

Sparseness of 4-cycle systems

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Abstract

An avoidance problem of configurations in 4-cycle systems is investigated by generalizing the notion of sparseness, which is originally from Erdős' r -sparse conjecture on Steiner triple systems. A 4-cycle system of order v , $4CS(v)$, is said to be r -sparse if for every integer j satisfying $2 \leq j \leq r$ it contains no configurations consisting of j 4-cycles whose union contains precisely $j+3$ vertices. If an r -sparse $4CS(v)$ is also free from copies of a configuration on two 4-cycles sharing a diagonal, called the double-diamond, we say it is strictly r -sparse. In this paper, we show that for every admissible order v there exists a strictly 4-sparse $4CS(v)$. We also prove that for any positive integer $r \geq 2$ and sufficiently large integer v there exists a constant number c such that there exists a strictly r -sparse 4-cycle packing of order v with $c \cdot v^2$ 4-cycles.

Keywords: 4-Cycle system, Configuration, Avoidance, r -Sparse

1 Introduction

A 4-cycle system of order v , denoted by $4CS(v)$, is an ordered pair (V, \mathcal{C}) , where $V = V(K_v)$, the vertex set of the complete graph K_v , and \mathcal{C} is a collection of edge-disjoint cycles of length four whose edges partition the edge set of the complete graph. It is well-known that a necessary and sufficient condition for the existence of a $4CS(v)$ is that $v \equiv 1 \pmod{8}$ (see, for example, Rodger [14]). Such orders are said to be *admissible*. Following the usual terminology of cycle systems, we call a cycle of length four a 4-cycle.

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A 4-cycle system is a natural generalization of the classical combinatorial design called a Steiner triple system, briefly STS, since an STS is just an edge-disjoint decomposition of a complete graph into triangles. A Steiner triple system of order v exists if and only if $v \equiv 1, 3 \pmod{6}$. In other words, the set of all admissible orders of an STS consists of all the positive integers $v \equiv 1, 3 \pmod{6}$.

As is the case with Steiner triple systems, various properties which may appear in a 4-cycle system have also been studied (see, for example, Mishima and Fu [13] and references therein). Such properties of cycle systems have also been investigated as a special graph design (see, for example, Jimbo and Kuriki [11]). Among many characteristics of STSs, the numbers of occurrences of particular substructures have been of interest to various areas (see Colbourn and Rosa [3]). In the current paper, we consider an extreme case for 4CSs, namely, avoidance of particular configurations. We first recall a long-standing conjecture on STSs posed by Erdős.

A (k, l) -configuration in an STS is a set of l triangles whose union contains precisely k vertices. In 1973, Erdős [4] conjectured that for every integer $r \geq 4$, there exists $v_0(r)$ such that if $v > v_0(r)$ and if v is admissible, then there exists a Steiner triple system of order v with the property that it contains no $(j+2, j)$ -configurations for any j satisfying $2 \leq j \leq r$. Such an STS is said to be r -sparse. Many results on the r -sparse conjecture and related problems have been since developed. In particular, after major progress due to Ling et al. [12] and earlier development found in their references, the simplest case when $r = 4$, as it is sometimes called the anti-Pasch conjecture, was eventually settled in the affirmative by Grannell et al. [10].

Theorem 1.1 (Grannell, Griggs and Whitehead [10]) *There exists a 4-sparse Steiner triple system of order v if and only if $v \equiv 1, 3 \pmod{6}$ and $v \neq 7, 13$.*

As far as the authors are aware, the r -sparse conjecture for $r \geq 5$ is still unsettled. In fact, no 7-sparse STS is realized for $v > 3$. Very recent results on sparseness and related problems are found in a series of papers: Forbes et al. [5], Wolf [15, 16] and the first author [6, 7, 8, 9]. For general background on configurations and sparseness in triple systems, the interested reader is referred to Colbourn and Rosa [3].

With regard to 4-cycle systems, the relating result is due to Bryant et al. [1], who investigated the numbers of occurrences of configurations consisting of two 4-cycles. They presented a formula for the number of occurrences of such configurations and studied avoidance and maximizing problems.

