = \begin{matrix} 1 & 1 & 2 & 3 \\ 4 & 2 & 4 & 1 \\ 2 & 5 & 3 & 2 \end{matrix} \qquad B = \begin{matrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 1 & 1 \end{matrix}$$

Then A and B are 3 by 4 arrays. Each symbol in A appears at most 3 times and each symbol in B appears at most 4 times. $T = \{(1,1),(2,2),(3,3)\}$ is a transversal of A. $P = \{(1,1),(2,2),(3,3)\}$ is a partial transversal of B of size 2. It is easy to check that B has no transversals. It is readily observed from the array B that $L(3,4) < 4$. P. Erdős and J. Spencer (1991) showed that an array of order n (n by n array) in which each symbol appears at most $(n-1)/16$ times has a transversal. This implies $L(n,n) \geq [(n-1)/16]$. Recently, S. Akbari et al. (2006) proved the following theorem:

Theorem 1.1. $L(m,n) = [(mn-1)/(m-1)]$ for $m \geq 2$ *and* $n \geq 2m^3 - 6m^2 + 6m - 1$.

In this article, we study the value $L(m,n)$ for certain pairs of positive integers m and n. First we use a probabilistic method to prove $L(n,n) \geq [(n+4e)/4e]$, which improves a known lower bound mentioned earlier. Then, by a more detailed argument, we determine $L(m,n)$ for certain m and n.

2. The Main Results

Before we use the probabilistic method, we review the idea that is used.

Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. Let $\bar{A}_i$ denote the complement of event A_i. Then the probability of A_1 given A_2 is $Pr(A_1|A_2) = \frac{Pr(A_1 \cap A_2)}{Pr(A_2)}$. If $Pr(A_1|A_2) = Pr(A_1)$, we say that A_1 and A_2 are mutually independent. Let S be a set of events. We say that A_i is mutually independent of S if $Pr(A_i | \bigcap_{A_j \in T} A_j) = Pr(A_i)$ for all $T \subseteq \{A_j | A_j \in S \text{ or } \bar{A}_i \in S\}$.

Definition 2.1. Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. A graph $G = (V, E)$ on the set of vertices $V = \{1, 2, ..., n\}$ is called a lopsidependency graph for the events $A_1, A_2, ..., A_n$ if $Pr(A_i | \bigcap_{j \in S} \bar{A}_j) \leq Pr(A_i)$ for each $i \in V$ and each $S \subseteq \{j |\ j \in V \backslash N_G[i]$ *and* $j \neq i\ \}$ where $N_G[i]$ is a set of vertices adjacent to i.

Theorem 2.2. ***[Lopsided Lovász Local Lemma]*** *Let* $A_1, A_2, ..., A_n$ *be events with lopsidependency graph G and suppose all the events have probability at most p and that each* $i \in G$ *has degree at most d. Assume* $4pd \leq 1$. *Then* $Pr(\bigcap_{i=1}^n \bar{A}_i) > 0$.

In order to prove our main result, we claim the assumption mentioned earlier, "$4pd \leq 1$," can be replaced by $ep(d+1) \leq 1$.

Theorem 2.3. *Let* $A_1, A_2, ..., A_n$ *be events with lopsidependency graph G and suppose all the events have probability at most p and that each* $i \in G$ *has degree at most d. Assume* $ep(d+1) \leq 1$ *where e is the base of the natural logarithm. Then* $Pr(\bigcap_{i=1}^n \bar{A}_i) > 0$.

Proof. First, if $d = 0$, then $Pr(\bigcap_{i=1}^n \bar{A}_i) = Pr(\bar{A}_1) \cdot Pr(\bar{A}_2|\bar{A}_1) \cdot \cdots \cdot Pr(\bar{A}_n | \bigcap_{i=1}^{n-1} \bar{A}_i) = (1 - Pr(A_1)) \cdot (1 - Pr(A_2|\bar{A}_1)) \cdot \cdots \cdot (1 - Pr(A_n | \bigcap_{i=1}^{n-1} \bar{A}_i)) \geq (1-p)^n > 0$. On the other hand, let $d \neq 0$.

Claim: $Pr(A_i | \bigcap_{j \in S} \bar{A}_j) \leq \frac{1}{d+1}$ for any $S \subseteq \{1, 2, ..., n\}$, $|S| = s < n$, and any $i \notin S$.

