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2-Cyclically Resolvable 4-Cycle Group Divisible Designs

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Abstract

A 4-Cycle group-divisible design $GDD(k, r; C_4)$ is an ordered triple $(\mathbb{V}, \mathbb{G}, \mathbb{B})$, where $\mathbb{V}$ is a set of rk elements, $\mathbb{G}$ is a partition of $\mathbb{V}$ into r groups of size k, and $\mathbb{B}$ is a collection of 4-cycles such that each pair of elements from the same group occur together in no 4-cycles, and each pair of elements from different groups occur together in exactly one 4-cycle. In his paper, we prove that for each pair of positive integers k and r satisfying (a) $4|k \cdot r$ and $2|k \cdot (r-1)$ there exists a 2-cyclically resolvable $GDD(k, r; C_4)$.

1. Introduction

An m-cycle, written $(c_0, c_1, ..., c_{m-1})$, consists of m distinct vertices c_0, c_1, ..., c_{m-1}, and m edges $\{c_i, c_{i+1}\}$, $0 \leq i \leq m-2$, and $\{c_0, c_{m-1}\}$. An m-cycle system of a graph G is a pair $(V, \mathbf{C})$ where V is the vertex set of G and $\mathbf{C}$ is a collection of m-cycles whose edges partition the edges of G.

The obvious necessary conditions for the existence of an m-cycle system of a graph G are that the value of m is not exceeding the order of G, m

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divides the number of edges in G, and the degree of each vertex in G is even. A graph G is called m-sufficient if the necessary conditions are met.

A graph G is said to be a complete r-partite graph $(r > 1)$ if its vertex set V can be partitioned into r disjoint non-empty sets $V_1, ..., V_r$ (called partite sets) such that there exists exactly one edge between each pair of vertices from different partite sets. If $|V_i| = n_i$ for $1 \leq i \leq r$, the complete r-partite graph is denoted by $K_{n_1,...,n_r}$. In particular, if $n_1 = ... = n_r = k$ (> 1) it is called balanced, and the graph will be simply denoted by $K_{r(k)}$.

A 4-cycle group-divisible design $GDD(k, r; C_4)$ is an ordered triple $(\mathbb{V}, \mathbb{G}, \mathbb{B})$, where $\mathbb{V}$ is a set of rk elements, $\mathbb{G}$ is a partition of $\mathbb{V}$ into r groups of size k, and $\mathbb{B}$ is a collection of 4-cycles such that

(1) each pair of elements that occur in the same group, occur together in no 4-cycles, and

(2) each pair of elements that occur in different groups, occur together in exactly one 4-cycle.

Therefore, it is easy to notice that a $GDD(k, r; C_4)$ exists if and only if a 4-cycle system of $K_{r(k)}$ exists.

The study of m-cycle systems of the complete graph and the complete multipartite graph has been one of the most interesting problems in graph decomposition. The existence question for m-cycle systems of K_n and $K_{r(2)}$ where $n = 2r$ has been completely settled by Alspach and Gavlas [1] in the case of m odd and by Šajna [14] in the even case. As to the existence for

m-cycle system of more general complete r-partite graph ($r > 3$), not much work is known; $m = 3$ was solved for some complete r-partite graphs [5], and $m = 4$ was completely settled by Billington et al. [4]. For existence surveys on cycle systems the reader may refer to [13, 14].

In what follows, we shall assume $G = (V, E)$ to be a simple graph with $|V| = v$. Given an m-cycle system $(V, \mathbf{C})$ of a graph G, let π be a permutation on V. For each cycle $(c_0, ..., c_{m-1})$ in $\mathbf{C}$ and a permutation π on V, let $C^\pi = \{(C_0{}^\pi, ..., C_{m-1}{}^\pi)\}|C \in \mathbf{C}$. If $C^\pi = \{C^\pi|C \in \mathbf{C}\} = \mathbf{C}$, then π is said to be an automorphism of $(V, \mathbf{C})$. If there is an automorphism π whose cycle representation is a cycle of length v, then the m-cycle system is called *cyclic*. For a cyclic m-cycle system, the vertex set V can be identified with Z_v and π can be represented by $\pi = (0, 1, ..., v-1)$ or $\pi : i \to i + 1 \pmod v$.

Assume $(V, \mathbf{C})$ to be an m-cycle system of G and let $C = (c_0, ..., c_{m-1})$ be a cycle in $\mathbf{C}$. The cycle orbit of C is defined by the set of distinct cycles

$$C + i = (c_0 + i, ..., c_{m-1} + i) \ (mod\ v)$$

for $i \in Z_v$. The length of a cycle orbit is its cardinality, i.e., the minimum positive integer k such that $C + k = C$. A cycle orbit of length v is said to be full, otherwise short. A base cycle of a cycle orbit O is a cycle in O that is chosen arbitrarily.

