

# On the Diameter of the Generalized Undirected De Bruijn Graphs

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## Abstract

The generalized de Bruijn digraph denoted by  $G_B(n, m)$  is the digraph  $(V, A)$  where  $V = \{0, 1, \dots, m-1\}$  and  $(i, j) \in A$  if and only if  $j \equiv ni + \alpha \pmod{m}$  for some  $\alpha \in \{0, 1, \dots, n-1\}$ . By replacing each arc of  $G_B(n, m)$  with an undirected edge and eliminating loops and multi-edges, we obtain a generalized undirected de Bruijn graph  $UG_B(n, m)$ . In this paper, we prove that the diameter of  $UG_B(n, m)$  is equal to 3 whenever  $n \geq 2$  and  $n^2 + (\frac{\sqrt{5}+1}{2})n \leq m \leq 2n^2$ .

Keywords: generalized de Bruijn graph, diameter.

## 1 Introduction

Throughout this paper, all graphs we consider are undirected, loopless and without multi-edges. For the terminologies in graph theory, we refer to [10].

For the sake of brevity, we define  $[a, b] = \{a, a+1, \dots, b\}$  for non-negative integers  $a < b$ .

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The well-known de Bruijn network  $B(n, m)$  was first generalized by Imase and Itoh [4] and then by Reddy, Pradhan and Kuhl [9]. The generalized de Bruijn digraph  $G_B(n, m)$  is the directed graph where the vertices are  $0, 1, \dots, m - 1$ , and the directed edges (arcs) are of the form

$$i \rightarrow in + \alpha \pmod{m}, \forall i \in [0, m - 1] \text{ and } \forall \alpha \in [0, n - 1].$$

The generalized undirected de Bruijn graph is the undirected graph, which is derived from the generalized de Bruijn digraph by replacing directed edges with undirected edges and omitting the loops and multi-edges. We denote such a graph by  $UG_B(n, m)$ . The set of neighbors of any vertex  $i$  in  $UG_B(n, m)$  is  $N(i) = R(i) \cup L(i)$ , where  $R(i) = \{in + \alpha \pmod{m} : \alpha \in [0, n - 1]\}$  and  $L(i) = \{j : jn + \beta \equiv i \pmod{m}, \text{ where } \beta \in [0, n - 1] \text{ and } j \in [0, m - 1]\}$ . Therefore, if  $j \in R(i)$ , then  $i \in L(j)$  in  $UG_B(n, m)$ .

Imase, Soneoka and Okada [5] proved that the generalized de Bruijn digraph  $G_B(n, m)$  is  $(n - 1)$ -connected and its diameter is bounded from above by  $\lceil \log_n m \rceil$ . Therefore,  $UG_B(n, m)$  is also  $(n - 1)$ -connected and its diameter is also bounded above by this value.

Since the study of the diameter of an interconnection network is to investigate the fault tolerance and transmission delay, it is interesting to determine the diameter of  $UG_B(n, m)$ . First, Nochefranca and Sy [8] have shown that the diameter of  $UG_B(n, n(n + 1))$  is 3. Later, Escuadro and

Muga [3] showed that  $UG_B(n, n^2)$  is  $2(n-1)$ -regular and has diameter 2, and Nochefranca and Sy [7] showed that the diameter of  $UG_B(n, n(n^2+1))$  is 4 for odd  $n \geq 3$ . Recently, Caro and Zeratsion [2] proved that the diameter of  $UG_B(n, m)$  is 2 for  $m$  in  $[n+1, n^2]$ , and 3 for  $m$  in  $[n^2+1, n^3]$  where  $n$  divides  $m$ . Furthermore, Caro et al. [1] also proved that the diameter of  $UG_B(n, n^2+1)$  is at most 4 for odd  $n \geq 5$ . In [6], the authors proved that the diameter of  $UG_B(n, m)$  is 3 whenever  $2n^2 \leq m \leq n^3$  or  $m = n^2+1$  and is 2 whenever  $m = n^2+2$  and  $n \geq 3$ . But, whenever  $n^2+2 < m < 2n^2$  the diameter of  $UG_B(n, m)$  is left unknown.

In this paper, we mainly prove that for each  $m$ ,  $n^2 + (\frac{\sqrt{5}+1}{2})n \leq m \leq 2n^2$  and  $n \geq 2$  the diameter of  $UG_B(n, m)$  is equal to 3.