By induction on s, the assertion is true for $s = 0$ since for each i, $Pr(A_i) \leq p \leq \frac{1}{e \cdot (d+1)} < \frac{1}{d+1}$. Suppose it holds for all $s' < s$. For some fixed i, we put $S_1 = \{j \in S | j \in$

$N_G[i]\}$ and $S_2 = S \backslash S_1$. If $S_1 = \emptyset$, then $Pr(A_i | \bigcap_{j\in S} \bar{A}_j) = Pr(A_i | \bigcap_{j\in S_2} \bar{A}_j) \le Pr(A_i) \le p \le \frac{1}{e\cdot(d+1)} < \frac{1}{d+1}$. Hence, we have the claim. Now, consider $S_1 = \{j_1, j_2, ..., j_r\}$ where $1 \le r \le d$. Then

$$Pr(A_i | \bigcap_{j\in S} \bar{A}_j) = \frac{Pr(A_i \cap (\bigcap_{j\in S_1} \bar{A}_j) | \bigcap_{l\in S_2} \bar{A}_l)}{Pr(\bigcap_{j\in S_1} \bar{A}_j | \bigcap_{l\in S_2} \bar{A}_l)}$$

By direct counting, we have the following two inequalities:

1. $Pr(A_i \cap (\bigcap_{j\in S_1} \bar{A}_j) | \bigcap_{l\in S_2} \bar{A}_l) \le Pr(A_i | \bigcap_{l\in S_2} \bar{A}_l) \le Pr(A_i) \le p$.
2. $Pr(\bigcap_{j\in S_1} \bar{A}_j | \bigcap_{l\in S_2} \bar{A}_l) = Pr(\bar{A}_{j_1} | \bigcap_{l\in S_2} \bar{A}_l) \cdot Pr(\bar{A}_{j_2} | \bar{A}_{j_1} \cap (\bigcap_{l\in S_2} \bar{A}_l)) \cdot \cdots \cdot Pr(\bar{A}_{j_r} | \bar{A}_{j_1} \cap \bar{A}_{j_2} \cap \cdots \cap \overline{A_{j_{r-1}}} \cap (\bigcap_{l\in S_2} \bar{A}_l)) = (1 - Pr(A_{j_1} | \bigcap_{l\in S_2} \bar{A}_l)) \cdot (1 - Pr(A_{j_2} | \bar{A}_{j_1} \cap (\bigcap_{l\in S_2} \bar{A}_l))) \cdot \cdots \cdot (1 - Pr(A_{j_r} | \bar{A}_{j_1} \cap \bar{A}_{j_2} \cap \cdots \cap \overline{A_{j_{r-1}}} \cap (\bigcap_{l\in S_2} \bar{A}_l))) \ge (1 - \frac{1}{d+1})^r \ge (1 - \frac{1}{d+1})^d > \frac{1}{e}$.

By (1) and (2), $Pr(A_i | \bigcap_S \bar{A}_j) \le p / \frac{1}{e} = e \cdot p \le \frac{1}{d+1}$. Hence, we have the claim. It follows that $Pr(\bigcap_{i=1}^n \bar{A}_i) = Pr(\bar{A}_1) \cdot Pr(\bar{A}_2 | \bar{A}_1) \cdot \cdots \cdot Pr(\bar{A}_n | \bigcap_{i=1}^{n-1} \bar{A}_i) = (1 - Pr(A_1)) \cdot (1 - Pr(A_2 | \bar{A}_1)) \cdot \cdots \cdot (1 - Pr(A_n | \bigcap_{i=1}^{n-1} \bar{A}_i)) \ge (1 - \frac{1}{d+1})^n > 0$. □

Now we are ready to prove our first main result.

Theorem 2.4. $L(n, n) \ge [(n + 4e)/4e]$.

Proof. Let $k = [(n + 4e)/4e]$.

Consider an n by n array A in which each symbol appears at most k times. Let S_n be the set of permutations on an n-set. Let $V = \{(s, t, u, v) | \ s < u, t \ne v \ and \ A(s, t) = A(u, v)\}$. For each $(s, t, u, v) \in V$, let $A_{stuv} = \{\sigma | \ \sigma \in S_n, \ \sigma(s) = t \ and \ \sigma(u) = v\}$. It is easy to check that $Pr(A_{stuv}) = (n-2)!/n! = 1/n(n-1)$. Then A has a transversal if and only if $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \ne 0$. Hence we will show that $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \ne 0$ in what follows by using Theorem 2.3.