A resolution class of an m-cycle system $(V, \mathbf{C})$ of a graph G is a collection of t $(= v/m)$ vertex disjoint m-cycle in $\mathbf{C}$. The m-cycle system is called resolvable if $\mathbf{C}$ can be partitioned into resolution classes $R_1, ..., R_s$ such that every vertex of V is contained in exactly one m-cycle of each class, and the set $\mathbf{R} = \{R_1, ..., R_s\}$ is called a resolution of the system. For any resolution class $R_j = \{C_{j,1}, ..., C_{j,t}\}$ in $\mathbf{R}$, let $R_j + i = \{C_{j,1} + i, ..., C_{j,t} + i\}$ for $i \in Z_v$. A resolution $\mathbf{R}$ of an m-cycle system $(V, \mathbf{C})$ of G is called i-cyclically resolvable if $V = Z_v$ and we have $R_j + i \in \mathbf{R}$ whenever $R_j \in \mathbf{R}$. If $i = 1$, it is simply called cyclically resolvable. Clearly, a cyclically resolvable m-cycle system of G is also an i-cyclically resolvable m-cycle system of G for each $i \geq 1$, and an i-cyclically resolvable m-cycle system of G is a cyclic m-cycle system of G.

The Oberwolfach problem was first formulated by Ringel and first mentioned in [2]. It is worth of noting that the Oberwolfach problem with factors of uniform length is equivalent to the existence problem for a resolvable m-cycle system of λK_v (the graph on v vertices in which each pair of vertices is joined by exactly λ edges). The spectrum for the Obserwolfach problem with particular restrictions on the length of factors was completely settled [2, 10].

It is well-known that a cyclic triple system of order v exists if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$ [12]. It is also well-known that a Kirkman triple system of order v exists if and only if $v \equiv 3$ (mod 6). As to cyclic m-cycle systems of λK_v, quite a few results have obtained by Burati et al recently, see [6,7,8] for references. On the other hand, resolvable m-cycle systems of λK_v for $m > 3$ do exist for some λ and m, see [?] for references. But, constructing a cyclically resolvable m-cycle system of λK_v is not easy at all. So far, only partial results in the case $\lambda = 1$ and $m = 3$ have been obtained in [?].

Since the study of the existence of m-cycle group divisible designs (GDD) extends the study of m-cycle system of the complete graphs, to obtain an i-cyclically resolvable m-cycle GDD is also difficult.

Recently, the present authors [15] have proved that there exists a maximum cyclic 4-cycle packing of the balanced complete multipartite graph $K_{r(k)}$ with specified leave. Of course, if the leave itself is the empty set, it is precisely a cyclic 4-cycle system of $K_{r(k)}$. This takes care of the cyclicity. On the other direction, Billington and Rodger investigate the existence of resolvable 4-cycle group divisible designs with two associate classes, and they are able to prove the case when each part size is even [3]. In this paper, we shall combine the cyclicity and resolvability together and the following result is proved.

Theorem 1.1 *There exists a 2-cyclically resolvable 4-cycle system of the complete multipartite graph G if and only if $G \cong K_{r(k)}$ for positive integers r and k satisfying $2|k(r-1)$ and $4|rk$.*

2. Preliminary results

Throughout this paper, all graphs we consider are defined on $\mathbb{Z}_v$ where $v = rk = 4q$ and use $\pm|a-b|$ to denote the difference of the edge $\{a, b\}$. The following lemma deserves to be mentioned first.

Lemma 2.1 *If exists exists a 2-cyclically resolvable m-cycle system of the complete multipartite graph* $G = K_{n_1,n_2,\ldots,n_r}$ *where* $V(G) = \mathbb{Z}_v$*, then (1)* $n_1 = n_2 = \ldots = n_r = k$*, (2) the partite sets of* G *are* $V_i = \{i, i+r, i+2r, \ldots, i+(k-1)r\}, i \in \{0, 1, 2, \ldots, r-1\}$ *and (3)* G *is m-sufficient and* $m|v$*.*

Proof. The first two conditions are the consequences of being a cyclic m-cycle system and the third condition follows from the existence of a resolvable m-cycle system.

Corollary 2.2 *If there exists a 2-cyclically resolvable 4-cycle system of* $K_{r(k)}$*, then* $2|(r-1)k$ *and* $4|rk$*.*

Note that if the above two conditions hold, then $K_{r(k)}$ is 4-sufficient.

