

Covering graphs with matchings of fixed size

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ABSTRACT

Let m be a positive integer and let G be a graph. We consider the question: can the edge set $E(G)$ of G be expressed as the union of a set \mathcal{M} of matchings of G each of which has size exactly m ? If this happens, we say that G is $[m]$ -coverable and we call \mathcal{M} an $[m]$ -covering of G . It is interesting to consider *minimum* $[m]$ -coverings, i.e. $[m]$ -coverings containing as few matchings as possible. Such $[m]$ -coverings will be called *excessive* $[m]$ -factorizations. The number of matchings in an excessive $[m]$ -factorization is a graph parameter which will be called the *excessive* $[m]$ -index and denoted by $\chi'_{[m]}$. In this paper we begin the study of this new parameter as well as of a number of other related graph parameters.

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1. Introduction

In this paper *graphs* are understood to be simple, finite and undirected. We use the term *multigraph* when multiple edges (but not loops) are allowed. Thus a graph is a particular type of multigraph. It is convenient, for our purposes, to exclude from our consideration graphs with no edges. Therefore we shall adopt the convention that, throughout this paper, the term “graph” has the meaning “graph with at least one edge”. The set of positive integers will be denoted by \mathbb{Z}^+ . If $\alpha \in \mathbb{Z}^+$, $\beta \in \mathbb{Z}^+ \cup \{\infty\}$, by $[\alpha, \beta]$ we shall denote the set

$$\{x \in \mathbb{Z}^+ : \alpha \leq x < \beta\}.$$

The notations (α, β) , $[\alpha, \beta]$, $(\alpha, \beta]$ are defined analogously.

If G is a graph, we denote by $V(G)$ the vertex set and by $E(G)$ the edge set of G . If $E_1 \subset E(G)$, by $G - E_1$ we denote the graph G with all edges in E_1 deleted. Similarly, if $V_1 \subset V(G)$, by $G - V_1$ we denote the graph G with all the vertices in V_1 (and all their incident edges) deleted. We use the notation K_n to denote the complete graph of order n , the notation $K_{p,q}$ to denote the complete bipartite graph whose partite classes contain p and q vertices, respectively, the notations C_n and P_n to denote, respectively, the cycle and path on n vertices, and the notation $K(n_1, n_2, n_3, \dots, n_t)$ to denote the complete multipartite graph whose partite sets consist, respectively, of n_1, n_2, \dots, n_t vertices, where we assume $n_1 \geq n_2 \geq \dots \geq n_t$ and $t \geq 3$. The symbol P will denote the *Petersen graph*.

A *matching* M in a multigraph G is a set of mutually nonincident edges. If M is a matching of G and $v \in V(G)$, we say that v is *saturated* by M if v is incident to an edge of M , and we say that v is *unsaturated* by M otherwise. For a reference book on *Matching Theory*, we refer the reader to Lovász and Plummer [6]. A *k-edge colouring* of a multigraph G is a map $\varphi : E(G) \rightarrow \mathcal{C}$, where \mathcal{C} is a set of cardinality k and $\varphi(e) \neq \varphi(f)$ for any pair of mutually incident edges e, f of G . If $\alpha \in \mathcal{C}$, the set of edges coloured α , i.e. the set $\varphi^{-1}(\{\alpha\})$, is called a *colour class* of φ . Clearly every colour class of an edge colouring is a matching.

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The least nonnegative integer k for which G admits a k -edge colouring is called the *chromatic index* of G and denoted by $\chi'(G)$. Notice that $\chi'(G)$ may be defined as the minimum number of matchings whose union is $E(G)$. It is sometimes useful to consider edge colourings whose colour classes have “approximately” the same size. More precisely, a k -edge colouring φ is called *equalized* if it has the property that, for every colour class C ,

$$\lfloor |E(G)|/k \rfloor \leq |C| \leq \lceil |E(G)|/k \rceil. \tag{1}$$

We shall often use the following lemma, due to de Werra [8] and (independently) McDiarmid [7].

Lemma 1. *Let G be a multigraph. Then G has an equalized k -edge colouring if and only if $k \geq \chi'(G)$.*

Thus, if G is k -edge colourable, we can always guarantee the existence of a k -edge colouring in which any two colour classes have either the same size or differ in size by at most 1. This may be a desirable situation for many practical purposes, e.g. in scheduling problems. However, it may be the case that the problem at hand demands “absolute uniformity”, i.e. that every matching corresponding to a colour class should have exactly the same size. This is obviously a very stringent restriction. By (1), this may be achieved by an edge colouring only if $|E(G)|$ is a multiple of the number of colour classes, in which cases such a colouring is called a *decomposition* of G into matchings of size m . For example, if $G = K_n$ and $m \mid \frac{n(n-1)}{2}$ then such a decomposition always exists (see [5]). In all other cases, in order to satisfy this requirement, we need to alter the definition of edge colouring, by allowing different colour classes to “overlap”. This motivates the following definition.

Let m be a positive integer. An $[m]$ -covering of a graph G is¹ a set $\mathcal{M} = \{M_1, M_2, \dots, M_k\}$ of matchings of G , where $|M_i| = m$ for each $i = 1, 2, \dots, k$ and $\bigcup_{i=1}^k M_i = E(G)$. The integer k will be called the *order* of the $[m]$ -covering.

Obviously, not all graphs admit an $[m]$ -covering. More precisely, a necessary and sufficient condition for a graph G to admit an $[m]$ -covering is that every edge of G belongs to a matching of size at least m (and hence to a matching of size exactly m). If a graph G admits an $[m]$ -covering, we say that G is $[m]$ -coverable. It is natural and of practical interest to consider $[m]$ -coverings of *smallest* order. Any such $[m]$ -covering will be called² an *excessive $[m]$ -factorization*. The order of any excessive $[m]$ -factorization of G will be denoted by $\chi'_{[m]}(G)$ and called *excessive $[m]$ -index*. In other words, we let

$$\chi'_{[m]}(G) = \min\{|\mathcal{M}| : \mathcal{M} \text{ is an } [m]\text{-covering of } G\},$$

where we adopt the standard set-theoretic convention that $\min \emptyset = \infty$. In such a way the parameter $\chi'_{[m]}(G)$ is now defined on all graphs.³

The purpose of the present paper is to begin the investigation of excessive $[m]$ -factorizations and of the parameter $\chi'_{[m]}$. The following fact, which we state in the form of a proposition, will be frequently used.