Our primary focus in the current paper is on existence of 4-cycle systems which are “sparse” in the sense that they do not contain configurations that consist of many 4-cycles on a small number of vertices in relative terms. In this sense, for

a given integer $w \leq v$ the “densest” configurations on w vertices in a $4CS(v)$ are ones that contain as many 4-cycles as possible. In terms of combinatorial design theory, such a configuration is said to be a maximum 4-cycle packing of order w . More formally, a *4-cycle packing of order w* is an ordered pair (W, \mathcal{D}) such that $|W| = w$ and \mathcal{D} is a set of 4-cycles sharing no common edges, where vertices of a 4-cycle in \mathcal{D} are elements of W . A 4-cycle packing is said to be *maximum* if no other 4-cycle packing of the same order contains a larger number of 4-cycles. Obviously, if w is admissible, a maximum 4-cycle packing of order w is just a $4CS(w)$.

The term (k, l) -configuration will also be used for substructures in $4CS$ s and is defined as a set of l 4-cycles on precisely k vertices where no pair of distinct 4-cycles share the same edge. We denote the set of vertices in a configuration \mathcal{A} by $V(\mathcal{A})$. Two configurations \mathcal{A} and \mathcal{B} are said to be *isomorphic*, denoted as $\mathcal{A} \cong \mathcal{B}$, if there exists a bijection $\phi : V(\mathcal{A}) \rightarrow V(\mathcal{B})$ such that for each 4-cycle $C \in \mathcal{A}$, the image $\phi(C)$ is a 4-cycle in \mathcal{B} .

In the case of STSs, sparseness is measured by lack of $(j+2, j)$ -configurations; one of reasons may be that they are possibly avoidable and form the essential portions of dense configurations (see Forbes, Grannell and Griggs [5]). Based on the following proposition and subsequent observation on $(j+3, j)$ -configurations, we propose an avoidance problem similar to the r -sparse conjecture on STSs.

Proposition 1 *For any positive integers j and d , any $(j+3, j+d)$ -configuration in a $4CS$ contains a $(j+3, j)$ -configuration as a substructure.*

Proof. If a $(j+3, j+d)$ -configuration contains a 4-cycle, say C , in which each vertex is also contained in another 4-cycle, then by discarding C we obtain a $(j+3, j+d-1)$ -configuration. We prove that for any positive integer d a $(j+3, j+d)$ -configuration contains such a 4-cycle. Suppose to the contrary that each 4-cycle in a given $(j+3, j+d)$ -configuration \mathcal{A} has at least one vertex appearing in no other 4-cycles. If $d \geq 4$, the total number of vertices exceeds $j+3$, a contradiction. Hence, we have $d = 1, 2$ or 3 . However, by counting the total number of vertices, it is easy to see that each case yields a contradiction. \square

Proposition 1 says that any denser configuration on $j+3$ vertices, including a $4CS$ or a maximum packing, contains a $(j+3, j)$ -configuration as its substructure. On the other hand, for $j = 2$ and $1 \leq e \leq 3$ every nontrivial $4CS(v)$ contains $(j+3+e, j)$ -configurations (see Bryant et al. [1]). However, as we will see in the next section, we can construct a $4CS$ containing no $(j+3, j)$ -configurations for any j satisfying $2 \leq j \leq 4$. Therefore, it may be natural to ask the following question similar to Erdős’ conjecture:

Problem 1.2 *Does there exist for every integer $r \geq 3$ a constant number $v_0(r)$ such that if $v > v_0(r)$ and v is admissible, then there exists a $4CS(v)$ containing no $(j+3, j)$ -configurations for any j satisfying $2 \leq j \leq r$?*

Remark. While for any positive integers e and j every nontrivial STS on a sufficiently large number of vertices contains a $(j+2+e, j)$ -configuration, we do not know in general the behavior of $(j+3+e, j)$ -configurations except for $j=2$. We briefly discuss in Section 3 the maximum number of 4-cycles of a 4-cycle packing avoiding $(j+3, j)$ -configurations.

