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$(4, 5)$-CYCLE SYSTEMS OF COMPLETE MULTIPARTITE GRAPHS

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Abstract. In 1981, Alspach conjectured that if $3 \leq m_i \leq v$, v is odd and $v(v-1)/2 = m_1+m_2+\cdots+m_t$, then the complete graph K_v can be decomposed into t cycles of lengths $m_1, m_2, \ldots, m_t$ respectively; if v is even, $v(v-2)/2 = m_1 + m_2 + \cdots + m_t$, then the complete graph minus a one-factor $K_v - F$ can be decomposed into t cycles of lengths $m_1, m_2, \ldots, m_t$ respectively. In this paper, we extend the study to the decomposition of the complete equipartite graph $K_{m(n)}$. For $m_i \in \{4, 5\}$, we prove that the trivial necessary conditions are also sufficient.

1. Introduction

An $\mathcal{H}$-decomposition of the graph G is a partition of $E(G)$ such that each element of the partition induces a subgraph isomorphic to a graph in $\mathcal{H}$. If $\mathcal{H}$ just contains a cycle C_k, such a decomposition is referred to as an k-*cycle decomposition* of G. k-cycle decomposition of various graph have been considered by many authors. Necessary and sufficient conditions for a complete graph of odd order, or for a complete graph of even order minus a one-factor, to have decomposition into cycles of some fixed length are now known; see [1,2,4,6,8,9,10,11,13] and references therein. Now, we extend the decomposition of K_n to that of the complete equipartite graph $K_{m(n)}$, with m parts of size n.

The obvious necessary conditions for the existence of a decomposition of the complete equipartite graph $K_{m(n)}$ into cycles $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_t$, of lengths $m_1, m_2, m_3, \ldots, m_t$, whose edges partition the edge set of $K_{m(n)}$ are

- $3 \leq m_i \leq mn$, for $i = 1, 2, \ldots, t$;
- the degree of every vertex in $K_{m(n)}$ is even;
- $m_1 + m_2 + \cdots + m_t = \frac{m(m-1)n^2}{2}$.

Here we prove that the above necessary conditions are sufficient when $m_i \in \{4, 5\}$, for $i = 1, 2, \ldots, t$.

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We start with some notations which will be used in what follows. A subgraph of graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; an induced subgraph H of G is a subgraph of G such $E(H)$ consists of all edges of G whose end points belong to $V(G)$. If S is a nonempty set of vertices of G, then the subgraph of G induced by S is the induced subgraph of G with vertex set S. This induced subgraph of G is denoted by $G[S]$. Similarly, if S_i, S_j, S_k are three disjoint subsets of $V(G)$, then the subgraph of G with vertex sets $S_i \cup S_j \cup S_k$ and the edge set contains all edges which are among the vertices in S_i, S_j and S_k, respectively is denoted by $G[S_i, S_j, S_k]$. An (m^r, n^s)-cycle system of a graph G is a set consisting of r m-cycles and s n-cycles whose edges partition $E(G)$. For any non-negative integer v, define $S_{m,n}(v) = \{(s, r)|ms+nr = v \text{ and } r, s \geq 0\}$ and for a given graph G, define $T_{m,n}(G)$ $= \{(r, s)|$ there exists an (m^r, n^s)-cycle system of $G\}$.

Let S be an n-element set. A *latin square* of order n based on S is an $n \times n$ array in which each cell contains a single element from S, such that each element occurs exactly once in each row and each column.

Before we consider $(4^r, 5^s)$-cycle system of $K_{m(n)}$, we need some 5-cycle packings of complete graphs and complete multipartite graphs.

Theorem 1.1. ([12]). *The minimum leaves of the maximum packings of K_v with 5-cycles are as follows in Table 1. Here, F is a 1-factor, C_i is a cycle of length i, $2C_3$ is a bowtie, F_i is a graph with $v/2 + i$ edges and each vertex has odd degree.*

Table 1. The minimum leaves of the maximum packings of K_v with 5-cycles

v (mod 10)	0	1	2	3	4	5	6	7	8	9
L (leave)	F	$\emptyset$	F	C_3	F_4	$\emptyset$	F_2	$2C_3$	F_4	$2C_3$