Proposition 1. *Let m, k be positive integers and let G be a graph. The following are equivalent conditions for G :*

- (i) $\chi'_{[m]}(G) \leq k$;
- (ii) G has a k -edge colouring whose colour classes are all of size at most m , and such that each colour class is contained in a matching of size m ;
- (iii) G is the underlying simple graph of a multigraph \tilde{G} such that $|E(\tilde{G})| = km$ and \tilde{G} is k -edge colourable.

Proof. Assume that $\chi'_{[m]}(G) \leq k$. Let $\mathcal{M} = \{M_1, M_2, \dots, M_k\}$ be an $[m]$ -covering (where, if necessary, we allow the same matching to appear several times as an element of \mathcal{M}). Define a function $\phi : E(G) \rightarrow [1, k]$ by

$$\phi(e) = \min\{i \in [1, k] : e \in M_i\}.$$

It is easily seen that ϕ is an edge colouring satisfying the desired conditions. Thus (i) implies (ii). Now, define the multigraph \tilde{G} to be the spanning supermultigraph of G satisfying the following property: any two distinct vertices x, y are joined in \tilde{G} by as many edges as there are matchings M_i in \mathcal{M} containing the edge xy if $xy \in E(G)$, and by no edges if $xy \notin E(G)$. It is easily seen that \tilde{G} satisfies the conditions stated in (iii). Hence (i) implies (iii). Conversely, if a multigraph \tilde{G} as stated in the proposition exists, and ϕ is an equalized k -edge colouring of \tilde{G} , then it is easily seen that the colour classes of ϕ (when viewed as matchings) constitute an $[m]$ -covering of G of order k , thus proving that (iii) implies (i). Similarly, if ϕ is an edge colouring satisfying the conditions stated in (ii), then, by simply extending the colour classes of ϕ to matchings of size exactly m , we obtain an $[m]$ -covering of G , proving that (ii) implies (i). \square

Clearly, if the graph G admits a decomposition into matchings of size m , then this will be an excessive $[m]$ -factorization and we have

$$\chi'_{[m]}(G) = |E(G)|/m.$$

¹ The use of square brackets here is motivated by the fact that other (not equivalent) notions of m -cover or m -covering are in common use in the literature.
² Also here the choice of the square brackets is imposed by the necessity of avoiding confusion with the term “ m -factorization”, which has another meaning in the literature.
³ We could conventionally define $\chi'_{[m]}(G) = \infty$ if $E(G) = \emptyset$, but, for our purposes, it is still convenient to consider only the case $E(G) \neq \emptyset$.

In this paper m is assumed to be a fixed positive integer and G is allowed to vary. When m is not independent of G but is assumed to be equal to the size of a 1-factor of G (i.e. $n/2$, where n is the number of vertices) the corresponding parameter, denoted by $\chi'_e(G)$, was introduced by Bonisoli and the first author in [1], and called *excessive index*. *Inter alia* it was proved in [1] that the problem of determining $\chi'_{[e]}(G)$ is NP-hard. The present authors recently computed the excessive index $\chi'_e(G)$ for all complete multipartite graphs G [3], and in [2] they extended the investigations to graphs of odd order and $[m]$ -coverings, where $m = (n - 1)/2$, i.e. the size of a near 1-factor. Thus the present work may be seen as an extension of the works cited above when m is allowed to assume any arbitrary (but fixed) integer value. We remark, however, that the NP-hardness of χ'_e does not seem to imply in an obvious way any fact about the complexity of computing $\chi'_{[m]}$ for fixed m , and the latter problem is presently open. In particular Rizzi (personal communication) has asked what is the minimum value of m for which the computation of $\chi'_{[m]}(G)$ is NP-complete (assuming that such an integer exists).

The paper is organized as follows. In Section 2 we discuss the *coverability index* of G , that is the maximum value of m for which G has an excessive $[m]$ -factorization. In Section 3 we study the *compatibility index*, that is the maximum value of m for which the identity

$$\chi'_{[m]}(G) = \max\{\chi'(G), \lceil |E(G)|/m \rceil\}$$

holds. In Section 4 we introduce the *augmentability index* and show its relation with the other parameters introduced. The augmentability index is the maximum value of m for which any matching of size less than m can be extended to a matching of size m . Further results concerning particular classes of graphs are considered in Section 5. Finally, in Section 6 we introduce a partition of the positive integers naturally associated to the above parameters and prove a useful theorem.

2. Coverability index

Let $\nu(G)$ denote the *matching number* of G , i.e. the size of a maximum matching. It is well known that $\nu(G)$ can be computed in polynomial time [4]. Clearly, if m is an integer and $m > \nu(G)$, then G cannot be $[m]$ -coverable. Let⁴ $\text{cov}(G)$ denote the maximum integer m with the property that G is $[m]$ -coverable. We call this number the *coverability index* of G . In other words we let

$$\text{cov}(G) = \max\{m \in \mathbb{Z}^+ : G \text{ is } [m]\text{-coverable}\}.$$

Notice that every graph G is $[1]$ -coverable. Since, if $m \geq 2$, every matching of size m contains a matching of size $m - 1$, we have that G is $[m]$ -coverable if and only if $1 \leq m \leq \text{cov}(G)$. The following proposition gives a formula for the coverability index in terms of the matching number.

Proposition 2. *For any graph G , we have*

$$\text{cov}(G) = \min_{xy \in E(G)} \{1 + \nu(G - x - y)\}. \quad (2)$$

Proof. Let $m = \text{cov}(G)$. By definition, G is $[m]$ -coverable but not $[m + 1]$ -coverable. By the fact that G is $[m]$ -coverable, for any edge $e = xy$, there exists a matching of size m containing e , i.e. there exists a matching of size $m - 1$ in $G - x - y$, thus proving that $\text{cov}(G)$ is greater than or equal to the right-hand side of (2). However, by the fact that G is not $[m + 1]$ -coverable, there also exists an edge $e = xy$ in G which is not contained in any matching of size $m + 1$. This implies that $G - x - y$ has matching number at most $m - 1$, proving the reverse inequality in (2). \square

Corollary 1. *For every graph G , $\nu(G) - 1 \leq \text{cov}(G) \leq \nu(G)$.*

Proof. The fact that $\text{cov}(G) \leq \nu(G)$ is obvious. By Proposition 2, in order to prove that $\text{cov}(G) \geq \nu(G) - 1$, it will suffice to prove that, for every $x, y \in V(G)$ such that $xy \in E(G)$, there exists a matching of $G - x - y$ of size at least $\nu(G) - 2$. If M is a matching of size $\nu(G)$, then M contains at most two edges incident with either x or y . Hence M contains a submatching M' of size at least $\nu(G) - 2$ which contains neither x nor y . This is the required matching. \square

Clearly $\text{cov}(G) = \nu(G)$ if and only if every edge of G belongs to a maximum matching. This is certainly the case, for example, if G is edge-transitive (e.g. $G = P$). The path P_3 provides an example of a graph G satisfying $\text{cov}(G) = \nu(G) - 1$.