## 2 The main results

Let  $d_G(x, y)$  denote the distance between two vertices  $x$  and  $y$  in a graph  $G$ , and  $diam(G)$  denote the diameter of the graph  $G$ . The following result is known.

**Theorem 2.1.** [5] The diameter of  $UG_B(n, m)$  is 2 or 3, for  $n^2 \leq m \leq n^3$ .

By definition, we know that  $N(i) = R(i) \cup L(i)$  for each  $i \in [0, m-1]$ . Let  $N[i] = N(i) \cup \{i\}$ . Thus, for any two distinct vertices  $x$  and  $y$  in  $G = UG_B(n, m)$ ,  $d_G(x, y) \geq 3$  if and only if  $N[x] \cap N[y] = \emptyset$  which is

equivalent to verifying the following six conditions:  $x \notin L(y)$ ,  $x \notin R(y)$ ,  $R(x) \cap L(y) = \emptyset$ ,  $L(x) \cap L(y) = \emptyset$ ,  $L(x) \cap R(y) = \emptyset$  and  $R(x) \cap R(y) = \emptyset$ .

**Proposition 2.2.**  $\text{diam}(UG_B(n, m)) = 3$  for  $n^2 + 2n \leq m \leq 2n^2$  and  $n \geq 7$ .

**Proof.** Let  $[0, m - 1]$  be the vertex set of  $G = UG_B(n, m)$ . By the discussion after Theorem 2.1, it suffices to show that there exists a pair of vertices  $x$  and  $y$  in  $[0, m - 1]$  such that  $d_G(x, y) \geq 3$ . Note that we shall let  $x = 1$  and find an element  $y \in Y = [n^2 - 2n, n^2 - 1]$  to satisfy the above inequality, i.e.,  $d_G(1, y) \geq 3$ . First, we claim the following six statements are true.

(1) For each  $y \in Y$ ,  $1 \notin L(y)$ .

This is a direct consequence of  $R(1) = \{n + \alpha \mid \alpha \in [0, n - 1]\} = [n, 2n - 1]$  and  $n \geq 4$ , since  $Y \cap [n, 2n - 1] = \emptyset$ .

(2) For each  $y \in Y$ ,  $R(1) \cap L(y) = \emptyset$ .

From (1),  $R(1) = [n, 2n - 1]$ . Therefore

$$\bigcup_{i \in R(1)} R(i) = [n^2, 2n^2 - 1] = [n^2, m - 1] \cup [0, 2n^2 - 1 - m]. \quad (2.1)$$

Now the proof follows from the fact that  $2n^2 - 1 - m \leq 2n^2 - 1 - n^2 - 2n < n^2 - 2n$ . Thus,  $Y \cap [n^2, 2n^2 - 1] = \emptyset$ .

(3) For each  $y \in Y$ ,  $L(1) \cap L(y) = \emptyset$ .

Suppose not. Then  $L(1) \cap L(y) \neq \emptyset$ . Therefore there exists a  $k \in [0, m-1]$  such that both 1 and  $y$  are in  $R(k)$ . This implies that there exist  $\alpha$  and  $\beta$ , where  $0 \leq \alpha, \beta \leq n-1$ , satisfying

$$\begin{cases} kn + \alpha \equiv 1 \pmod{m}, \\ kn + \beta \equiv y \pmod{m}. \end{cases} \quad (2.2)$$

This implies that  $\beta - \alpha \equiv y - 1 \pmod{m}$  and  $\beta - \alpha \in [0, n-1] \cup [m - n + 1, m - 1]$ . But  $y - 1 \in [n^2 - 2n - 1, n^2 - 2]$ . Thus system (2.2) has no solutions for  $(\alpha, \beta)$ . Hence, we have (3).

(4) There exists a set  $Y' \subset Y$  with at most four elements, such that for each  $y \in Y \setminus Y'$ ,  $R(1) \cap R(y) = \emptyset$ .

Observe that

$$\bigcup_{y \in Y} R(y) = \bigcup_{y \in Y} \{yn + \alpha \mid \alpha \in [0, n-1]\} = [n^3 - 2n^2, n^3 - 1]$$

and  $R(1) = [n, 2n - 1]$ .