Theorem 1.2. ([5]). *If v is odd then $T_{m,n}(K_v) = S_{m,n}(|E(K_v)|)$, and if v is even then $T_{m,n}(K_v - F) = S_{m,n}(|E(K_v - F)|)$, where F is a 1-factor of K_v.*

Theorem 1.3. ([7]). *Let m be an odd integer. Then the minimum leaves of the maximum packings of $K_{m(n)}$ with 5-cycles are as follows: m is taken to be the number modulo 10, n is considered to be modulo 5.*

Table 2. The minimum leaves of the maximum packings of $K_{m(n)}$ with 5-cycles

m / n	0	1	2	3	4
1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
3	$\emptyset$	C_3	$C_3 \cup C_4$	$C_3 \cup C_4$	C_3
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	$\emptyset$	$2C_3$	C_4	C_4	$2C_3$
9	$\emptyset$	$2C_3$	C_4	C_4	$2C_3$

Lemma 1.4. ([7]). *Let $n \geq 2$, and $C_{5(n)}$ denote the graph with vertex set $Z_n \times Z_5$ and edge set $E(C_{5(n)})$, where $\{(i_1, j_1), (i_2, j_2)\} \in E(C_{5(n)})$ if and only if $j_2 \equiv j_1 + 1$ (mod 5). Then $C_{5(n)}$ can be decomposed into 5-cycles.*

It is easy to see that $C_{5(2)}$ can be decomposed into $5C_4$ or $4C_5$, and $C_{5(3)}$ can be decomposed into $9C_5$ or $5C_4 \cup 5C_5$ or $10C_4 \cup C_5$, i.e. $T_{4,5}(C_{5(n)}) = S_{4,5}(|E(C_{5(n)}|)$, when $n = 2, 3$.

Lemma 1.5. ([7]). *There is a 5-cycle packing of $K_{n,n,n}$ with leave (i) $\emptyset$ when $n \equiv 0$ (mod 5) (ii) C_3 when $n \equiv 1$ or 4 (mod 5) and (iii) $C_3 \cup C_4$ when $n \equiv 2$ or 3 (mod 5).*

By the same technique, we have

Lemma 1.6. *There is a 5-cycle packing of $K_{n,n,n}$ with leave (i) C_3 when $n \equiv 1$ or 4 (mod 5) and (ii) $4C_3$ when $n \equiv 2$ or 3 (mod 5).*

Theorem 1.7. ([3]).
Let H_1, H_2 and H_3 be the graphs of , ,  respectively. Then (1) $H_1|K_m$ if and only if $n \equiv 0$ or 1 (mod 5), (2) $H_2|K_m$ if and only if $n \equiv 0$ or 1 (mod 5), $n > 6$, and (3) $H_3|K_m$ if and only if $n \equiv 0$ or 1 (mod 5), $n \neq 5$.

For convenience, let $(v_0; v_1, v_3; v_2, v_4)$ denote the graph H_1, where $\{v_i | i \in Z_5\}$ is the vertex set of H_1 and v_0, v_1, v_2 adjacent to each other, v_3, v_4 adjacent to v_1, v_2, respectively; let $(v_0, v_1, v_2; v_3, v_4)$ denote the graph H_2, where $\{v_i | i \in Z_5\}$ is the vertex set of H_2 and v_0, v_1, v_2 adjacent to each other, v_3, v_4 adjacent to v_2, together; finial, let $(v_0; v_1, v_3; v_2, v_3)$ denote the graph H_3, where $\{v_i | i \in Z_4\}$ is the vertex set of H_3 and v_0, v_1, v_2 adjacent to each other, v_3 adjacent to v_1, v_2. Let $\mathcal{H} = \{H_1, H_2, H_3, H_4 (= C_5)\}$. Before we consider the 5-cycle packing of complete equipartite graph $K_{m(n)}$, we first study an $\mathcal{H}$-packing of complete graph K_m.

2. $\mathcal{H}$-Packing of Complete Graph K_m

Let $H_{1(n)}$ and $H_{2(n)}$ be the 5-partite graphs with vertex set $Z_n \times Z_5$ and $\{(i_1, j_1), (i_2, j_2)\} \in E(H_{i(n)})$ if and only if $\{j_1, j_2\} \in E(H_i)$, $i = 1, 2$. Similarly, let $H_{3(n)}$ be the 4-partite graph with vertex set $Z_n \times Z_4$ and $(i_1, j_1), (i_2, j_2)$ are adjoined if and only if j_1, j_2 are adjoined in H_3. By the following lemmas, $H_{i(2n)}$ can be decomposed into a combination of 5-cycles and 4-cycles, for $i = 1, 2, 3$.