In the following proposition, whose easy proof is left to the reader, we express the coverability index and matching number of a few simple classes of graphs.

Proposition 3. *We have*

1. $\text{cov}(K_n) = \lfloor n/2 \rfloor = \nu(K_n)$;
2. $\text{cov}(K_{p,q}) = \min\{p, q\} = \nu(K_{p,q})$;
3. $\text{cov}(C_n) = \lfloor n/2 \rfloor = \nu(C_n)$;

⁴ Here the implicit assumption that $E(G) \neq \emptyset$ guarantees that $\text{cov}(G)$ always exists and is a positive integer.

- 4. $\text{cov}(P_n) = \lfloor (n - 1)/2 \rfloor \leq \nu(P_n) = \lceil (n - 1)/2 \rceil$.
- 5. $\text{cov}(P) = 5 = \nu(P)$, where P is the Petersen graph.

As a further example, we consider the class of complete multipartite graphs.

Proposition 4. *We have*

- (a) $\nu(K(n_1, n_2, \dots, n_t)) = \min\{\lfloor n/2 \rfloor, n - n_1\}$;
- (b) $\text{cov}(K(n_1, n_2, \dots, n_t)) = \min\{\lfloor n/2 \rfloor, n - 1 - n_1\}$,

where $n = \sum_{i=1}^t n_i$.

Proof. Let $G = K(n_1, n_2, \dots, n_t)$ and let $n = \sum_{i=1}^t n_i$ be the order of G . Assume first

$$\lfloor n/2 \rfloor > \sum_{i=2}^t n_i = n - n_1$$

i.e.

$$n_1 > (n - n_1).$$

Then, by matching exactly $n - n_1$ vertices of the largest partite set V_1 with all the remaining vertices, we construct a matching which is clearly maximum, therefore confirming the truth of (a) in this case. On the other hand, if

$$n_1 \leq (n - n_1)$$

then there exists (see [3] for details) a 1-factor if n is even, or a near 1-factor if n is odd, thereby completing the proof of (a).

We now prove (b). Using Proposition 2, we have

$$\text{cov}(G) = \min_{i < j} \{\nu(G_{ij})\} + 1,$$

where

$$G_{ij} = K(n_1, n_2, \dots, n_i - 1, \dots, n_j - 1, \dots, n_t).$$

But, by (a), we have

$$\nu(G_{ij}) = \min\{\lfloor (n - 2)/2 \rfloor, n - 2 - n_1 + \delta_{ii}\},$$

where $\delta_{ii} = 1$ if $i = 1$ and $\delta_{ii} = 0$ if $i > 1$. It follows that

$$\min_{1 \leq i < j \leq t} \nu(G_{ij}) = \min\{\lfloor (n - 2)/2 \rfloor, n - 2 - n_1\}$$

and hence

$$\text{cov}(G) = \min\{\lfloor (n - 2)/2 \rfloor, n - 2 - n_1\} + 1 = \min\{\lfloor n/2 \rfloor, n - 1 - n_1\},$$

which proves (b). \square

3. Compatibility index

We now establish a fundamental lower bound on the excessive $[m]$ -index of a graph.

Theorem 1. *For any graph G and any positive integer m , we have*

$$\chi'_{[m]}(G) \geq \max\{\chi'(G), \lceil |E(G)|/m \rceil\}. \tag{3}$$

Proof. We can clearly assume that $\chi'_{[m]}(G) = k < \infty$. Let $\mathcal{M} = \{M_1, M_2, \dots, M_k\}$ be an excessive $[m]$ -factorization of G . Then

$$|E(G)| = \left| \bigcup_{i=1}^k M_i \right| \leq \sum_{i=1}^k |M_i| = km,$$

so that (using the fact that k is an integer) we have

$$k \geq \lceil |E(G)|/m \rceil. \tag{4}$$

Now, let $\varphi : E(G) \rightarrow [1, k]$ be defined by

$$\varphi(e) = \min\{i \in [1, k] : e \in M_i\}.$$

It is straightforward to verify that φ is an edge colouring of G . Hence $k \geq \chi'(G)$, which, combined with (4), proves the theorem. \square

The inequality of [Theorem 1](#) can be strict, even if $\chi'_{[m]}(G)$ is finite. For example, as proved in [1], the Petersen graph P satisfies $\chi'_{[5]}(P) = 5$, whereas the quantity $\max\{\chi'(P), \lceil |E(P)|/5 \rceil\}$ equals 4. It will be useful to distinguish those graphs for which the inequality (3) holds strictly from those for which it does not, for any fixed value of m . This distinction turns out to be crucial in the present context. Accordingly, we introduce the following notation, which will be used extensively in the sequel. Namely, for any positive integer m , we let the parameter $\Lambda_m(G)$ be defined as follows:

$$\Lambda_m(G) = \max\{\chi'(G), \lceil |E(G)|/m \rceil\}. \quad (5)$$

We say that a graph G is m -compatible if $\chi'_{[m]}(G) = \Lambda_m(G)$, i.e. if inequality (3) holds as an equality. Notice that, knowing that a given graph G is m -compatible reduces the task of computing $\chi'_{[m]}(G)$ to the task of computing $\chi'(G)$, which (despite the fact that computing $\chi'(G)$ is still NP-hard in general) is a substantial simplification.

It is natural to ask (and it is not obvious *a priori* that this should be the case) whether any graph G which is m -compatible is also m' -compatible for any positive integer $m' < m$. This question is answered affirmatively by the following theorem.