Following the terminology of STSs, we say that a 4CS is r -sparse if it contains no $(j+3, j)$ -configuration for any j satisfying $2 \leq j \leq r$. Every r -sparse 4CS is also $(r-1)$ -sparse for $r \geq 3$. Since no $(5, 2)$ -configuration can appear in a 4CS, every 4CS is 2-sparse. Up to isomorphism, there are two kinds of $(6, 3)$ -configuration described by three 4-cycles (a, b, c, d) , (a, e, c, f) and (b, e, f, d) , and (a, b, c, d) , (a, e, c, f) and (b, e, d, f) respectively. A routine argument proves that any $(7, 4)$ -configuration is isomorphic and can be described by four 4-cycles (a, b, c, d) , (a, e, b, f) , (c, f, d, g) and (a, c, e, g) . Hence, a 4CS is 3-sparse if it lacks the two types of $(6, 3)$ -configuration, and it is 4-sparse if it also avoids the unique type of $(7, 4)$ -configuration simultaneously.

Our results presented in the next section give resolution for the existence problem of a 4-sparse $4CS(v)$.

Theorem 1.3 *There exists a 4-sparse $4CS(v)$ if and only if $v \equiv 1 \pmod{8}$.*

Up to isomorphism, there are four possible configurations formed by two 4-cycles in a 4CS, the numbers of vertices ranging from six to eight. While there are two kinds of $(6, 2)$ -configuration, both $(7, 2)$ - and $(8, 2)$ -configurations are unique. A $(6, 2)$ -configuration sharing a common diagonal, described by two 4-cycles (a, b, c, d) and (a, e, c, f) , is called the *double-diamond* configuration. A 4-cycle system is said to be *D-avoiding* if it contains no double-diamond configurations.

Bryant et al. [1] showed that for every admissible order v there exists a D -avoiding $4CS(v)$.

Theorem 1.4 (Bryant et al.) [1] *There exists a D -avoiding $4CS(v)$ for all $v \equiv 1 \pmod{8}$.*

Since a double-diamond configuration appears in both types of $(6, 3)$ -configuration, every D -avoiding 4CS is 3-sparse but the converse does not hold. In fact, for every small admissible order v one can easily find a 3-sparse $4CS(v)$ which is not D -avoiding. On the other hand, Bryant et al. [1] showed that the other type of

(6, 2)-configuration appears constantly depending only on the order v , that is, the number of occurrences is unique between 4CSs of the same order. Considering these facts, we say that a 4CS is *strictly r -sparse* if it is both r -sparse and D -avoiding.

In Section 2, we give a proof of existence of a strictly 4-sparse 4CS(v) for every admissible order v .

Theorem 1.5 *There exists a strictly 4-sparse 4CS(v) if and only if $v \equiv 1 \pmod{8}$.*

We also study in Section 3 the maximum number of 4-cycles of a 4-cycle packing avoiding $(j+3, j)$ -configurations.

Let $ex(v, r)$ be the maximum number of 4-cycles of a 4-cycle packing of order v containing neither double-diamond configurations nor $(j+3, j)$ -configurations for every $2 \leq j \leq r$. By probabilistic methods, we prove that for any positive integer $r \geq 2$ the maximum number $ex(v, r) = O(v^2)$.

2 Strictly 4-sparse 4-cycle systems

In this section, we present a proof of Theorem 1.5. Obviously, the proof also verifies Theorem 1.3. To show Theorem 1.5, we first prove two lemmas.

A *jointed-diamond* configuration in a 4CS is a $(7, 3)$ -configuration described by three 4-cycles (a, b, c, d) , (a, e, b, g) and (c, f, d, g) ; the 4-cycle (a, b, c, d) is referred to as a *joint 4-cycle*. Every $(7, 4)$ -configuration contains a jointed-diamond configuration as its substructure.