By Lemma 2.1, without mentioning otherwise, we shall let V_i denote the ith partite set of $K_{r(k)}$ in what follows. Thus, the set of distinct differences of edges in $K_{r(k)}$ is $D[K_{r(k)}] = \mathbb{Z}_v \setminus \pm\{0, r, \ldots, \lfloor k/2 \rfloor r\}$. The set of distinct even differences of edges in $K_{r(k)}$ is denoted by $D^*[K_{r(k)}]$. A bit of reflection, in order to construct a cyclic 4-cycle system of $\mathbb{Z}_v$, we have to partition $D[K_{r(k)}]$ into subsets such that each subset of differences induces a subgraph of $K_{r(k)}$ in which a cyclic 4-cycle system exists. Since we also need resolvability, the arrangement of base cycles is crucial in constructing a cyclically resolvable 4-cycle system.

Let $\Omega \subseteq D[K_{r(k)}]$ such that $\Omega = -\Omega$. Let $G[\Omega]$ denote the subgraph of $K_{r(k)}$ which contains the edges $\{a, a+b\}$ with $a \in \mathbb{Z}_v$ and $b \in \Omega$. Then we have

Lemma 2.3 *For any integer $p \in \{1, 2, ..., \frac{v}{2} - 1\}$ and $\Omega_p = \pm\{p, \frac{v}{2} - p\}$, there exists a cyclic 4-cycle system of $G[\Omega_p]$. Furthermore, if p is odd, then there exists a cyclically resolvable 4-cycle system of $G[\Omega_p]$.*

Proof. The proof of the first part follows by using $C = (0,\ p,\ \frac{p}{2},\ p + \frac{v}{2})$ as the base cycle. On the second part, the set of v cycles generated by C can be partitioned into two resolution classes R_0 and R_1 where

$$R_j = \{(j+2i,\ j+p+2i,\ j+\frac{v}{2}+2i,\ j+p+\frac{v}{2}+2i) | i = 0, 1, 2, ..., \frac{v}{4}-1\},\ j = 0, 1. \blacksquare$$

It should be noted that if p is even, then $G[\Omega_p]$ is not necessarily resolvable. However, if $\frac{v}{4} \not\equiv 0 \pmod{r}$, then a cyclically resolvable 4-cycle system of $G[\Omega_{\frac{v}{4}}]$ does exist. This follows by letting $R = \{(i, i + \frac{v}{4}, i + \frac{v}{2}, i + \frac{3}{4}v) | i = 0, 1, 2, ..., \frac{v}{4} - 1\}$.

Before proving the essential lemmas we need to introduce a couple of new sequences. A set of n integers such that every integer is congruent to exactly one integer of $\{0, 1, 2..., n-1\}$ is said to be a *complete residue system modulo n*. For instance, the set $\{0, 2, 3, 5, 7, 10\}$ is a complete residue system modulo 6.

A type A $2t$-residue sequence on $\mathbb{Z}_{r \cdot k}$ is a set $S = \{(a_i, b_i) | a_i, b_i \in \mathbb{Z}_{r \cdot k}$ and $a_i < b_i$ for $1 \leq i \leq t\}$ satisfying the following conditions:

(a) $2t | \frac{r \cdot k}{2}$;

(b) $\{a_i, b_i | (a_i, b_i) \in S\}$ is a complete residue system modulo $2t$.

(c) If $d_i = b_i - a_i$ is even for some $(a_i, b_i) \in S$, then $d_i < \frac{r \cdot k}{4}$; and

(d) For any two distinct elements (a_i, b_i) and (a_j, b_j) in S, $(b_i - a_i) + (b_j - a_j) \neq \frac{r \cdot k}{2}$ and $(b_i - a_i) \neq (b_j - a_j)$.

For convenience, we use $D(S)$ (respectively $D^*(S)$) to denote the set $\pm\{b_i - a_i | (a_i, b_i) \in S\}$ (respectively $\pm\{b_i = a_i | (a_i, b_i) \in S, b_i - a_i$ is even and $1 < b_i - a_i < \frac{r \cdot k}{4}\}$). As an example, $\{(0,8),(1,7),(2,6),(3,5),(4,9)\}$ is a type A 2·5 residue sequence on $\mathbb{Z}_{10\cdot 4}$ with $D(S) = \pm\{2,4,5,6,8\}$ and $D^*(S) = \pm\{2,4,6,8\}$.

A type B $2t$-residue sequence on $\mathbb{Z}_{r\cdot k}$ is a set $S = \{(a_i, b_i) | a_i, b_i \in \mathbb{Z}_{r\cdot k}, a_i < b_i$ for $1 \leq i \leq t\}$ satisfying the conditions

(a) $2t | \frac{r \cdot k}{2}$;

(b) $\{a_i, b_i | (a_i, b_i) \in S\}$ is a complete residue system modulo $2t$;

(c) If $d_i = b_i - a_i$ is even for some $(a_i, b_i) \in S$, then $d_i < \frac{r \cdot k}{4}$;

(d) There exist at most two elements (a_i, b_i) and (a_j, b_j) in S with $b_i - a_i = b_j - a_j$.