Theorem 2. *Let $m \geq 2$ be an integer. Let G be an m -compatible graph. Then G is $(m - 1)$ -compatible.*

Proof. Let $m \geq 2$ and assume G is m -compatible. Let $\Lambda_{m-1}(G)$ be defined as in (5), i.e.

$$\Lambda_{m-1}(G) = \max\{\chi'(G), \lceil |E(G)|/(m - 1) \rceil\}. \quad (6)$$

We split the proof into two cases.

Case 1. $\Lambda_{m-1}(G) = \chi'(G)$. Let

$$k = \chi'(G) = \Lambda_{m-1}(G).$$

We have, by assumption,

$$\chi'(G) \geq \lceil |E(G)|/(m - 1) \rceil \geq \lceil |E(G)|/m \rceil, \quad (7)$$

and hence (since G is m -compatible) we have

$$\chi'_{[m]}(G) = \chi'(G) = k. \quad (8)$$

Let \tilde{G} be a multigraph obtained from G as in the statement of [Proposition 1\(iii\)](#), which exists by [Proposition 1](#). We claim that there exists a multigraph \widehat{G} such that

1. $G \subset \widehat{G} \subset \tilde{G}$;
2. $|E(\widehat{G})| = k(m - 1)$.

To see this, it suffices to notice that, by [Proposition 1](#), $|E(\tilde{G})| = km$, and G satisfies, by (7) and (8), the condition

$$\chi'(G) = \chi'_{[m]}(G) = k \geq \lceil |E(G)|/(m - 1) \rceil \geq |E(G)|/(m - 1),$$

which implies that

$$|E(G)| \leq k(m - 1) = |E(\tilde{G})| - k.$$

To obtain \widehat{G} , we delete k arbitrary edges from $\tilde{G} - E(G)$. Notice that, by [Proposition 1](#), G is an underlying simple graph of \tilde{G} , and, by construction, G is also an underlying simple graph of \widehat{G} . Since \tilde{G} is k -edge colourable, \widehat{G} is k -edge colourable. Therefore, by [Proposition 1](#) applied to the integers k and $m - 1$ and to the multigraph \widehat{G} , we have

$$\chi'_{[m-1]}(G) \leq k = \chi'(G).$$

The reverse inequality follows directly from [Theorem 1](#). Therefore

$$\chi'_{[m-1]}(G) = k = \Lambda_{m-1}(G),$$

i.e. G is $(m - 1)$ -compatible.

Case 2. $\Lambda_{m-1}(G) = \lceil |E(G)|/(m - 1) \rceil > \chi'(G)$. Let $k = \chi'(G)$. Write $|E(G)|$ in the form

$$|E(G)| = (m - 1)x + y,$$

where x, y are integers with $0 \leq x, 0 \leq y < m - 1$. By assumption, we have

$$x = \lfloor |E(G)|/(m - 1) \rfloor \geq \chi'(G) = k,$$

where the inequality can hold as an equality only if $y > 0$. If $y = 0$ and ϕ is an equalized $\Lambda_{m-1}(G)$ -edge colouring, then the colour classes of ϕ provide the required excessive $[m - 1]$ -factorization of G , thus proving that $\chi'_{[m-1]}(G) = \Lambda_{m-1}(G)$, i.e. that G is $(m - 1)$ -compatible. If $y > 0$, adding to G an arbitrary matching of size $m - 1 - y$ yields a multigraph \widehat{G} with the following properties:

1. G is the underlying simple graph of \tilde{G} ;
2. $\chi'(\tilde{G}) \leq \chi'(G) + 1 = k + 1$;
3. $|E(\tilde{G})| = (m - 1)(x + 1)$.

Therefore, by Proposition 1 and the assumption, we conclude that

$$\chi'_{[m-1]}(G) \leq x + 1 \leq \lceil |E(G)| / (m - 1) \rceil = \Lambda_{m-1}(G),$$

which, by Theorem 1, implies that G is $(m - 1)$ -compatible. \square

Since, for every graph G , trivially

$$\chi'_{[1]}(G) = |E(G)|,$$

every graph is 1-compatible. Therefore it follows from Theorem 2 that, for any graph G , there exists an integer m^* such that G is m -compatible if and only if $1 \leq m \leq m^*$. Such integer m^* will be called the *compatibility index* of G and denoted by $\text{com}(G)$. By the above remark, we have

$$\text{com}(G) = \max\{m \in \mathbb{Z}^+ : G \text{ is } m\text{-compatible}\}.$$

Obviously $\text{com}(G) \leq \text{cov}(G)$ for any graph G . If this inequality holds as an equality, i.e. if $\text{com}(G) = \text{cov}(G)$, we say that G is *fully compatible*. We will show later that complete graphs, complete bipartite graphs, cycles and paths are fully compatible.

4. Augmentability index

An auxiliary concept, which can be very useful when trying to calculate the $[m]$ -excessive index, is the following. Recall that a graph G is $[m]$ -coverable if and only if every edge belongs to a matching of G of size m . This may be rephrased by saying that every matching of size 1 can be extended to a matching of size m . But suppose that G has the (stronger) property that every matching of size less than m can be extended to a matching of size m . We then say that G is *m -augmentable*. Clearly, if G is m -augmentable and $m \geq 2$, then G is also $(m - 1)$ -augmentable. Hence there always exists an integer m^+ such that G is m -augmentable if and only if $1 \leq m \leq m^+$. Such integer will be called the *augmentability index* of G and denoted by $\text{aug}(G)$. By the above remark, we have

$$\text{aug}(G) = \max\{m \in \mathbb{Z}^+ : G \text{ is } m\text{-augmentable}\}.$$

Clearly $\text{aug}(G) \leq \text{cov}(G)$ for every graph G . If $\text{aug}(G) = \text{cov}(G)$, we say that G is *fully augmentable*. The following proposition expresses the augmentability index in a different fashion. We say that a matching is *maximal* if it is not properly contained in another matching.

Proposition 5. *For any graph G , $\text{aug}(G)$ is the size of a smallest maximal matching of G .*

Proof. Let M be a maximal matching of smallest size and let $m = |M|$. Clearly M cannot be extended to a larger matching, hence $\text{aug}(G) \leq m$. On the other hand, any matching of size smaller than m is contained in a matching of size at least m , proving that $\text{aug}(G) \geq m$. \square

Rizzi (personal communication) informed us that the problem of finding the size of a smallest maximal matching in a graph is, in general, *APX-hard* and, in particular, NP-hard. This fact indicates that it may be very hard to determine $\text{aug}(G)$ in general. However, it appears that computing $\text{aug}(G)$ is still simpler than computing the excessive $[m]$ -index of G , or even the compatibility index of G . The following theorem justifies the introduction of the concept of augmentability.