Lemma 2.1 *Let q be a prime power satisfying $q \equiv 1 \pmod{8}$ and not a power of three. Then there exists a strictly 4-sparse 4CS(q).*

Proof. Let q be a prime power satisfying $q \equiv 1 \pmod{8}$ and not a power of three. Let χ be a multiplicative character of order four of $\text{GF}(q)$ such that $\chi(x)$ has possible values $1, -1, i, -i$ for $x \neq 0$. Then there exists a 4-cycle $(0, x, x-1, x^2)$, $x \in \text{GF}(q)$, such that $\chi(x^2) = -1$, $\chi((x^2 - x + 1)^2) = -1$, and $\chi(x(x^2 - x + 1)) = 1$ (see Bryant et al. [1]). Considering these conditions, we have either $\chi(x) = i$, $\chi(x^2 - x + 1) = -i$, and $\chi(x(x-1)) = i \cdot \chi(x-1)$, or $\chi(x) = -i$, $\chi(x^2 - x + 1) = i$, and $\chi(x(x-1)) = -i \cdot \chi(x-1)$. Also, since $q \equiv 1 \pmod{8}$, we have $\chi(-1) = 1$. Let α be a primitive element of $\text{GF}(q)$ and V the set of all elements of $\text{GF}(q)$. Define a set \mathcal{C} of 4-cycles as $\{y, x \cdot \alpha^{4n} + y, (x-1) \cdot \alpha^{4n} + y, x^2 \cdot \alpha^{4n} + y : y \in \text{GF}(q), 0 \leq n \leq \frac{q-1}{8} - 1\}$. Then (V, \mathcal{C}) forms a D -avoiding 4CS(q). In fact, \mathcal{C} is developed from the 4-cycle $(0, x, x-1, x^2)$ by the group $G = \{z \mapsto z \cdot \alpha^{4n} +$

$y : y, z \in \text{GF}(q), 0 \leq n \leq \frac{q-1}{8} - 1\}$. To prove that (V, \mathcal{C}) is strictly 4-sparse, it suffices to show that (V, \mathcal{C}) contains no jointed-diamond configurations. Suppose to the contrary that it contains a jointed-diamond configuration J described by three 4-cycles (a, b, c, d) , (a, e, b, g) and (c, f, d, g) . Since every 4-cycle in \mathcal{C} can be obtained from $(0, x, x-1, x^2)$ by the group G , considering the joint 4-cycle (a, b, c, d) , we have $\chi(a-b) = -\chi(c-d)$. However, since the edges $\{a, b\}$ and $\{c, d\}$ lie in diagonals of (a, e, b, g) and (c, f, d, g) respectively, we have $\chi(a-b) = \chi(c-d), i \cdot \chi(c-d)$ or $-i \cdot \chi(c-d)$, a contradiction. The proof is complete. \square

Lemma 2.2 *There exists a strictly 4-sparse 4CS(9).*

Proof. Let $V = \{0, 1, 2, \dots, 8\}$ be the set of elements of the cyclic group \mathbf{Z}_9 . Define a set \mathcal{C} of 4-cycles as $\{(0+a, 1+a, 8+a, 5+a) : a \in \mathbf{Z}_9\}$. The pair (V, \mathcal{C}) forms a 4CS(9) under the transitive action of \mathbf{Z}_9 on the vertex set V . Since \mathcal{C} has only one 4-cycle orbit, (V, \mathcal{C}) is D -avoiding, and hence it is 3-sparse.

Suppose to the contrary that (V, \mathcal{C}) is not 4-sparse and contains a jointed-diamond. Take a representative, say $C = (0, 1, 8, 5)$, of the 4-cycle orbit. The two differences of the vertices in a diagonal of C are ± 1 and ∓ 4 respectively. Hence, the joint 4-cycle in a jointed-diamond lying in \mathcal{C} has the form (a, b, c, d) , where the differences $a-b$ and $c-d$ are each 1, -1 , 4 or -4 . However, considering the four differences of the adjacent vertices in C , this is a contradiction. \square

We now return to the proof of Theorem 1.5. The proof employs a special decomposition of the complete graph into smaller complete graphs.