(e) For any two distinct elements (a_i, b_i) and (a_j, b_j) in S, $(b_i - a_i) + (b_j - a_j) \neq \frac{r \cdot k}{2}$;

(f) If $(a_i, b_i) \in S$ satisfying $b_i - a_i$ is even, then exactly one of $(a_i + 1, b_i + 1)$ and $(a_i - 1, b_i - 1)$ is also in S;

(g) If $b_i - a_i = b_j - a_j$ is an odd integer, then $a_i + a_j$ or $b_i + b_j$ is odd; and

(h) $D^*[K_{r(k)}] \setminus D^*(S)$ contains no even differences.

For example, $S = \{(1,2),(6,7),(14,16),(15,17),(5,8),(4,9),(12,18),$ $(13,19),(3,10),(0,11)\}$ is a type B $2{\cdot}10$-residue sequence on $\mathbb{Z}_{4\cdot 10}$. Now, we are ready for the essential tools of construction. Note that if the 4-cycle systems we construct are indeed cyclically resolvable, we won't emphasize that they are 2-cyclically resolvable.

Lemma 2.4 *Suppose that S is a type A $2t$-residue sequence on $\mathbb{Z}_{r\cdot k}$ with $D(S) = \pm\{d_1, d_2, ..., d_t\}$. Then there exists a cyclically resolvable 4-cycle system of $G[\cup_{i=1}^{t}\Omega_{d_i}]$.*

Proof. Let $S = \{(a_i, b_i) | i = 1, 2, \ldots, t\}$ be a type A $2t$-residue sequence on $\mathbb{Z}_{r\cdot k}$ and $b_i - a_i = d_i$. By Lemma 2.3, $G[\Omega_{d_i}]$ is generated by $(0, d_i, \frac{rk}{2}, d_i + \frac{rk}{2})$. Hence, let $C_i = (a_i, b_i, a_i + \frac{rk}{2}, b_i + \frac{rk}{2})$ which is also a base cycle of $G[\Omega_{d_i}]$. This implies that $G[\cup_{i=1}^{t}\Omega_{d_i}]$ can be generated by $\{(a_i, b_i, a_i + \frac{rk}{2}, b_i + \frac{rk}{2}) | (a_i, b_i) \in S\}$. Now, the proof follows by letting $R_i = \{(a_x, b_x, a_x + \frac{rk}{2}, b_x + \frac{rk}{2}) + 2jt + i | (a_x, b_x) \in S, j = 0, 1, ..., p-1 \text{ and } rk = 4pt\}$, $i = 0, 1, 2..., 2t-1$. ■

The following two lemmas will be used repeatedly, we call them Construction A and B respectively.

Lemma 2.5 *(Construction A)*

Suppose that there exist q type A $2t$-residue sequences $S_i (1 \le i \le q)$ on $\mathbb{Z}_{r\cdot k}$ satisfying (a) for $1 \le i \ne j \le q$, $D(S_i) \bigcap D(S_j) = \varnothing$ and there does not exist differences $\pm x \in D(S_i)$ and $\pm y \in D(S_j)$ such that $x + y = \frac{rk}{2}$, and (b) $D^*[K_{r(k)}] \setminus \bigcup_{i=1}^{q} D^*(S_i)$ contains no even differences. Then there exists a cyclically resolvable 4-cycle system of $K_{r(k)}$.

Proof. It is a direct consequence of Lemma 2.3 and 2.4. ∎

Lemma 2.6 *(Construction B)*

Suppose that there is a type B $2t$-residue sequence S on $\mathbb{Z}_{r\cdot k}$. Then there exists a 2-cyclically resolvable 4-cycle system of $K_{r(k)}$.

Proof. Without loss of generality, we may assume that the elements in the type B $2t$-residue sequence with repeated even differences are in $\{(a_i, b_i), (a_i+1, b_i+1) : i = 1, \ldots, p\}$, the elements with repeated odd differences are in $\{(e_i, f_i), (e_i^{'}, f_i^{'}) : i = 1, \ldots, q\}$, and the remaining elements with odd differences are in $\{(g_i, h_i) : i = 1, \ldots, s\}$. For $i = 1, \ldots, s$, let $h_i - g_i = d_i$ and the other differences of the sequence are $d_{s+1}, d_{s+2}, \ldots, d_t$. Note that $2p + 2q + s = t = rk/4$ and $e_i + e_i^{'}$ is odd, so is $f_i + f_i^{'}$.