Define a graph G with vertex set V and (s, t, u, v) adjacent to (x, y, z, w) if and only if $\{s, u\} \cap \{x, z\} \ne \emptyset$ or $\{t, v\} \cap \{y, w\} \ne \emptyset$. Note here that we shall delete the loops. The deletion of a loop in a lopsidependency graph results in a lopsidependency graph. Then we can count the maximal degree of G. Given $(s, t, u, v) \in V$, there are at most $4n - 4$ choices of (x, y) with either $x \in \{s, u\}$ or $y \in \{t, v\}$ and at most $k - 1$ choices for (z, w) with $A(x, y) = A(z, w)$. Either (x, y, z, w) or (z, w, x, y) is adjacent to (s, t, u, v) but not both. Since G has no loops, the maximal degree is at most $(4n - 4)(k - 1) - 1$. Therefore, $e \cdot (1/n(n-1)) \cdot ((4n - 4)(k - 1) - 1 + 1) \le e \cdot (1/n(n-1)) \cdot (4(n-1)(n/4e)) = 1$.

To show that G is a lopsidependency graph, by symmetry, it suffices to show $Pr(A_{1122} | \bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}) \le 1/n(n-1)$ where $S \subseteq \{(s, t, u, v) | \ (s, t, u, v) \in V \ and \ s, t, u, v \notin \{1, 2\}\ \}$.

Let $N_{ij} = \{\sigma \,|\, \sigma(1) = i, \sigma(2) = j \ and \ \sigma \in \bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}\}$.

Claim: $|N_{12}| \le |N_{ij}|$ for all $i \ne j$.

First, consider $i, j > 2$. Let $\sigma \in N_{12}$. There exist a, b with $\sigma(a) = i$, $\sigma(b) = j$. Define σ^* by $\sigma^*(1) = i$, $\sigma^*(2) = j$, $\sigma^*(a) = 1$, $\sigma^*(b) = 2$, and $\sigma^*(x) = \sigma(x)$ for all $x \ne 1, 2, a, b$. Since $(1, i)$, $(2, j)$, $(a, 1)$, and $(b, 2)$ cannot be part of any (s, t, u, v) in S, σ^* is in N_{ij}. Then $f : N_{12} \to N_{ij}$ is injective. Thus, $|N_{12}| \le |N_{ij}|$.

Second, consider $i = 1$ and $j > 2$. Let $\sigma \in N_{12}$. There exists a with $\sigma(a) = j$. Define σ^* by $\sigma^*(1) = 1$, $\sigma^*(2) = j$, $\sigma^*(a) = 2$, and $\sigma^*(x) = \sigma(x)$ for all $x \neq 1, 2, a$. Since $(1, 1)$, $(2, j)$, and $(a, 2)$ cannot be part of any (s, t, u, v) in S, σ^* is in N_{1j}. Then $f : N_{12} \to N_{1j}$ is injective. Thus, $|N_{12}| \leq |N_{1j}|$. This is similar for $i = 2$ and $j > 2$.

Third, consider $i > 2$ and $j = 1$. Let $\sigma \in N_{12}$. There exists a with $\sigma(a) = i$. Define σ^* by $\sigma^*(1) = i$, $\sigma^*(2) = 1$, $\sigma^*(a) = 2$, and $\sigma^*(x) = \sigma(x)$ for all $x \neq 1, 2, a$. Since $(1, i)$, $(2, 1)$, and $(a, 2)$ cannot be part of any (s, t, u, v) in S, σ^* is in N_{i1}. Then $f : N_{12} \to N_{i1}$ is injective. Thus, $|N_{12}| \leq |N_{i1}|$. This is similar for $i > 2$ and $j = 2$.

Final, consider $i = 2$ and $j = 1$. Let $\sigma \in N_{12}$. Define σ^* by $\sigma^*(1) = 2$, $\sigma^*(2) = 1$, and $\sigma^*(x) = \sigma(x)$ for all $x \neq 1, 2$. Since $(1, 2)$ and $(2, 1)$ cannot be part of any (s, t, u, v) in S, σ^* is in N_{21}. Then $f : N_{12} \to N_{21}$ is injective. Thus $|N_{12}| \leq |N_{21}|$. Hence, $|N_{12}| \leq |N_{ij}|$ for all $i \neq j$.

Now, by using the claim,

$$Pr(A_{1122} | \bigcap_{(s,t,u,v)\in S} \bar{A}_{stuv}) = |N_{12}| / \sum_{i\neq j} |N_{ij}| \leq |N_{12}| / \sum_{i\neq j} |N_{12}| = 1/n(n-1)$$

Hence, by Theorem 2.3, $Pr(\bigcap_{(s,t,u,v)\in V} \bar{A}_{stuv}) \neq 0$ holds and thus A has a transversal. This concludes the proof. □

The following results are obtained by direct argument. First, we need several lemmas. Since they are either known or easy to see, we omit their proofs.