Since  $m \leq |\bigcup_{y \in Y} R(y)| = 2n^2 < 2m$ , there are at most two elements in  $[n^3 - 2n^2, n^3 - 1]$  which are congruent to  $n$  modulo  $m$ . Let the smaller  $y \in Y$  such that  $n \in R(y)$  be  $y_1$  and the larger one such that  $n \in R(y)$  be  $y_2$  (if exists). Note that if  $y_1 n \not\equiv n \pmod{m}$  then  $R(y_1 + 1) \cap R(1) \neq \emptyset$ . On the other hand, if  $y_1 n \equiv n \pmod{m}$  then  $R(y_1) = [n, 2n - 1] = R(1)$  and  $R(y_1 + 1) \cap R(1) = \emptyset$ . The argument for  $y_2$  is similar.

For example, in Figure 1, if  $R(y_1) \cap R(1) \neq \emptyset$ ,  $R(y_1 + 1) \cap R(1) \neq \emptyset$  and  $R(y_2) = R(1)$ , then there are exactly three elements of  $Y$  satisfying  $R(1) \cap R(y) \neq \emptyset$ . If  $R(y_1) \neq R(1)$  and  $R(y_2) \neq R(1)$  (if  $y_2$  exists), then there are at most four elements of  $Y$  satisfying  $R(1) \cap R(y) \neq \emptyset$ . Hence, by letting  $Y'$  be  $\{y_1, y_1 + 1, y_2, y_2 + 1\}$ , we conclude the proof. We remark here that  $|y_2 n - y_1 n - m| < n$ .

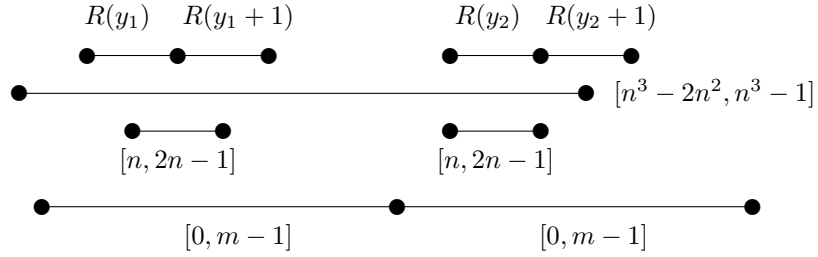


Figure 1:  $R(y_1)$ ,  $R(y_1 + 1)$ ,  $R(y_2)$ ,  $R(y_2 + 1)$ , and  $[n^3 - 2n^2, n^3 - 1]$ .

(5) There exist at least three elements  $y$  in  $Y$  such that  $L(1) \cap R(y) = \emptyset$ .

Observe that  $\bigcup_{y \in Y} R(y) = [n^3 - 2n^2, n^3 - 1]$ . Thus,

$$\bigcup_{i \in [n^3 - 2n^2, n^3 - 1]} R(i) = [n^4 - 2n^3, n^4 - 1]$$

which has  $2n^3$  consecutive positive integers. Therefore, by taking modulo  $m \geq n^2 + 2n$ , we have at most  $2n - 3$  integers which are congruent to 1 modulo  $m$ , since  $2n^3 = (n^2 + 2n)(2n - 4) + 8n$  and  $n \geq 7$ . This implies that there are at least three elements, say  $y_3, y_4, y_5$ , in  $Y$ , such that  $L(1) \cap R(y_i) = \emptyset$ , for  $i = 3, 4, 5$ .

(6) For each  $y \in Y$  satisfying (4) and (5),  $1 \notin R(y)$ .

Suppose not. Then  $1 \in R(y)$ , i.e.,  $yn + \alpha \equiv 1 \pmod{m}$  for some  $0 \leq \alpha \leq n - 1$ . First, if  $\alpha = 0$ , by the fact that  $yn + (n - 1) \equiv n \pmod{m}$ ,  $n \in R(1) \cap R(y)$ , a contradiction. On the other hand, if  $\alpha \neq 0$  then  $yn + (\alpha - 1) \equiv 0 \pmod{m}$  which implies that  $0 \in L(1) \cap R(y)$ , a contradiction. This concludes the proof of (6).