Now, let $C_i (1 \leq i \leq t)$ be 4-cycles given as

$C_i = (a_i, b_i, a_i + rk/2, b_i + rk/2)$ *for* $i = 1, \ldots, p$;

$C_{p+i} = (a_i + 1, b_i + 1, a_i + 1 + rk/2, b_i + 1 + rk/2)$ *for* $i = 1, \ldots, p$;

$C_{2p+i} = (e_i, f_i, e_i + rk/2, f_i + rk/2)$ *for* $i = 1, \ldots, q$;

$C_{2p+q+i} = (e_i^{'}, f_i^{'}, e_i^{'} + rk/2, f_i^{'} + rk/2)$ *for* $i = 1, \ldots, q$; *and*

$C_{2p+2q+i} = (g_i, h_i, g_i + rk/2, h_i + rk/2)$ *for* $i = 1, \ldots, s$.

For $i = 0, \ldots, rk/4 - 1$, set $R_i = \{C_1 + 2i, \ldots, C_t + 2i\}$. Clearly, each R_i is a resolution class and $R = \{R_0, \ldots, R_{rk/4-1}, R_{d_1,-}, \ldots, R_{d_s,-}\}$ constitutes a 2-cyclically resolvable resolution of $G[\bigcup_{i=1}^{t} \Omega_{d_i}]$, where $R_{d_i,-}$ denote the set of two resolution classes in $G[\Omega_{d_i}]$ obtained in Lemma 2.3.

To conclude the proof, let $U = \pm\{c_1, \ldots, c_m\} = D[K_{r(k)}] \setminus D(S)$. Also by Lemma 2.3, there exists a cyclically resolvable 4-cycle system of $G[\Omega_{c_i}]$ for $i = 1, \ldots, m$ since c_i is odd. Moreover, if $rk/4 \not\equiv 0 (\mathrm{mod}\ r)$, we have a cyclically resolvable 4-cycle system of the circulant graph $G[\Omega_{\frac{rk}{4}}]$ by the note following Lemma 2.3. Therefore, combining the results stated above, the proof then follows. ∎

3. The main result

By Construction A, the existence of a cyclically resolvable 4-cycle system of $K_{r(k)}$ can be verified by constructing a set of type A $2t$-residue sequences on $\mathbb{Z}_{r\cdot k}$ satisfying a couple of conditions. On the other hand, if there exists a type B $2t$-residue sequence on $\mathbb{Z}_{r\cdot k}$, then by Construction B we have a 2-cyclically resolvable 4-cycle system of $K_{r(k)}$. Hence, to prove our main theorem, it suffices to construct suitable $2t$-residue sequences on $\mathbb{Z}_{r\cdot k}$ depending on the cases. Nevertheless, combining both results, we are able to construct a 2-cyclically resolvable 4-cycle system of $K_{r(k)}$ in general.

The proof of Theorem 1.1:

The necessity part follows by Corollary 2.2. Now, assume that $2|(r-1)k$ and $4|rk$. For convenience, let $rk = 4q$. We claim that a 2-cyclically resolvable 4-cycle system of $K_{r(k)}$ does exist. For clearness, we split the proof into 7 cases. Note that we shall use Construction A for the first case and all the others use Construction B.

Case 1. $r \equiv 2\ (mod\ 4)$ and $k \equiv 0\ (mod\ 4)$

Let $t = r/2$. For $j = 0, 1, 2, \dots, k/4 - 1$, let S_j be type A $2t$-residue sequence on $\mathbb{Z}_{rk}$ given by

$$S_j = \{(t-2-i, t+jr+i), (t-1, r-1+jr) | i = 0, 1, 2, \dots, t-2\}.$$

Since the set of sequences $\{S_j\}$ satisfies the conditions in Construction A, the proof follows.

For the rest of proof, since the sequences S we construct are of type B, we shall use three subsequences S_1, S_2 and S_3 to form S depending on $D(S_i), i = 1, 2, 3$. In which, $D(S_1) = \pm\{d | d \equiv 0\ (mod\ 4)\}$, $D(S_2) = \pm\{d | d \equiv 2\ (mod\ 4)\}$ and $D(S_3) = \pm\{d | d\ is\ odd\}$.

Case 2. $r, k \equiv 0\ (mod\ 4)$

Subcase 2.1: $r = 4$.

The type B $2t$-residue sequence S on $\mathbb{Z}_{rk}$ is defined as

$\{(2i, k-2-2i), (2i+1, k-1-2i) |\ i = 0, \dots, k/4-1\} \cup \{(k+i, 2k-1-i) |\ i = 0,$

$\dots, k/2-1\}$.

Subcase 2.1: $r > 4$.