Lemma 2.5. *Let A be an m by n array such that A has a transversal. Then the new array A' obtained by the following three operations also has a transversal: (1) a permutation of rows; (2) a permutation of columns; and (3) a permutation of symbols.*

Lemma 2.6. *(Stein and Szabó 2006) (1) $L(m+1, n) \leq L(m, n)$ and (2) $L(m, n) \leq L(m, n + 1)$.*

Theorem 2.7. *(Stein and Szabó 2006) If $n \leq 2m - 2$, then $L(m, n) \leq n - 1$.*

Lemma 2.8. $L(m, n) \leq [(mn - 1)/(m - 1)]$.

Proof. Suppose $mn = k(m - 1) + r$ where $k, r \in \mathbb{Z}$, $0 \leq r < m - 1$. First, if $r = 0$, $L(m, n) < mn/(m - 1) = k$ and thus $L(m, n) \leq k - 1 = [(mn - 1)/(m - 1)]$. On the other hand, if $1 \leq r < m - 1$, then $L(m, n) < mn/(m - 1) = k + r/(m - 1) < k + 1$. Therefore $L(m, n) \leq k = [(mn - 1)/(m - 1)]$. □

By the preceding lemma, if we can show that $L(m, n) \geq [(mn - 1)/(m - 1)]$, then $L(m, n) = [(mn - 1)/(m - 1)]$.

Theorem 2.9. *For $n \geq 43$, $L(4, n) = [(4n - 1)/3]$.*

Proof. Consider a 4 by n array A in which each symbol appears at most $[(4n - 1)/3]$ times. Since $L(3, n) = [(3n - 1)/2] \geq [(4n - 1)/3]$ follows from Theorem 1.1, the 3 by n array consisting of the first three rows of A has a transversal. Suppose that A has no transversal. Then A is equivalent to the following array:

1				x_1	x_2	x_3	x_4	x_5	......	x_{n-4}
	2								......	
		3							......	
			1	x_{n-3}	x_{n-2}	x_{n-1}	x_n	x_{n+1}	......	x_{2n-8}

where $x_i \in \{1, 2, 3\}$, for all $1 \leq i \leq 2n - 8$.

Hence, there are at least $2(n - 4) + 2$ cells containing x_i or 1. Since 1 appears at most $[(4n - 1)/3]$ times in A and $2(n - 4) + 2 > [(4n - 1)/3]$, there must be a 2 or 3 in some cells marked x_i. Without loss of generality, we take x_1 to be 2. Then we have the following array:

1				2	x_2	x_3	x_4	x_5	......	x_{n-4}
y_1	2				y_2	y_3	y_4	y_5	......	y_{n-4}
		3							......	
			1	x_{n-3}	x_{n-2}	x_{n-1}	x_n	x_{n+1}	......	x_{2n-8}

where $x_i, y_j \in \{1, 2, 3\}$, for all $2 \leq i \leq 2n - 8$ and $1 \leq j \leq n - 4$.

This implies that there are at least $3(n - 4) + 3$ cells containing x_i, y_j, 1, or 2. Since 1 and 2 appear at most $2[(4n - 1)/3]$ times in A and $3(n - 4) + 3 > 2[(4n - 1)/3]$, there must be a 3 in some cell marked x_i or y_j. Now, we have five inequivalent cases to consider: $x_2 = 3$, $x_{n-3} = 3$, $x_{n-2} = 3$, $y_1 = 3$, and $y_2 = 3$.

If $x_2 = 3$, then we have the following array:

1				2	3	x_3	x_4	x_5	......	x_{n-4}
y_1	2				y_2	y_3	y_4	y_5	......	y_{n-4}
z_1		3		z_2		z_3	z_4	z_5	......	z_{n-4}
			1	x_{n-3}	x_{n-2}	x_{n-1}	x_n	x_{n+1}	......	x_{2n-8}

where $x_i, y_j, z_k \in \{1, 2, 3\}$, for all $3 \leq i \leq 2n - 8$ and $1 \leq j \leq n - 4$ and $1 \leq k \leq n - 4$. Deleting the first six columns and deleting the last row we get a 3 by $n - 6$ array B in which each symbol appears at most $[(4n - 1)/3] - 2$ times. Since $[(3(n - 6) - 1)/2] \geq [(4n - 1)/3] - 2$, B has a transversal T. Note that the symbols that occur in T are 1, 2, 3. Hence $A(4, 1), A(4, 2), A(4, 3) \in \{1, 2, 3\}$. Otherwise, A has a transversal. Similarly, it can be shown that all the $4n$ cells of A should be filled with $1, 2, 3$. But $4n > 3[(4n - 1)/3]$, a contradiction. Hence, A has a transversal. Since the argument of the other cases are similar, we omit the details. In fact, for each case, we can get a 4 by $n - 6$ array consisting of the last $n - 6$ columns of A in which all symbols in the array are 1, 2, 3.