Now, we are ready to find the pair  $(1, y)$  satisfying  $d_G(1, y) \geq 3$ . Clearly, if there exists a  $y \in \{y_3, y_4, y_5\} \setminus \{y_1, y_1+1, y_2, y_2+1\}$  in (4) and (5), then this  $y$  satisfies the conditions from (1) to (6). This implies that  $N[1] \cap N[y] = \emptyset$  and the proof follows. On the other hand, if  $\{y_3, y_4, y_5\} \setminus Y' = \emptyset$ , then  $\{y_3, y_4, y_5\} \subseteq Y'$ . Without loss of generality, let  $y_3 = y_1$  and  $y_4 = y_3 + 1 = y_1 + 1$ . Then by (5),  $L(1) \cap R(y_1) = \emptyset$  and  $L(1) \cap R(y_1 + 1) = \emptyset$ . This implies that  $L(1) \cap [R(y_1) \cup R(y_1 + 1)] = \emptyset$ . But,

$$\bigcup_{i \in R(y_1) \cup R(y_1+1)} R(i) = \bigcup_{i \in [y_1 n^2, y_1 n^2 + 2n^2 - 1]} R(i) = [y_1 n^2, y_1 n^2 + 2n^2 - 1]$$

which is a set of  $2n^2$  consecutive integers and thus  $[y_1 n^2, y_1 n^2 + 2n^2 - 1] \pmod{m} \supseteq [0, m - 1] \supseteq \{1\}$ , a contradiction. So there exists a  $y \in Y$  such that  $d_G(1, y) \geq 3$ . This concludes the proof.  $\blacksquare$

**Proposition 2.3.**  $\text{diam}(UG_B(n, m)) = 3$  for  $n^2 + (\frac{\sqrt{5}+1}{2})n \leq m \leq n^2 + 2n - 1$  and  $n \geq 3$ .

**Proof.** Let  $[0, m-1]$  be the vertex set of  $G = UG_B(n, m)$ . By Theorem 2.1, it suffices to show that there exists a pair of vertices  $x$  and  $y$  in  $[0, m-1]$  such that  $d_G(x, y) \geq 3$ . Here, we let  $x = 0$  and try to find an element  $y \in Y = [n^2, m-n]$  to satisfy the inequality  $d_G(0, y) \geq 3$ . Now, we show that the following six statements are true. The proof uses a similar argument as in the above Proposition.

(1) For each  $y \in Y$ ,  $0 \notin L(y)$ .

This is a direct consequence of  $R(0) = \{0n + \alpha | \alpha \in [0, n-1]\} = [1, n-1]$  and  $n \geq 3$ , since  $Y \cap [1, n-1] = \emptyset$ .

(2) For each  $y \in Y$ ,  $R(0) \cap L(y) = \emptyset$ .

From (1),  $R(0) = [1, n-1]$ , and so we have

$$\bigcup_{i \in R(0)} R(i) = \bigcup_{i \in [1, n-1]} R(i) = [n, n^2 - 1]. \quad (2.3)$$

Therefore,  $Y \cap [n, n^2 - 1] = \emptyset$  for  $n \geq 3$  and (2) is true.

(3) For each  $y \in Y$ ,  $L(0) \cap L(y) = \emptyset$ .

Suppose not. Then  $L(0) \cap L(y) \neq \emptyset$ . Therefore, there exists a  $k \in [0, m-1]$  such that both 0 and  $y$  are in  $R(k)$ . This implies that there exist  $\alpha$  and  $\beta$ , where  $0 \leq \alpha, \beta \leq n-1$ , satisfying

$$\begin{cases} kn + \alpha \equiv 0 \pmod{m}, \\ kn + \beta \equiv y \pmod{m}. \end{cases} \quad (2.4)$$



This implies that  $y \equiv \beta - \alpha \pmod{m}$  and  $\beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$ .

But  $Y \cap ([0, n-1] \cup [m-n+1, m-1]) = \emptyset$ . Thus, the system (2.4) has no solutions for  $(\alpha, \beta)$ . Hence, (3) is true.

(4) There exists an element  $y \in Y$  such that  $L(0) \cap R(y) = \emptyset$ .

It suffices to show that there exists an element  $y \in Y$  such that  $0 \notin \bigcup_{i \in R(y)} R(i)$ . First, let  $t = m - n^2 - n$  and  $A_i = [(i-1)(m-n^2) + 1, i(m-n^2)]$ , where  $i = 1, 2, \dots, t+1$ . Therefore,

$$\left| \bigcup_{i=1}^{t+1} A_i \right| = (t+1)(m-n^2) = (t+1)(n+t).$$

Since  $t \geq (\frac{\sqrt{5}-1}{2})n$ ,

$$(t+1)(n+t) = t^2 + t + nt + n \geq (\frac{\sqrt{5}-1}{2}n)^2 + (\frac{\sqrt{5}-1}{2}n)n^2 + t + n = m.$$

By the fact that  $\{A_i\}_{i=1}^{t+1}$  is a collection of disjoint sets and  $\bigcup_{i=1}^{t+1} A_i = [1, (t+1)(m-n^2)]$ , there exists an  $i_0$  such that  $n^4 \pmod{m} \in A_{i_0}$ .