Let $p = k(r-4)/8$. As stated previously, we partition the type B $2t$-residue sequence S on $\mathbb{Z}_{rk}$ into three subsequences $S_i (1 \le i \le 3)$:

$S_1 = \{(2j - 2 + i(r-4)/2, q - 2 - 2j - i(r+4)/2), (2j - 1 + i(r-4)/2, q -$

$1-2j-i(r+4)/2)|\ i=0,\ldots,k/4-1$ and $j=1,\ldots,(r-4)/4\}$

$S_2=\{(2p+k+2i,2q-2-2i),(2p+k+1+2i,2q-1-2i)|\ i=0,\ldots,q/4-1\}$.

If $k\equiv 4\ (mod\ 8)$, then

$S_3=\{(p,q-1),(p+1,q-2),(p+r/2+j+i(r+4)/2,q-r/2-1-j-i(r+4)/2)|\ i=0,\ldots,(k-12)/8$ and $j=0,1,2,3\}$.

If $k\equiv 0\ (mod\ 8)$, then

$S_3=\{(p,q-1),(p+1,q-2),(3p/2+k/2-2,3p/2+k/2+1),(3p/2+k/2-1,3p/2+k/2),(p+r/2+j+i(r+4)/2,q-r/2-1-j-i(r+4)/2)|\ i=0,\ldots,k/8-2$ and $j=0,1,2,3\}$.

By routine verification, it can be shown that $S=S_1\cup S_2\cup S_3$ is a type B $2t$-residue sequence on $\mathbb{Z}_{rk}$.

Case 3. $r\equiv 0\ (mod\ 4)$ and $k\equiv 2\ (mod\ 4)$

Subcase 3.1: $k=2$.

If $r=4$, the proof follows directly from Lemma 2.3. If $r=8$, then $\{(1,2),(0,3),(4,6),(5,7)\}$ is the type B complete $2t$-residue sequence on $\mathbb{Z}_{8\cdot 2}$. When $r>8$, the sequence S on $\mathbb{Z}_{rk}$ is given as

$S=\{(r/4-1+\epsilon,r/2-2+2\epsilon),(r/4-2+\epsilon,r/2-1+2\epsilon)\}\cup\{(r/4-4+\epsilon-2i,r/4+\epsilon+2i),(r/4-3+\epsilon-2i,r/4+1+\epsilon+2i)|\ i=0,\ldots,\lfloor(r-12)/8\rfloor\}\cup\{(3r/4-2+\epsilon-2i,3r/4+\epsilon+2i),(3r/4-1+\epsilon-2i,3r/4+1+\epsilon+2i)|\ i=0,\ldots,\lfloor(r-8)/8\rfloor\}$, where $\epsilon=1$ or 0 according to whether $r\equiv 4$ or $0\ (mod\ 8)$.

Subcase 3.2: $k>2$.

If $r=4$, then

$S = \{(3k/2-1-2j, 3k/2+1+2j), (3k/2-2j, 3k/2+2+2j) : j = 0, \ldots, (k-6)/4\} \cup \{(1,2), (k/2+1, k/2+2), (0, k+1), (k/2-j, k/2+3+j) : j = 0, \ldots, k/2-3\}$.

On the other hand, if $r > 4$, then let $p = k(r-4)/8$ and let $\epsilon = 0$ or 1 according to whether $r \equiv 0$ or 4 $(mod\ 8)$. The sequence S can be obtained by the union of S_1, S_2, and ~~either $S_{3,1}$ or $S_{3,2}$ as the cases may be:~~

$S_1 = \{(p-3+\epsilon-2j-i(r-4)/2, p+1+\epsilon+2j+i(r+4)/2), (p-2+\epsilon-2j-i(r-4)/2, p+2+\epsilon+2j+i(r+4)/2) |\ i = 0, \ldots, (k-6)/4$ and $j = 0, \ldots, r/4-2\} \cup \{(2i, q-4+2\epsilon-2i), (1+2i, q-3+2\epsilon-2i) |\ i = 0, \ldots, \lfloor (r-12)/8 \rfloor\}$.

$S_2 = \{(q+2\epsilon+2i, 2q-2-2i), (q+1+2\epsilon+2i, 2q-1-2i) | i = 0, \ldots, \lfloor r/8 \rfloor k/2 + (k-2)\epsilon/4 - 1\}$.

~~If $k \equiv 2$ $(mod\ 8)$, then~~

$S_{3,1} = \{(p-1+\epsilon, q-1+2\epsilon), (p+\epsilon, q-2+2\epsilon)\} \cup \{(p+r/2-1+\epsilon+i(r+4)/2+j, q-r/4-1+\epsilon-i(r+4)/2-j) |\ i = 0, \ldots, \lfloor k/8 \rfloor - 1$ and $j = 0, 1, 2, 3\}$.

~~If $k \equiv 6$ $(mod\ 8)$, then~~

$S_{3,2} = \{(r(3k/2+1)/8 - (k-6)/4 - 4 + \epsilon, r(3k/2+1)/8 - (k-6)/4 - 1 + \epsilon)\} \cup \{(r(3k/2+1)/8 - (k-6)/4 - 4 + \epsilon, r(3k/2+1)/8 - (k-6)/4 - 1 + \epsilon)\}$.