A *group divisible design* with *index one* is a triple $(V, \mathcal{G}, \mathcal{B})$, where

- (i) V is a finite set of elements called *points*,
- (ii) \mathcal{G} is a family of subsets of V , called *groups*, which partition V ,
- (iii) \mathcal{B} is a collection of subsets of V , called *blocks*, such that every pair of points from distinct groups occurs in exactly one blocks,
- (iv) $|G \cap B| \leq 1$ for all $G \in \mathcal{G}$ and $B \in \mathcal{B}$.

When all blocks are of the same size k and the number of groups of size n_i is t_i , one refers to the design as a k -GDD of type $n_0^{t_0} n_1^{t_1} \dots n_{g-1}^{t_{g-1}}$, where $t_0 + t_1 + \dots + t_{g-1} = |\mathcal{G}|$. We need 4-GDDs and the required types are of 12^t ($t \geq 4$), 4^{3t+1} ($t \geq 1$), 8^{3t+1} ($t \geq 1$), and $2^{3t} 5^1$ ($t \geq 3$). For their existence, we refer the reader to Colbourn and Dinitz [2].

Proof of Theorem 1.5. A strictly 4-sparse $4CS(v)$ is necessarily D -avoiding. We follow a part of the proof of existence of a D -avoiding $4CS(v)$ by Bryant et al. [1] and consider four cases:

Case (1) : $v \equiv 1 \pmod{24}$. Lemma 2.1 gives a strictly 4-sparse $4CS(v)$ for $v \leq 73$ and $v \equiv 1 \pmod{24}$. We consider the case $v > 73$. Take a 4-GDD $(V, \mathcal{B}, \mathcal{G})$ of type 12^t for $t \geq 4$. For each group $G \in \mathcal{G}$, take $(G \times \{0, 1\}) \cup \{\infty\}$ by replacing each point by two new points and adding a new point ∞ . Let \mathcal{H}_G be a copy of the strictly 4-sparse $4CS(25)$ given in Lemma 2.1 on $(G \times \{0, 1\}) \cup \{\infty\}$. For each block $B = \{a, b, c, d\} \in \mathcal{B}$, construct a 4-cycle decomposition \mathcal{C}_B of a copy of $K_{2,2,2,2}$ on $B \times \{0, 1\}$ by developing a 4-cycle $((a, 0), (b, 0), (c, 1), (d, 0))$ under the group $\langle (d)(a b c) \rangle \times \mathbb{Z}_2$. Let $W = (V \times \{0, 1\}) \cup \{\infty\}$ and $\mathcal{D} = (\bigcup_{G \in \mathcal{G}} \mathcal{H}_G) \cup (\bigcup_{B \in \mathcal{B}} \mathcal{C}_B)$. Then (W, \mathcal{D}) forms a $4CS(24t + 1)$. Since no pair of 4-cycles in \mathcal{D} shares a common diagonal, (W, \mathcal{D}) is D -avoiding.

It remains to establish that the $4CS$ contains no $(7, 4)$ -configuration. Suppose to the contrary that (W, \mathcal{D}) contains a $(7, 4)$ -configuration. Then it contains a jointed-diamond configuration J . If the joint 4-cycle in J lies in \mathcal{H}_G , the other two 4-cycles in J are also in \mathcal{H}_G . Since \mathcal{H}_G is a copy of a strictly 4-sparse $4CS(25)$, this is a contradiction. If the joint 4-cycle in J lies in \mathcal{C}_B , again the other two 4-cycles in J are in \mathcal{C}_B . A routine argument proves that \mathcal{C}_B contains no jointed-diamond configuration.