Lemma 2.1. *$H_{1(t)}$, $H_{2(t)}$, and $H_{3(t)}$ can be decomposed into $t^2 H_1$, $t^2 H_2$, $t^2 H_3$ respectively.*

Proof. Let $Z_t \times Z_5$ be the vertex set of $H_{1(t)}$ and $H_{2(t)}$, and $Z_t \times Z_4$ be the vertex set of $H_{3(t)}$. Let M be a latin square of order t base on Z_t. For $(i, j, M(i, j)), 0 \leq i, j \leq t-1$,

$H_{1(t)}$ can be decomposed into t^2H_1 as $((i,0);(j,1),(M(i,j),3);\,(M(i,j),2),(j,4))$, $H_{2(t)}$ can be decomposed into t^2H_2 as $((i,0),(j,1),(M(i,j),2);\,(j,3),(j,4))$, and $H_{3(t)}$ can be decomposed into t^2H_3 as $((i,0);(j,1),(i,3);(M(i,j),2),\,(i,3))$. ■

Lemma 2.2. *$H_{i(2)}$, $i = 1,2,3$ can be decomposed into $4C_5$'s.*

Proof. $H_{1(2)}$ can be decomposed into four 5-cycles as: $((0,0),(0,1),(0,3),(1,1),(0,2))$, $((0,0),(1,1),(1,3),(0,1),(1,2))$, $((1,0),(0,1),(0,2),(0,4),(1,2))$, $((1,0),\,(1,1),(1,2),\,(1,4),(0,2))$, $H_{2(2)}$ can be decomposed into four 5-cycles as : $((0,0),(0,1),(0,2),(0,3),\,(1,2))$, $((0,0),(1,1),(1,2),(1,3),(0,2))$, $((1,0),(0,1),(1,2),(0,4),(0,2))$, $((1,0),(1,1),\,(0,2),(1,4),(1,2))$, and $H_{3(2)}$ can be decomposed into four 5-cycles: $((0,0),\,(0,1),\,(0,3),\,(1,1),\,(0,2))$, $((0,0),\,(1,1),\,(1,3),\,(0,1),\,(1,2))$, $((1,0),\,(0,1),\,(0,2),\,(0,3),\,(1,2))$, $((1,0),\,(1,1),\,(1,2),\,(1,3),\,(0,2))$. ■

Lemma 2.3. *K_{12}, K_{14} can be packed with graphs in $\mathcal{H}$ which has leave a bowtie.*

Proof. (1) Let Z_{12} be the vertex set of K_{12}. Then K_{12} can be packed with $K_6 \cup 6H_2 \cup 3H_3$ as the following : $K_6 = K_{12}[\{0,1,2,3,4,5\}]$, $6H_2$: $(7,11,2;6,9)$, $(3,7,8;2,11)$, $(6,11,3;9,10)$, $(7,9,4;6,10)$, $(4,8,10;2,9)$, $(5,8,6;9,10)$, $3H_3$: $(1;6,0;7,0)$, $(1;8,\,0;9,0)$, $(1;10,0;11,0)$, which has leave a bowtie : $(5,7,10)$, $(5,9,11)$. By theorem 1.7, K_6 can be decomposed into $3H_2$, and K_{12} can be packed with H_2 and H_3 which has leave a bowtie. (2) Let Z_{14} be the vertex set of K_{14}. Then K_{14} can be packed with $2H_1 \cup 9H_2 \cup 6H_3$ as following: $2H_1$: $(2;6,11;10,9)$, $(3;6,9;1,12)$, $9H_2$: $(1,9,5;8,0)$, $(3,8,2;5,11)$, $(3,7,4;2,5)$, $(7,10,5;6,11)$, $(6,7,12;5,8)$, $(7,11,8;6,9)$, $(11,12,3;9,10)$, $(12,4,9;2,\,11)$, $(4,11,10;1,12)$, $6H_3$: $(5;3,0;13,0)$, $(13;7,0;9,0)$, $(13;1,0;11,0)$, $(13;4,0;6,0)$, $(13;8,0;10,0)$, $(13;2,0;12,0)$ which has leave a bowtie : $(1,2,7)$, $(1,4,8)$. ■