Case 4. $r, k \equiv 2$ $(mod\ 4)$

First, if $r = 2$, then the proof follows from Lemma 2.3. Thus, we consider $r \geq 6$.

Subcase 4.1: $k = 2$.

Let $\epsilon = 0$ or 4 according to whether $r \equiv 6$ or 2 $(mod\ 8)$. Then S can be obtained by taking the union of the following sequences.

$S_1 = \{(3r-2-\epsilon)/4-2-2i, (3r+2-\epsilon)/4+1+2i, (3r-2-\epsilon)/4-1-2i, (3r+2-\epsilon)/4+2+2i|\ i = 0, \ldots, (r-14+\epsilon)/8\}$.

$S_2 = \{(2i, r/2-1-\epsilon/2-2i), (2i+1, r/2-\epsilon/2-2i)|\ i = 0, \ldots, (r-6-\epsilon)/8\}$.

$S_3 = \{(3r-2-\epsilon)/4, (3r-2-\epsilon)/4+1\}$.

Subcase 4.2: $k = 6$.

Let $p = (r-2)/4$ and let $\epsilon = 0$ or 1 according to whether $r \equiv 2$ or 6 $(mod\ 8)$. Then the followings give the sequence S.

$S_1 = \{(9p+2-\epsilon-2i, 9p+6-\epsilon+2i), (9p+3-\epsilon-2i, 9p+7-\epsilon+2i)|\ i = 0, \ldots, (3p-2+\epsilon)/2\}$.

$S_2 = \{(2i, 6p+2-2\epsilon-2i), (2i+1, 6p+3-2\epsilon-2i)|\ i = 0, \ldots, (p-2-\epsilon)/2\} \cup \{(3p-\epsilon-2i, 3p+2-\epsilon+2i), (3p+1-\epsilon-2i, 3p+3-\epsilon+2i)|\ i = 0, \ldots, p-1\}$.

$S_3 = \{(9p+4-\epsilon, 9p+5-\epsilon), (p+1-\epsilon, 5p+2-\epsilon), (p-\epsilon, 5p+3-\epsilon)\}$.

Subcase 4.3: $k > 6$.

First, if $r = 6$, then let $p = (3k-14)/4$ and let $\epsilon = 0$ or 3 according to whether $k \equiv 6$ or 2 $(mod\ 8)$. By S_1, S_2 and S_3 defined as follows we have the sequence S.

$S_1 = \{(3p+6+\epsilon-2j-6i, 3p+10+\epsilon+2j+6i), (3p+7+\epsilon-2j-6i, 3p+11+\epsilon+2j+6i)|\ i = 0, \ldots, (k-6-4\epsilon/3)/8$ and $j = 0, 1\}$.

$S_2 = \{(p+\epsilon, p+2+\epsilon), (p+1+\epsilon, p+3+\epsilon)\} \cup \{(p-4+\epsilon-2j-6i, p+6+\epsilon+2j+6i), (p-3+\epsilon-2j-6i, p+7+\epsilon+2j+6i)|\ i = 0, \ldots, (k-14-4\epsilon/3)/8$ and $j = 0, 1\}$.

$S_3 = \{(3p+8+\epsilon, 3p+9+\epsilon), (0, 3k/2-4+2\epsilon), (3k/2-3+2\epsilon, 3k-1)\} \cup \{(p-1+\epsilon-6i, p+4+\epsilon+6i), (p-2+\epsilon-6i, p+5+\epsilon+6i) \mid i = 0, \ldots, (k-14+4\epsilon/3)/8\} \cup \{(3p+3+\epsilon-6i, 3p+14+\epsilon+6i), (3p+2+\epsilon-6i, 3p+15+\epsilon+6i) \mid i = 0, \ldots, (k-14+4\epsilon/3)/8\}$.

On the other hand, if $r > 6$, then it is more complicate to obtain S. Let $p = (rk-12)/8$ and let $\epsilon = 0$ or 1 according to whether $r \equiv 2$ or 6 $(mod\ 8)$. Now, if $r \equiv 2\ (mod\ 8)$ and $k \equiv 2\ (mod\ 8)$ or $r \equiv 6\ (mod\ 8)$ and $k \equiv 6\ (mod\ 8)$, then let

$S_1 = \{(3p+1-2j-ir, 3p+5+2j+ir), (3p+2-2j-ir, 3p+6+2j+ir) \mid j = 0, \ldots, (r-4)/2$ and $i = 0, \ldots, (k-10-4\epsilon)/8\} \cup \{(q-1+2i, 2q-2-2i), (q+2i, 2q-1-2i) \mid i = 0, \ldots, ((1+2\epsilon)r-10)/8\}$,