Now, let  $\mathbf{y} = n^2 + i_0 - 1$ . Then we have

$$\begin{aligned} \mathbf{y}n^2 &\equiv (n^2 + i_0 - 1)n^2 \equiv n^4 + (i_0 - 1)(n^2) \pmod{m} \\ &\in [(i_0 - 1)(m - n^2) + 1 + (i_0 - 1)(n^2), i_0(m - n^2) + (i_0 - 1)(n^2)] \pmod{m} \\ &= [1 + (i_0 - 1)m, m - n^2 + (i_0 - 1)m] \pmod{m} = [1, m - n^2] \pmod{m}. \end{aligned}$$

This implies that

$$\bigcup_{i \in R(\mathbf{y})} R(i) = \{ni + \alpha \mid i \in R(\mathbf{y}) \text{ and } \alpha \in [0, n-1]\}$$

$$\begin{aligned}
&= \{ni + \alpha \mid i \in [n(n^2 + i_0 - 1), n(n^2 + i_0 - 1) + (n - 1)] \text{ and } \alpha \in [0, n - 1]\} \\
&\subseteq [1, m - n^2 + (n^2 - 1)] = [1, m - 1], \text{ then } 0 \notin \bigcup_{i \in R(\mathbf{y})} R(i).
\end{aligned}$$

This concludes the proof of (4).

(5) For  $y \in Y$  satisfying (4),  $R(0) \cap R(y) = \emptyset$ .

Suppose not. Then  $R(0) \cap R(y) \neq \emptyset$  and thus  $yn + \alpha \equiv 0n + \beta \pmod{m}$  has solutions for some  $\alpha$  and  $\beta$  where  $0 \leq \alpha, \beta \leq n - 1$ . Therefore,

$$yn \equiv \beta - \alpha \in [0, n - 1] \cup [m - n + 1, m - 1], \text{ and} \quad (2.5)$$

$$yn^2 \in \{0, n, 2n, \dots, (n - 1)n\} \cup \{2n + t, 3n + t, \dots, n^2 + t\}. \quad (2.6)$$

But, since  $y \in Y$  satisfies (4),  $yn^2 \in [1, m - n^2] = [1, n + t]$ . This implies that

$$yn^2 \in (\{0, n, 2n, \dots, (n - 1)n\} \cup \{2n + t, 3n + t, \dots, n^2 + t\}) \cap [1, n + t] = \{n\}.$$

From equations 2.5 and 2.6,  $yn \equiv 0, 1, \dots, n - 1, m - n + 1, \dots, m - 1 \pmod{m}$  and the corresponding  $yn^2 \equiv 0, n, \dots, n^2 - n, 2n + t, 3n + t, \dots, n^2 + t \pmod{m}$ , we get  $yn \equiv 1 \pmod{m}$ , since  $yn^2 \equiv n \pmod{m}$ . Now, by letting  $y = n^2 + s$ ,  $0 \leq s \leq t$ , we have

$$yn \equiv (n^2 + s)n \equiv (-n - t + s)n \equiv -tn + sn + n + t \equiv 1 \pmod{m}.$$

It follows that  $t - 1 \equiv (t - s - 1)n \pmod{m}$ . Clearly, this equation has no solutions for  $(\frac{\sqrt{5}-1}{2})n \leq t \leq n - 1$ . This concludes the proof of (5).

(6) For  $y \in Y$  satisfying (4) and (5),  $0 \notin R(y)$ .

Suppose not. Then  $yn + \alpha \equiv 0 \pmod{m}$ . If  $\alpha = 0$ , then  $yn^2 \equiv (yn)n \equiv 0 \pmod{m}$ , a contradiction to (4),  $L(0) \cap R(y) = \emptyset$ . If  $\alpha \neq 0$ , then  $yn^2 \equiv -\alpha n \notin [1, m - n^2]$ , a contradiction to (4) again. Hence,  $0 \notin R(y)$ .

Since we can always find a pair of vertices  $x = 0$  and  $y \in [n^2, m - n]$  that satisfy properties (1) to (6),  $N[x] \cap N[y] = \emptyset$ . We have the proof. ■

Then we have the following main theorem.