Case (2) : $v \equiv 9 \pmod{24}$. Lemma 2.2 gives a strictly 4-sparse $4CS(9)$. Take a 4-GDD $(V, \mathcal{B}, \mathcal{G})$ of type 4^{3t+1} for $t \geq 1$. As in *Case (1)*, construct a $4CS(24t + 9)$ on $(V \times \{0, 1\}) \cup \{\infty\}$ by placing a copy of the strictly 4-sparse $4CS(9)$ given in Lemma 2.2 and decomposing $K_{2,2,2,2}$ s into 4-cycles. By following the argument in *Case (1)*, the resulting $4CS(24t + 9)$ is strictly 4-sparse.

Case (3) : $v \equiv 17 \pmod{48}$. Employing the strictly 4-sparse $4CS(17)$ constructed in Lemma 2.1 and a 4-GDD of type 8^{3t+1} for $t \geq 1$, we obtain the required strictly 4-sparse $4CS$ s by the same technique as in *Case (1)*.

Case (4) : $v \equiv 41 \pmod{48}$. Lemma 2.1 gives a strictly 4-sparse $4CS(v)$ for $v \leq 137$ and $v \equiv 41 \pmod{48}$. We consider the case $v > 137$. Take a 4-GDD $(V, \mathcal{B}, \mathcal{G})$ of type $2^{3t}5^1$ for $t \geq 3$. For each block $B = \{a, b, c, d\} \in \mathcal{B}$, replace each point in B by four new points and define $A_i = \{i\} \times \{0, 1, 2, 3\}$ for $i \in B$. The points and lines of an affine space over $\text{GF}(2^2)$ of dimension 2 form a 4-GDD of type 4^4 . For each $B \in \mathcal{B}$, place a 4-GDD of type 4^4 on $B \times \{0, 1, 2, 3\}$ such that the set of groups is $\{A_i : i \in B\}$ and let \mathcal{C}_B be the resulting blocks of the 4-GDD on $B \times \{0, 1, 2, 3\}$. For each \mathcal{C}_B , $B \in \mathcal{B}$, construct a 4-cycle decomposition $\mathcal{D}_{\mathcal{C}_B}$ of a copy of $K_{2,2,2,2}$ on $\mathcal{C}_B \times \{0, 1\}$ by developing a 4-cycle $((a, i, 0), (b, j, 0), (c, k, 1), (d, l, 0))$ under the group $\langle ((d, l))((a, i) (b, j) (c, k)) \rangle \times$

Z₂. For each group $G \in \mathcal{G}$, take $(G \times \{0, 1, 2, 3\} \times \{0, 1\}) \cup \{\infty\}$ and let \mathcal{H}_G be a copy of either the strictly 4-sparse 4CS(17) or 4CS(41) given in Lemma 2.1 on $(G \times \{0, 1, \dots, 7\}) \cup \{\infty\}$ according to the group size $|G|$, that is, place a copy of the 4CS(17) if $|G| = 2$, otherwise put a copy of the 4CS(41). Let $W = (V \times \{0, 1, 2, 3\} \times \{0, 1\}) \cup \{\infty\}$ and $\mathcal{E} = (\bigcup_{G \in \mathcal{G}} \mathcal{H}_G) \cup (\bigcup_{B \in \mathcal{B}} \mathcal{D}_{C_B})$. It is straightforward to see that (W, \mathcal{E}) forms a 4CS($48t + 41$). The same argument as in *Case (1)* proves that (W, \mathcal{E}) is strictly 4-sparse. \square

3 r -Sparse 4-cycle packing

In this section, we consider the maximum number of 4-cycles in a 4-cycle packing of order v avoiding $(j+3, j)$ -configurations. As with a 4CS, a 4-cycle packing is said to be r -sparse if it contains no $(j+3, j)$ -configuration for any j satisfying $2 \leq j \leq r$. Also if it is r -sparse and D -avoiding, we say that it is *strictly* r -sparse. We prove that for any positive integer $r \geq 2$ and sufficiently large integer v there exists a constant number c such that there exists a strictly r -sparse 4-cycle packing of order v with $c \cdot v^2$ 4-cycles. It is notable that a resolution for the analogous problem to the r -sparse conjecture on STSs would prove that $c \sim \frac{1}{8}$.