Lemma 2.4. *K_8 can be packing with $\mathcal{H}$ which has leave a 3-cycle.*

Proof. Let Z_8 be the vertex set of K_8. Then K_8 can be decomposed into $5H_1 \cup C_3$ as following: $5H_1$: $(2;3,4;7,5)$, $(2;6,4;1,7)$, $(4;0,3;7,6)$, $(5;3,1;6,0)$, $(5;4,1;2,0)$, and C_3 : $(0,1,5)$. ■

Lemma 2.5. *$K_{5,5,t}$ has an $\mathcal{H}$-decomposition for $t = 2,4$ or 8.*

Proof. (1) Let $(Z_2 \times \{0\}) \cup (Z_5 \times \{1,2\})$ be the vertex set of $K_{2,5,5}$. Then $K_{2,5,5}$ can be decomposed into $4H_1 \cup 5H_2$ as the following :
$4H_1$: $((4,2);(0,1),(1,2);(1,0),(3,2))$, $((0,2);(1,1),(2,2);(1,0),(4,1))$, $((1,0);(2,1),(3,2);(1,2),(4,1))$, $((1,0);(3,1),(4,2);(2,2),(4,1))$, and $5H_2$: $((0,0),(0,2),(0,1);\,(2,2),(3,2))$, $((0,0),(1,2),(1,1);(3,2),(4,2))$, $((0,0),(2,2),(2,1);\,(4,2),(0,2))$, $((0,0),\,(3,2),(3,1);(1,2),(0,2))$, $((0,0),(4,2),(4,1);(3,2),(0,2))$.

(2) Let $(Z_4 \times \{0\}) \cup (Z_5 \times \{1,2\})$ be the vertex set of $K_{4,5,5}$. Then $K_{4,5,5}$ can be decomposed into $6H_1 \cup 7H_2$ as the following : $6H_1$: $((2,2);(2,1),(3,2);(0,0),$

$(0,2))$, $((3,2);(3,1),(4,2);(0,0),(1,2))$, $((3,2);(0,1),(1,2);(2,0),(0,2))$, $((4,2);(1,1),(2,2);\ (2,0),(1,2))$, $((1,2);(4,1),(0,2);(3,0),(0,2))$, $((2,2);(0,1),(0,2);(3,0),(3,1))$, and $7H_2$: $((4,1),(4,2),(0,0);(0,1),(1,1))$, $((0,1),(4,2),(1,0);(0,2),(1,2))$, $((1,0),(2,2),\ (3,1);(1,2),(0,2))$, $((4,1),(3,2),(1,0);(1,1),(2,1))$, $((4,1),(2,2),(2,0);(2,1),(3,1))$, $((3,0),(3,2),(1,1);(1,2),(0,2))$, $((3,0),(4,2),\ (2,1);(0,2),(1,2))$.

(3) Let $(Z_5 \times Z_2) \cup (Z_8 \times \{2\})$ be the vertex set of $K_{5,5,8}$. Then $K_{5,5,8}$ can be decomposed into $9H_1 \cup 6H_2 \cup 6H_3$ as the following : $9H_1$: $((1,2);(1,1),(5,2);(0,0),(6,2))$, $((2,2);(2,1),(5,2);(0,0),(7,2))$, $((4,2);(0,1),(6,2);(1,0),(5,2))$, $((2,2);(3,1),(6,2);\ (1,0),(7,2))$, $((3,2);(4,1),(6,2);(1,0),(0,2))$, $((3,2);(0,1),(7,2);(2,0),(5,2))$, $((2,2);\ (4,1),(7,2);(2,0),(0,2))$, $((1,2);(4,1),(4,0);(3,0),(0,2))$, $((2,2);(2,1),(1,0);(4,0),\ (6,2))$, $6H_2$: $((0,0),(3,2),(3,1);(3,0),(5,2))$, $((0,0),(4,2),(4,1);(0,2),(5,2))$, $((0,1),\ (2,2),(3,0);(5,2),(6,2))$, $((3,0),(3,2),(1,1);(0,2),(7,2))$, $((0,1),(1,2),(4,0);(5,2),\ (7,2))$, $((4,0),(3,2),(2,1);(0,2),(2,0))$, and $6H_3$: $((0,2);(0,1),(5,2);(0,0),(5,2))$, $((1,2);(2,1),(6,2);(1,0),(6,2))$, $((4,2);(1,1),(6,2);(2,0),(6,2))$, $((1,2);(3,1),(7,2);\ (2,0),(7,2))$, $((4,2);(2,1),\ (7,2);(3,0),(7,2))$, $((4,2);(3,1),(0,2);(4,0),(0,2))$. ∎

Now, we have the following theorem.