**Theorem 2.4.**  $diam(UG_B(n, m)) = 3$  for  $n^2 + (\frac{\sqrt{5}+1}{2})n \leq m \leq 2n^2$  and  $n \geq 2$ .

**Proof.** This follows from Propositions 2.2 and 2.3 and the Appendix. ■

### 3 Concluding Remarks

Caro et al. [1] proved the diameter of  $UG_B(n, n^2 + 1)$  is at most 4 for odd  $n \geq 5$  and proved  $diam(UG_B(n, m)) = 3$  when  $n^2 + 1 \leq m \leq n^3$  and  $n|m$  in [2]. Kuo and Fu [6] proved  $diam(UG_B(n, m)) = 3$  when  $2n^2 \leq m \leq n^3$  and  $n \geq 2$ . In this paper, we have proved that for each  $n^2 + (\frac{\sqrt{5}+1}{2})n \leq m \leq 2n^2$  and  $n \geq 2$  the diameter of  $UG_B(n, m)$  is equal to 3. But, for  $n^2 + 2 < m < n^2 + (\frac{\sqrt{5}+1}{2})n$ , the diameters of  $UG_B(n, m)$  are still unknown. We expect that their diameters are 3 from the research we have done so far. Hopefully, this can be verified in the near future.

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## References

- [1] J. D. L. Caro, L. R. Nochefranca and, P. W. Sy, On the diameter of the generalized de Bruijn graphs  $UG_B(n, n^2 + 1)$ , 2000 International Symposium on Parallel Architectures, Algorithms, and Networks (ISPAN '00), 57-61.
- [2] J. D. L. Caro and T. W. Zeratsion, On the diameter of a class of the generalized de Bruijn graphs, 2002 International Symposium on Parallel Architectures, Algorithms, and Networks (ISPAN '02), 197-202.
- [3] H. E. Escudro, and F. P. Muga II, Wide-diameter of generalized undirected de Bruijn graph  $UG_B(n, n^2)$ , 1997 International Symposium on Parallel Architectures, Algorithms, and Networks (ISPAN '97), 417-420.
- [4] M. Imase and M. Itoh, Design to minimize a diameter on building-block network, IEEE Trans. on Computers, C-30 (1981), 439-442.

- [5] M. Imase, T. Soneoka and K. Okada, Connectivity of regular directed graphs with small diameters, *IEEE Trans. on Computers*, C-34, 3 (1985), 267-273.
- [6] J. Kuo and H. L. Fu, On the diameter of the generalized undirected de Bruijn graphs  $UG_B(n, m)$ ,  $n^2 < m \leq n^3$ , preprint.
- [7] L. R. Noche Franca and P. W. Sy, The diameter of the generalized de Bruijn graph  $UG_B(n, n(n^2 + 1))$ , 1997 International Symposium on Parallel Architectures, Algorithms, and Networks (ISPA '97), 421-423.
- [8] L. R. Noche Franca and P. W. Sy, The diameter and hamiltonian cycle of the generalized de Bruijn graphs  $UG_B(n, n(n + 1))$ , *International Journal of Foundations of Computer Science*, Vol. 9, No. 1 (1998) 13-23.
- [9] S. M. Reddy, D. K. Pradhan and J. G. Kuhl, Directed graphs with minimal diameter and maximal connectivity, School of Engineering, Oakland Univ. Tech. Rep. 1980.
- [10] D. B. West, *Introduction to Graph Theory*, Prentice-Hall, Upper Saddle River, NJ 07458, 2001.

## Appendix

The following program generates the results in Table 1 for  $d_G(0, y) \geq 3$ .

For  $n = 2$  to 6

For  $m = n^2 + 2n$  to  $2n^2$

For  $y = n^2 - 2n$  to  $n^2 - 1$

If  $distance_G(x = 0, y) \geq 3$

    print  $m, n, x, y$

Endif

Endfor  $y$

Endfor  $m$

Endfor  $n$

n	m	y
2	8	5
3	15,16,18	9
3	17	10
4	25	16
4	24,27,29,30	17
4	31	18
4	26,28,32	23
5	39,44,47,48,50	25
5	36,38,40,45,46,49	26
5	35,37,42	27
5	41,43	28
6	56,58,61,64,67,68	36
6	49,51,53,57,60,63,66,69,70,72	37
6	52,54,59,62,65,71	38
6	48,50	39
6	55	40

Table 1:  $\text{diam}(UG_B(n, m)) = 3$  for small  $n$  and  $m$