Theorem 3. *Every m -augmentable graph is m -compatible.*

Proof. Let G be an m -augmentable graph. Then, by definition,

$$\Lambda_m(G) \geq \chi'(G),$$

and hence G is $\Lambda_m(G)$ -edge colourable. Thus G has an equalized $\Lambda_m(G)$ -edge colouring φ . We claim that

$$\lceil |E(G)| / \Lambda_m(G) \rceil \leq m. \tag{9}$$

This follows immediately from

$$\Lambda_m(G) \geq \lceil |E(G)| / m \rceil \geq |E(G)| / m,$$

together with the fact that m is an integer.

Since φ is an equalized $\Lambda_m(G)$ -edge colouring, every colour class has size at most

$$\lceil |E(G)| / \Lambda_m(G) \rceil,$$

and hence at most m by (9).

By assumption, each such colour class can be extended to a matching of size exactly m , which yields an $[m]$ -covering of G of order $\Lambda_m(G)$. Hence $\chi'_{[m]}(G) \leq \Lambda_m(G)$, where we must necessarily have equality by Theorem 1. This concludes the proof. \square

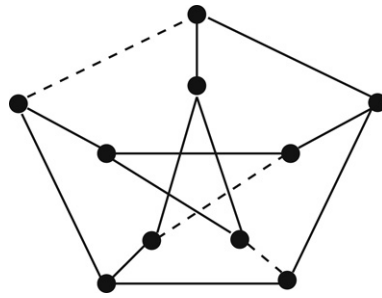


Fig. 1. A maximal matching of P of size 3.

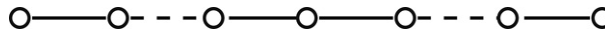


Fig. 2. A maximal matching of size 2 in P_7 .

Theorem 3 implies the inequality $\text{aug}(G) \leq \text{com}(G)$ for all graphs G . This is not an equality in general. For example, if P is the Petersen graph, then, by **Proposition 8** below, $\chi'_{[4]}(P) = \chi'(P) = 4$, which implies that $\text{com}(P) \geq 4$, but it is easy to see that $\text{aug}(P) = 3$ (see **Fig. 1**).

An easy lower bound for $\text{aug}(G)$ is obtained as follows. We first recall some standard definitions. A *dominating set* in a graph G is a set $S \subset V(G)$ with the property that every vertex of G is either in S or adjacent to an element of S . The *domination number* of G , denoted by $\gamma(G)$, is the cardinality of a smallest dominating set. The *line graph* of G , denoted by $L(G)$, is the graph whose vertex set is $E(G)$ and where two vertices are adjacent if and only if the corresponding edges of G are incident.

Theorem 4. For any graph G , $\text{aug}(G) \geq \gamma(L(G))$.

Proof. It is an elementary observation that, if I is a maximal independent set in G , then I is a dominating set. Hence, if k is the size of a smallest maximal independent set, we have

$$\gamma(G) \leq k.$$

Applying the above inequality to the line graph $L(G)$ and using **Proposition 5**, we obtain

$$\gamma(L(G)) \leq \text{aug}(G),$$

which is the required inequality. \square

A corollary of the above theorem is the following.

Corollary 2. Let G be a graph. Then $\text{aug}(G) \geq \lceil |E(G)| / (2\Delta(G) - 1) \rceil$.

Proof. Let E_1 be a dominating set for $L(G)$. We claim that

$$|E_1|(2\Delta(G) - 1) \geq |E(G)|. \tag{10}$$

Indeed, if $e \in E_1$, then e is incident with at most $2\Delta(G) - 2$ distinct edges and hence e dominates at most $2\Delta(G) - 1$ edges. It follows that E_1 dominates at most $|E_1|(2\Delta(G) - 1)$ edges and, since E_1 is a dominating set for $L(G)$, inequality (10) follows. Now, taking E_1 to be a smallest dominating set for $L(G)$ and using **Theorem 4** and (10), we obtain the desired inequality. \square

We now calculate $\text{aug}(G)$ for some simple classes of graphs.

Proposition 6. We have

- (i) $\text{aug}(K_n) = \lfloor n/2 \rfloor = \text{cov}(K_n)$, i.e. K_n is fully augmentable;
- (ii) $\text{aug}(K_{p,q}) = \min\{p, q\} = \text{cov}(K_{p,q})$, i.e. $K_{p,q}$ is fully augmentable;
- (iii) $\text{aug}(C_n) = \lceil n/3 \rceil$;
- (iv) $\text{aug}(P_n) = \lceil (n - 1)/3 \rceil$;
- (v) $\text{aug}(P) = 3$, where P is the Petersen graph.

Proof. (i) and (ii) are immediate. (v) follows from the fact that P is, by **Corollary 2**, 3-augmentable, and by the existence of the maximal matching of P displayed in **Fig. 1**. For the cases (iii) and (iv), the \geq part of the equality follows from **Corollary 2**. A maximal matching of size $\lceil (n - 1)/3 \rceil$ for P_n is shown in **Fig. 2** for $n = 7$. A similar idea works for any value of n and can be used to prove also that $\text{aug}(C_n) = \lceil n/3 \rceil$, thus proving the \leq part of the equality in (iii) and (iv). \square

Proposition 7. *The augmentation number of $G = K(n_1, n_2, \dots, n_t)$ is the integer*

$$\text{aug}(G) = \max\{\lceil (n - n_1)/2 \rceil, n_2\}.$$

Proof. Let M be a maximal matching of G of minimum size. Then M cannot leave two vertices in two different partite sets unsaturated, because it is maximal. Hence all the unsaturated vertices must be in the same partite set, and thus there are at most n_1 unsaturated vertices. This proves that

$$2|M| \geq n - n_1.$$

Suppose that M leaves only nodes in the partite set V_i unsaturated, where $1 \leq i \leq t$. We claim that M consists of:

- a maximum matching M_0 of $G - V_i$;
- as many edges as possible joining V_i to $G - V_i - V(M_0)$;

Indeed, let M^* be the matching obtained from M by deleting all the edges incident vertices in V_i . We prove that M^* is a maximum matching of $G - V_i$. For, if there was a matching M^+ of $G - V_i$ of size larger than M^* , then M^+ could be extended (by adding as many edges as possible from V_i to $G - V_i - V(M^+)$) to a matching of G which is clearly maximal and has size smaller than M , contradicting the assumption that M is a smallest maximal matching. Therefore M^* is a maximum matching of $G - V_i$.