Theorem 2.6. *The minimum leaves of the maximum packings of* K_v *with* $\mathcal{H}$*-set are as follows:*

Table 3. The minimum leaves of the maximum packings of K_v with $\mathcal{H}$-set

v (mod 10)	0	1	2	3	4	5	6	7	8	9
L (leave)	$\emptyset$	$\emptyset$	e	C_3	e	$\emptyset$	$\emptyset$	e	C_3	e

Proof. (i) If the order $v \equiv 0$, 1, 5, 6 $(mod\ 10)$, by theorem 1.8, K_v can be decomposed into H_1. (ii) If $v \equiv 3$, 7, or 9 $(mod\ 10)$, by theorem 1.2, K_v can be packed with $H_4 (= C_5)$ which has leave C_3, $2C_3$, and $2C_3$, respectively. $2C_3 = H_2 \cup \{e\}$. So we can get the above results. (iii) If $v \equiv 2$, 4, or 8 $(mod\ 10)$, let $G = K_{10s+t}$, $t = 2$, 4, or 8, G can be viewed as a graph which contains $2s$ parts of K_5 and one part of K_t, and every parts join to the other part. Then if $s = 3p$, G can be decomposed into $6pK_5$, $1K_2$, $3pK_{5,5,t}$ and $p(6p-2)K_{5,5,5}$. If $s = 3p+1$, then G can be decomposed into $(6p+2)K_5$, $1K_t$, $(3p+1)K_{5,5,t}$ and $2p(3p+1)K_{5,5,5}$. If $s = 3p+2$(i.e. G contains $6p+4$ parts of K_5 and one part of K_t and every parts join to the other parts), $p \geq 1$, G can be decomposed into $(6p+4)K_5$, $1K_t$, $(3p+2)K_{5,5,t}$, $(6p(p+1)-2)K_{5,5,5}$, and $K_{5,5,5,5,5}$. By the above lemmas, we know that the minimum leaves of the maximum packings of K_{10s+t} with $\mathcal{H}$-set are the same as the minimum leaves of the maximum packings of K_t with $\mathcal{H}$-set. So, there exists an $\mathcal{H}$-packing of K_v which has the leave as the above table. ∎

By the above discussion, we have the following proposition:

Proposition 2.7. *There exists an $\mathcal{H}$-packings of K_v with the following leaves.*

Table 4. The leaves of an $\mathcal{H}$-packing of K_v

v (mod 10)	0	1	2	3	4	5	6	7	8	9
L (leave)	$\emptyset$	$\emptyset$	$2C_3$	C_3	$2C_3$	$\emptyset$	$\emptyset$	$2C_3$	C_3	$2C_3$

Combine proposition 2.7 and Lemma 1.4, we have

Theorem 2.8. *The minimum leaves of the maximum packings of $K_{m(n)}$ with $\mathcal{H}$-set are as follows: m, n are considered to be the number modulo 10, 5 respectively; e is one edge, C_i is a cycle of length i.*

Table 5. The minimum leaves of the maximum packings of $K_{m(n)}$ with $\mathcal{H}$-set

$n \setminus m$	0	1	2	3	4	5	6	7	8	9
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$	e	C_3	e	$\emptyset$	$\emptyset$	e	C_3	e
2	$\emptyset$	$\emptyset$	$4e$	$2e$	$4e$	$\emptyset$	$\emptyset$	$4e$	$2e$	$4e$
3	$\emptyset$	$\emptyset$	$4e$	$2e$	$4e$	$\emptyset$	$\emptyset$	$4e$	$2e$	$4e$
4	$\emptyset$	$\emptyset$	e	C_3	e	$\emptyset$	$\emptyset$	e	C_3	e

Theorem 2.9. $T_{4,5}(K_{m(n)}) = S_{4,5}(|E(K_{m(n)}|)$.