Let \mathcal{F} be a set of configurations of 4-cycles and $ex(v, \mathcal{F})$ the largest positive integer n such that there exists a set \mathcal{C} of n 4-cycles on a finite set V of cardinality v having property that \mathcal{C} contains no configuration which is isomorphic to a member $F \in \mathcal{F}$.

Theorem 3.1 *For any positive integer $r \geq 2$ and sufficiently large integer v there exists a constant number c such that there exists a strictly r -sparse 4-cycle packing of order v with $c \cdot v^2$ 4-cycles.*

Proof. Let V be a finite set of cardinality v . Define \mathcal{F}' as the set of all nonisomorphic $(j+3, j)$ -configurations for $2 \leq j \leq r$ and \mathcal{F}'' as the set of all nonisomorphic $(4, 2)$ - and $(6, 2)$ -configurations. Let $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$. It is easy to see that if $ex(v, \mathcal{F}) \geq c \cdot v^2$ for some constant c , then the assertion of Theorem 3.1 follows.

Pick uniformly at random 4-cycles from V with probability $p = \frac{c'}{v^2}$, independently of the others, where c' satisfies $0 < c' < \frac{1}{44}$. Let b_C be a random variable counting the number of configurations isomorphic to C in the resulting set of 4-cycles and $E(b_C)$ its expected value. Then

$$\begin{aligned}
E\left(\sum_{C \simeq F \in \mathcal{F}} b_C\right) &\leq \binom{v}{4} \cdot \binom{3 \cdot \binom{4}{2}}{2} \cdot p^2 + \binom{v}{6} \cdot \binom{3 \cdot \binom{6}{2}}{2} \cdot p^2 \\
&\quad + \sum_{j=2}^r \binom{v}{j+3} \cdot \binom{3 \cdot \binom{j+3}{4}}{j} \cdot p^j \\
&\leq \left[\binom{v}{4} \cdot \binom{3}{2} + \binom{v}{6} \cdot \binom{45}{2} \right] p^2 \\
&\quad + \sum_{j=2}^r \left(\frac{e \cdot v}{j+3} \right)^{j+3} \cdot \left(\frac{e \cdot (j+3)^3}{8} \right)^j \cdot p^j \\
&= \frac{11 \cdot c'^2}{8} \cdot v^2 + f(v),
\end{aligned}$$

where $f(v) = O(v)$. By Markov's Inequality,

$$P\left(\sum_{C \simeq F \in \mathcal{F}} b_C \geq 2 \cdot E\left(\sum_{C \simeq F \in \mathcal{F}} b_C\right)\right) \leq \frac{1}{2}.$$

Hence,

$$P\left(\sum_{C \simeq F \in \mathcal{F}} b_C \leq \frac{11 \cdot c'^2}{4} \cdot v^2 + 2 \cdot f(v)\right) \geq \frac{1}{2}.$$

Let t be a random variable counting the number of 4-cycles and $E(t)$ its expected value. Then

$$E(t) = p \cdot 3 \cdot \binom{v}{4} = \frac{c'}{8} \cdot v^2 - g(v),$$

where $g(v) = O(v)$. Since t is a binomial random variable, we have for sufficiently large v

$$P\left(t < \frac{E(t)}{2}\right) < e^{-\frac{E(t)}{8}} < \frac{1}{2}.$$

Hence, if v is sufficiently large, then we have, with positive probability, a set \mathcal{S} of 4-cycles with the property that $|\mathcal{S}| \geq \frac{E(t)}{2}$ and the number of configurations in \mathcal{S} which are isomorphic to a member of \mathcal{F} is at most $\frac{11 \cdot c'^2}{4} \cdot v^2 + 2 \cdot f(v)$. Since $f(v), g(v) = O(v)$, by deleting a 4-cycle from each configuration isomorphic to a member of \mathcal{F} , we have

$$ex(v, \mathcal{F}) \geq \frac{c'(1 - 44 \cdot c')}{16} \cdot v^2 - h(v),$$

where $h(v) = O(v)$. The proof is complete. \square

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