It follows from this claim that the size of M is easily computed as

$$|M| = |M_0| + (|V(G - V_i)| - 2|M_0|) = n - n_i - |M_0|. \tag{11}$$

Since

$$|M_0| = \nu(G - V_i) = \nu(K(n_1, n_2, \dots, n_{i-1}, n_{i+1}, \dots, n_t)),$$

we have, by Proposition 4,

$$|M_0| = \min\{\lfloor (n - n_i)/2 \rfloor, n - n_i - n_x\}, \tag{12}$$

where $x = 2$ if $i = 1$ and $x = 1$ if $i \neq 1$. Hence, using (11) and (12), we see that

$$|M| = \max\{\lceil (n - n_i)/2 \rceil, n_x\}. \tag{13}$$

Clearly the left-hand side of (13) has the minimum value when $i = 1$, i.e. when

$$|M| = \max\{\lceil (n - n_1)/2 \rceil, n_2\}. \tag{14}$$

Thus the proposition will be proved if we can convince ourselves that there exists a maximal matching M of G which leaves some nodes on V_1 unsaturated. Such a matching is easy to construct, e.g. taking a maximum matching of $G - V_1$ and following the procedure described above. By (14), this matching has the required size. \square

5. Further results

Using the results proven so far, we can now express the excessive $[m]$ -index for some simple classes of graphs.

Proposition 8. *We have*

- (i) $\chi'_{[m]}(K_n) = \Lambda_m(K_n)$ for $1 \leq m \leq \lfloor n/2 \rfloor$ and ∞ otherwise, i.e. K_n is fully compatible;
- (ii) $\chi'_{[m]}(K_{p,q}) = \Lambda_m(K_{p,q})$ for $1 \leq m \leq \min\{p, q\}$ and ∞ otherwise, i.e. $K_{p,q}$ is fully compatible;
- (iii) $\chi'_{[m]}(C_n) = \Lambda_m(C_n)$ for $1 \leq m \leq \lfloor n/2 \rfloor$ and ∞ otherwise, i.e. C_n is fully compatible;
- (iv) $\chi'_{[m]}(P_n) = \Lambda_m(P_n)$ for $1 \leq m \leq \lfloor (n - 1)/2 \rfloor$ and ∞ otherwise, i.e. P_n is fully compatible.
- (v) $\chi'_{[m]}(P) = \Lambda_m(P)$ for $1 \leq m \leq 4$, $\chi'_{[5]}(P) = 5$ and $\chi'_{[m]}(P) = \infty$ for $m > 5$.

Proof. (i) and (ii) follow immediately by Proposition 6 and Theorem 3. To prove (iii), since $\text{aug}(C_n) = \lceil n/3 \rceil$ by Proposition 6, we can restrict ourselves, without loss of generality, to the case

$$\lceil n/3 \rceil < m \leq \lfloor n/2 \rfloor.$$

Under these conditions it is easily seen that

$$\Lambda_m(C_n) = \max\{\chi'(C_n), \lceil |E(C_n)|/m \rceil\} = \begin{cases} 2 & \text{if } n \text{ is even and } m = n/2; \\ 3 & \text{otherwise.} \end{cases}$$

In the first case C_n is clearly m -compatible, since a 1-factorization of C_n is an excessive $[m]$ -factorization of order 2. In the latter case, in order to prove that $\chi'_{[m]}(C_n) \leq 3$ we shall use Proposition 1 and prove that C_n has a 3-edge colouring whose

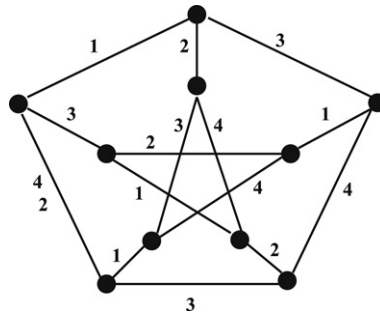


Fig. 3. An excessive [4]-factorization of the Petersen graph.

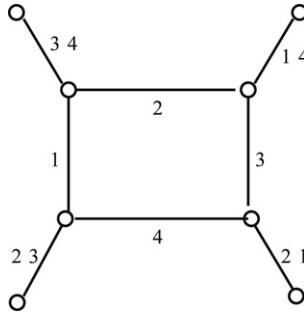


Fig. 4. An example of a [3]-coverable graph which is not 3-compatible.

colour classes have size at most m , and such that each colour class can be extended to a matching of size m . To see this, let e_1, e_2, \dots, e_n be the edges of C_n in circular order. Define $\varphi : E(C_n) \rightarrow \{\alpha, \beta, \gamma\}$ by letting

$$\varphi(e_i) = \begin{cases} \alpha & \text{if } i = 1, 3, 5, \dots, 2m - 1 \\ \beta & \text{if } i = 2m + 1, 2m + 3, \dots, n^* \\ \gamma & \text{if } i = 2s + 2, 2s + 4, \dots, n^+ \end{cases}$$

where n^*, n^+ are, respectively, the largest odd and even integer less than or equal to n , and s is chosen in such a way that the β - colour class has size precisely m . It is easily seen that the above colouring is a proper 3-colouring of G satisfying the required properties. This terminates the proof of (iii). In a similar way one can prove that P_n is fully compatible, i.e. (iv). To prove (v), notice that, by Proposition 6 and Theorem 3, P is m -compatible, for $m \leq 3$. Since $\text{cov}(P) = 5$ by Proposition 3, we are left only with the cases $m = 4$ and $m = 5$ of (v). The case $m = 5$ was established in [1]. Thus, we are only left with the proof that $\chi'_{[4]}(P) = \Lambda_4(P) = 4$. By Theorem 1, we only need to prove that $\chi'_{[4]}(P) \leq 4$. A [4]-covering of P of order 4 is shown in Fig. 3. This completes the proof. \square

The determination of the parameter $\chi'_{[m]}$, when G is a complete multipartite graph and m is an arbitrary integer, seems to be an interesting unsolved problem (as mentioned above, we solved this problem completely in [3] only when G has even order n and $m = n/2$). For no classes of graphs, other than those listed in Proposition 8, we have evaluated $\chi'_{[m]}$ for all possible values of m . We shall now restrict our attention to particular values of m , in the attempt to be able to say something more specific about the parameter $\chi'_{[m]}$, at least for some classes of graphs.