Thus, $L(4, n) \geq [(4n - 1)/3]$. By Lemma 2.10, $L(4, n) \leq [(4n - 1)/3]$. So, $L(4, n) = [(4n - 1)/3]$. □

We remark here that for $n \geq 3$, $L(2, n) = 2n - 1$ and for $n \geq 5$, $L(3, n) = [(3n - 1)/2]$, as proved in stein and Szabó (2006). We can use the same technique for a more general case.

Theorem 2.10. *For $m \geq 2$ and $n \geq 2m^3 - 8m^2 + 12m - 5$, $L(m, n) = [(mn - 1)/(m - 1)]$.*

Proof. We use induction on m to prove the assertion.

If $m = 2$, then $n \geq 3$. It can be found in Stein and Szabó (2006) that $L(2, n) = 2n - 1$ for $n \geq 3$. Assume that this is true for $m - 1$. That is, $L(m - 1, n) = [((m - 1)n - 1)/(m - 2)]$ for $n \geq 2(m - 1)^3 - 8(m - 1)^2 + 12(m - 1) - 5 = 2m^3 - 14m^2 + 34m - 27$.

For $n \geq 2m^3 - 8m^2 + 12m - 5$, consider an m by n array A in which each symbol appears at most $[(mn - 1)/(m - 1)]$ times. Since $2m^3 - 8m^2 + 12m - 5 \geq 2m^3 - 14m^2 + 34m - 27$ and $[((m - 1)n - 1)/(m - 2)] \geq [(mn - 1)/(m - 1)]$, the $m - 1$ by n array consisting of the first $m - 1$ rows of A has a transversal. Suppose that A has no transversal. Then A is equivalent to the following array:

1					a	a	a	a	.....	a
	2									
		$\ddots$								
			$m - 1$							
				1	a	a	a	a	.....	a

where $a \in \{1, 2, ..., m - 1\}$.

There are at least $2(n - m) + 2$ cells containing a or 1. Since 1 appears at most $[(mn - 1)/(m - 1)]$ times and $2(n - m) + 2 > [(mn - 1)/(m - 1)]$, there must be an element in $\{2, 3, ..., m - 1\}$ occuring in some cells marked a. Without loss of generality, we take the symbol to be 2. Then A is equivalent to the following array:

1					2	a	a	a	.....	a
a	2					a	a	a	.....	a
		$\ddots$								
			$m - 1$							
				1	a	a	a	a	.....	a

where $a \in \{1, 2, ..., m - 1\}$.

If $k(n - m) + k > (k - 1)[(mn - 1)/(m - 1)]$ for $2 \leq k \leq m - 1$, then we can continue the argument. It suffices to show that $k(m - 1)(n - m) + k(m - 1) > (k - 1)(mn - 1)$. By direct counting, it is equivalent to show that $k(m - 1)(n - m) + k(m - 1) - (k - 1)(mn - 1) = (m - k)n - km^2 + 2km - 1 > 0$. By the fact $m - k \geq 1$, it suffices to prove that $n > m(m - 2)(m - 1) + 1 \geq km^2 - 2km + 1 = m(m - 2)k + 1$. This follows from $2m^3 - 8m^2 + 12m - 5 > (m - 1)m^2 - 2(m - 1)m + 1$. We can get an m by $n - (2m - 2)$ array B consisting of the last $n - (2m - 2)$ columns of A. The symbols in B are from the set $\{1, 2, ..., m - 1\}$. Furthermore, each symbol in B appears at most $[(mn - 1)/(m - 1)] - 2$ times. Since $[((m - 1)(n - 2m + 2) - 1)/(m - 2)] \geq [(mn - 1)/(m - 1) - 2]$, the array obtained from deleting any row in B has a transversal T. Note that the symbols that occur in T are in $\{1, 2, ..., m - 1\}$. Otherwise, A has a transversal. Therefore, all the mn cells of A

should be filled with $1, 2, ..., m-1$. But $mn > (m-1)[(mn-1)/(m-1)]$, a contradiction. Thus, A has a transversal. Hence $L(m,n) \geq [(mn-1)/(m-1)]$.

By Lemma 2.8, we conclude the proof. □

Acknowledgments

The authors express their gratitude to the referees for their helpful comments in revising this article. Research supported in part by NSC 97-2115-M-009-011-MY3.

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