Proof. If a complete equipartite graph $K_{m(n)}$ is (4,5)-sufficient then n is even or m, n are both odd. (i) If n is even, say $n = 2s$. View two vertices in the same partite set of $K_{m(2s)}$ as a point, then $K_{m(2s)}$ can be viewed as a complete multipartite graph $K'_{m(s)}$, and each edge e' in $K'_{m(s)}$ is a C_4 in $K_{m(2s)}$. By the theorem 2.8, $K'_{m(s)}$ can be decomposed into $\beta_1 H'_1$, $\beta_2 H'_2$, $\beta_3 H'_3$, $\beta_4 H'_4$, and a leave L' with $|E(L')| = \alpha < 4$. This implies that $K_{m(2s)}$ can be decomposed into $\beta_1 H_{1(2)}$, $\beta_2 H_{2(2)}$, $\beta_3 H_{3(2)}$, $\beta_4 H_{4(2)}$, and αC_4. Because $H_{i(2)}$, $i = 1, 2, 3, 4$ can be decomposed into $5C_4$'s or $4C_5$'s, discretionarily, in the other word, if the size of a complete equipartite graph $K_{m(2s)}$ is equal to $4r+5s$, then the graph can be decomposed into r 4-cycles and s 5-cycles.

(ii) Let m, n are both odd, say $m = 2s + 1$, $n = 2t + 1$. Let $V(K_{m(n)}) = (\{\infty\} \cup Z_{2t}) \times Z_m$ then $K_{m(n)} - (\{\infty\} \times Z_m)$ is isomorphic to $K_{m(2t)}$. By Theorem 1.1, if $m \equiv 1$ or 5 (mod 10), $K_{m(2t)}$ can be decomposed into $C_{5(2t)}$'s; if $m \equiv 3$ (mod 10), $K_{m(2t)}$ can be packing with $C_{5(2t)}$'s which has leave a $C_{3(2t)}$; if $m \equiv 7$ or 9 (mod 10), $K_{m(2t)}$ can be packing with $C_{5(2t)}$'s which has leave $2C_{3(2t)}$'s. $C_{5(2t)}$ can be decomposed into $t^2 C_{5(2)}$'s. W.L.O.G. assume the five partite sets of $C_{5(2)}$ are $\{j_i | i \in Z_5\}$. Let $\bar{G}$ be the graph with vertex set $V(C_{5(2)}) \cup \{(\infty, j_i) | i \in Z_5\}$ and edge set $E(\bar{G}) = E(C_{5(2)}) \cup \{((l, j_i), (\infty, j_{i+1})) | l = \infty, 0, 1; i \in Z_5\}$.

Then $\bar{G}$ is isomorphic to $C_{5(3)}$. Because $T_{4,5}(C_{5(3)} - C_5) = S_{4,5}(|E(C_{5(3)} - C_5)|)$, where $C_5 = ((\infty, j_0), (\infty, j_1), (\infty, j_2), (\infty, j_3), (\infty, j_4))$. Then $T_{4,5}(K_{m(n)} - K_m) = S_{4,5}(|E(K_{m(n)} - K_m)|)$, where $V(K_m) = \{(\infty, j)|j \in Z_m\}$. By theorem 1.3, $T_{4,5}(K_{m(n)}) = S_{4,5}(|E(K_{m(n)})|)$, when $m \equiv 1$, or 5 (mod 10). Similarly, $T_{4,5}(C_{3(3)} - C_3) = S_{4,5}(|E(C_{3(3)} - C_3)|)$, $T_{4,5}(K_{m(2t+1)} - K_m) = S_{4,5}(|E(K_{m(2t+1)} - K_m)|)$, where $V(K_m) = \{(\infty, j)|j \in Z_m\}$, $m \equiv 3, 7$ or 9 (mod 10). By theorem 1.3, $T_{4,5}(K_{m(n)}) = S_{4,5}(|E(K_{m(n)})|)$, when m, n are odd. ■

Corollary 2.10. *Alspach's conjecture is true if the cycle set just contains only 4-cycle and 5-cycle.*

Proof. Let $n = 1$ and 2, respectively. ■

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