We have already noticed that every graph G is 1-compatible. However, not all graphs are 2-compatible, since there are obvious examples of graphs which are not even 2-coverable. But this is the only limitation in this respect, i.e. we have the following.

Proposition 9. Every [2]-coverable graph is 2-compatible.

Proof. The truth of the statement of the proposition follows immediately from the observation that every [2]-coverable graph is 2-augmentable and by Theorem 3. \square

An obvious question at this point is whether every [3]-coverable graph is also 3-compatible. The answer to this question is negative, as shown by the graph G depicted in Fig. 4.

If this graph was 3-compatible, then we would have $\chi'_{[3]}(G) = 3$, but it is easy to see that any [3]-covering of G contains at least 4 matchings, since no pair of the edges of the 4-cycle of G can be included in the same matching of size 3. Similar examples can be easily produced for any $m \geq 3$. Thus the question “is every $[m]$ -coverable graph m -compatible?” is better rephrased as “which $[m]$ -coverable graphs are m -compatible?”. This seems to be a fundamental question for which we do not have a satisfactory answer at present. However, we can prove the following.

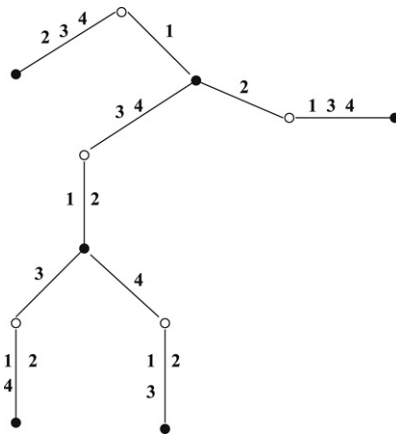


Fig. 5. A [5]-coverable tree which is not 5-compatible. An excessive [5]-factorization is shown.

Theorem 5. Every [3]-coverable tree is 3-compatible.

Proof. Let T be a [3]-coverable tree. If T is 3-augmentable, then we are done by Theorem 3. Hence we can assume, without loss of generality, that T is not 3-augmentable, and hence (since T is necessarily 2-augmentable by the fact that it is [2]-coverable) T contains a matching $\{e, f\}$ which is not contained in a matching of size 3. Let $k = \Delta_3(T)$. By Proposition 1 and (9), it will suffice to prove the existence of an equalized k -edge colouring of T such that every colour class can be extended to a matching of size 3. Notice that every edge in T is incident or coincident to either e or f , otherwise the matching $\{e, f\}$ would be 3-augmentable. Let $e = x_e y_e$ and let $f = x_f y_f$, where we assume that y_e and x_f are at the shortest distance in T (it is easily seen that such distance must be either 1 or 2). Since the matchings $\{e\}$ and $\{f\}$ are, by the [3]-coverability of T , contained in matchings of size 3, there must be, for each of x_e, y_e, x_f, y_f , one incident edge, which is different from the edges e, f . We claim the following:

Claim. The only maximal matching in T of size 2 is $\{e, f\}$. To see this, suppose that $\{\lambda, \mu\}$ is a maximal matching of T . Let $\{x_e, y_e, x_f, y_f\}$ be the endpoints of e, f . It is easily seen that λ and μ together must be incident to all the vertices in $\{x_e, y_e, x_f, y_f\}$, otherwise (by the above remark) the matching $\{\lambda, \mu\}$ would not be a maximal matching of T . But, since T is a tree, the only possibility is that $\{\lambda, \mu\} = \{e, f\}$, thus proving our claim.

Let now φ be an equalized k -edge colouring of T . Notice that, by (9), every colour class of φ has size at most 3. If φ does not contain $\{e, f\}$ as a colour class, then, by the Claim and Proposition 1, we are done. Hence we can assume that one of the colour classes is $\{e, f\}$. Now, we can easily obtain a new equalized k -edge colouring of T which does not contain $\{e, f\}$ as a colour class as follows. Starting with the colouring φ , we simply exchange the colours between f and one of the other edges incident with x_f (more precisely, we choose any edge not on the path joining y_e and x_f if such an edge exists, or the only edge incident with x_f and different from f otherwise). In this way we always obtain the desired colouring of T , and hence, arguing as above, we can claim that T is 3-compatible. \square

An open problem is to see whether the statement of Theorem 5 remains true when m is assumed to be 4, i.e. to prove (or disprove) that every [4]-coverable tree is 4-compatible. However, for $m = 5$ the same statement is certainly false, as the example of Fig. 5 shows. As proved in [2] such tree T has excessive [5]-index equal to 4, whereas the quantity $\max\{\chi'(T), \lceil |E(T)|/5 \rceil\}$ equals 3, proving that T is not 5-compatible.

Now, for any class \mathcal{G} of graphs, one may ask the following question: “what is the maximum positive integer m such that every graph $G \in \mathcal{G}$ which is $[m]$ -coverable is also m -compatible?”

We propose to call such integer the *compatibility level* of the class \mathcal{G} and denote it by $m(\mathcal{G})$. For example, we have seen earlier that, if \mathcal{G} denotes the class of all graphs, we have

$$m(\mathcal{G}) = 2,$$

By what we have just shown, if \mathcal{T} denotes the class of trees, we have

$$m(\mathcal{T}) = 3 \text{ or } 4,$$

according to whether there exists a [4]-coverable tree which is not 4-compatible or not, respectively.

Since the graph of Fig. 4 is bipartite, we also have

$$m(\mathcal{B}) = 2,$$

where \mathcal{B} denotes the class of bipartite graphs. Indeed, we also have

$$m(\mathcal{Q}) = 2,$$

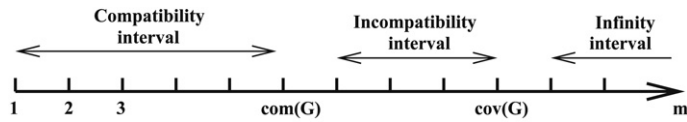


Fig. 6. The three intervals associated to a graph G.

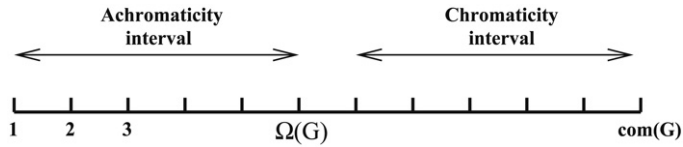


Fig. 7. The subdivision of the compatibility interval in the intervals \mathcal{I}_a and \mathcal{I}_c .

where \mathcal{Q} denotes the class of complete multipartite graphs, since, using theorem, one can easily see that the complete multipartite graph $K(2, 1, 1, 1, 1)$ is [3]-coverable but not 3-compatible.

For the class of paths, cycles, complete graphs and complete bipartite graphs, or (more generally) for any class of graphs \mathcal{H} with the property that all its members are fully compatible, we obviously have

$$m(\mathcal{H}) = \infty.$$

6. Intervals of integers

As a consequence of Theorem 2, for any graph G, one can subdivide the set of positive integers m into three intervals (see Fig. 6):

1. The interval $[1, \text{com}(G)]$, which we call the *compatibility interval*, consisting of those values of m such that G is m -compatible, i.e.

$$\chi'_{[m]}(G) = \Lambda_m(G).$$

2. The interval $(\text{com}(G), \text{cov}(G)]$, which we call the *incompatibility interval*, consisting of those values of m such that G is $[m]$ -coverable but not m -compatible, i.e.

$$\Lambda_m(G) < \chi'_{[m]}(G) < \infty.$$

3. The interval $(\text{cov}(G), \infty)$, which we call the *infinity interval*, consisting of those values of m such that G is not $[m]$ -coverable, i.e.

$$\chi'_{[m]}(G) = \infty.$$

Notice that the incompatibility interval may be empty (when G is fully compatible), but the other two intervals are always non-empty.

We may further subdivide the compatibility interval into two subintervals, namely the interval where the parameter $\Lambda_m(G)$ takes the value $\lceil |E(G)|/m \rceil$ and the interval where it takes the value $\chi'(G)$. Strictly speaking, however, these two intervals may not always be disjoint, so we introduce a convention. Let the integer $\Omega(G)$ be defined by

$$\Omega(G) = \lfloor |E(G)|/\chi'(G) \rfloor.$$

We define *achromaticity interval*, denoted by \mathcal{I}_a , the interval $[1, \Omega(G)] \cap [1, \text{com}(G)]$ and *chromaticity interval*, denoted by \mathcal{I}_c , the interval $(\Omega(G), \text{com}(G)]$ (see Fig. 7).

Notice that \mathcal{I}_c is empty if and only if the identity

$$\lfloor |E(G)|/\chi'(G) \rfloor = \text{com}(G)$$

holds, which may sometimes occur (e.g. when $G = C_4$). However, \mathcal{I}_a is never empty since it always contains the integer 1.

It is easy to see that

$$\chi'_{[m]}(G) = \begin{cases} \lceil |E(G)|/m \rceil & \text{if } m \in \mathcal{I}_a; \\ \chi'(G) & \text{if } m \in \mathcal{I}_c. \end{cases}$$

Thus the partition of the compatibility interval in the intervals \mathcal{I}_a and \mathcal{I}_c satisfies the desired condition. Notice that a positive integer m belongs to \mathcal{I}_a if and only if the following two conditions are satisfied:

- (a) $m \leq \text{com}(G)$;
- (b) $m \leq \Omega(G)$.

However, as we shall prove now, condition (a) is redundant.

Theorem 6. *Let G be a graph and let m be a positive integer. If $|E(G)|/m \geq \chi'(G)$, then G is m -compatible.*

Proof. Let G and m be as in the statement of the theorem. We first prove that G is $[m]$ -coverable. By Lemma 1 we can edge-colour G with colour classes of size $\lfloor |E(G)|/\chi'(G) \rfloor$ and $\lceil |E(G)|/\chi'(G) \rceil$. Since $\lfloor |E(G)|/\chi'(G) \rfloor \geq m$ by assumption, we have in particular that every edge belongs to a matching of size m , and hence G is $[m]$ -coverable. Now, assume that

$$|E(G)|/m = \lceil |E(G)|/m \rceil = \chi'(G).$$

Then G has a decomposition in matchings of size m , and hence is m -compatible. We may then assume

$$\lceil |E(G)|/m \rceil > \chi'(G). \tag{15}$$

Let $k = \Lambda_m(G) = \lceil |E(G)|/m \rceil$. We need to prove that $\chi'_{[m]}(G) = k$. By Theorem 1, it will suffice to prove that $\chi'_{[m]}(G) \leq k$. By (15) and the assumption,

$$k = \Lambda_m(G) = \lceil |E(G)|/m \rceil > \chi'(G). \tag{16}$$

Write

$$|E(G)| = mx + y, \tag{17}$$

where $0 \leq y < m$. By (16) and (17) we have

$$k = x + \lceil y/m \rceil = \begin{cases} x & \text{if } y = 0 \\ x + 1 & \text{otherwise.} \end{cases} \tag{18}$$

If $y = 0$, we have $|E(G)| = km$ by (18). Since, by (16), $k \geq \chi'(G)$, there exists an equalized k -edge colouring of G and hence a decomposition into matchings of size m , and we are done. Thus we can assume, without loss of generality, that $y > 0$. Let M be a matching of G of size $m - y$. By duplicating precisely those edges in G which are in M , we obtain a multigraph \tilde{G} with the property that:

- (i) G is the underlying simple graph of \tilde{G} ;
- (ii) $|E(\tilde{G})| = |E(G)| + m - y = mx + y + (m - y) = m(x + 1) = mk$;
- (iii) $\chi'(\tilde{G}) \leq \chi'(G) + 1 \leq k$.

Therefore, by Proposition 1, $\chi'_{[m]}(G) \leq k$, whence the proof is completed. \square

Corollary 3. $\text{com}(G) \geq \lfloor |E(G)|/\chi'(G) \rfloor = \Omega(G)$.

Notice that the condition of Theorem 6, which is equivalent to $m \leq \Omega(G)$, cannot be relaxed to $m \leq \lceil |E(G)|/\chi'(G) \rceil$, since the graph of Fig. 4 provides (for $m = 3$) a counterexample. This fact (together with Theorem 6) provides an additional justification for the choice of $\Omega(G)$ as the integer delimiting the intervals I_a and I_c .

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