$S_2 = \{(p-1-2i, p+1+2i), (p-2i, p+2+2i) \mid i = 0, \ldots, (r-6)/4\} \cup \{(p-r/2-2-2j-ir, p+r/2+2+2j+ir), (p-r/2-1-2j-ir, p+r/2+3+2j+ir) \mid j = 0, \ldots, (r-4)/2$ and $i = 0, \ldots, (k-18+4\epsilon)/8\} \cup \{(2i, q-3-2i), (2i+1, q-2-2i) \mid i = 0, \ldots, ((3-2\epsilon)r-14)/8\}$, and

$S_3 = \{(3p+3, 3p+4)\} \cup \{(3p+4-r-ir, 3p+3+r+ir), (3p+3-r-ir, 3p+4+r+ir) \mid i = 0, \ldots, (k-10-4\epsilon)/8\} \cup \{(p-r/2+1-ir, p+r/2+ir), (p-r/2-ir, p+r/2+1+ir) \mid i = 0, \ldots, (k-10+4\epsilon)/8\}$.

On the other hand, if $r \equiv 2\ (mod\ 8)$ and $k \equiv 2\ (mod\ 8)$, then let

$S_1 = \{(3p+2-2j-ir, 3p+6+2j+ir), (3p+3-2j-ir, 3p+7+2j+ir) \mid j = 0, \ldots, (r-4)/2$ and $i = 0, \ldots, (k-14+4\epsilon)/8\} \cup \{(q+1+2i, 2q-2-2i), (q+2+2i, 2q-1-2i) \mid i = 0, \ldots, ((3-2\epsilon)r-14)/8\}$,

$S_2 = \{(p-2i, p+2+2i), (p+1-2i, p+3+2i) | i = 0, \ldots, (r-6)/4\} \cup \{(p-$

$r/2-1-2j-ir, p+r/2+3+2j+ir), (p-r/2-2j-ir, p+r/2+4+2j+ir) |\ j = 0, \ldots, (r-4)/2$ and $i = 0, \ldots, (k-14-4\epsilon)/8\} \cup \{(2i, q-1-2i), (2i+1, q-2i) |\ i = 0, \ldots, ((1+2\epsilon)r-10)/8\}$, and

$S_3 = \{(3p+4, 3p+5)\} \cup \{(3p+5-r-ir, 3p+4+r+ir), (3p+4-r-ir, 3p+5+r+ir) |\ i = 0, \ldots, (k-14+4\epsilon)/8\} \cup \{(p-r/2+2-ir, p+r/2+1+ir), (p-r/2+1-ir, p+r/2+2+ir) |\ i = 0, \ldots, (k-6-4\epsilon)/8\}$.

Case 5. $r = 3$ and $k \equiv 0\ (mod\ 4)$

Let $\epsilon = 0$ or 1 according to whether $k \equiv 0$ or $4\ (mod\ 8)$ and

$S_1 = \{(3k/4+2\epsilon+3i, 3k/2-2-\epsilon-3i), (3k/4+1+2\epsilon+3i, 3k/2-1-\epsilon-3i) |\ i = 0, \ldots, (k-8-4\epsilon)/8\}$,

$S_2 = \{(1-\epsilon+3i, 3k/4-3+2\epsilon-3i), (2-\epsilon+3i, 3k/4-2+2\epsilon-3i) |\ i = 0, \ldots, (k-8+4\epsilon)/8$, and

$S_3 = \{(3k/4+2-\epsilon-3i, 3k/4-3+2\epsilon-3i) |\ i = 0, \ldots, (k-8+4\epsilon)/8\} \cup \{(2\epsilon+3i, 3k/4-1-\epsilon-3i) |\ i = 0, \ldots, (k-8-4\epsilon)/8$.

Case 6. $r \equiv 1\ (mod\ 2)$ and $k \equiv 4\ (mod\ 8)$

Starting from here, let $p = (rk-4)/8$.

Subcase 6.1: $k = 4$.

Let $\epsilon = 1$ or 0 depending on wether $r \equiv 1$ or $3\ (mod\ 4)$ and

$S_1 = \{(r+1-2\epsilon+2i, 2r-2-2i), (r+2-2\epsilon+2i, 2r-1-2i) |\ i = 0, \ldots, (r-7+2\epsilon)/4\}$,

$S_2 = \{(2i, r-1-2\epsilon_1-2i), (2i+1, r-2\epsilon_1-2i) |\ i = 0, \ldots, (r-3-2\epsilon)/4\}$, and

$S_3 = \{3p+1-\epsilon, 3p+2-\epsilon\}$.

Subcase 6.2: $k > 4$.

If $r \equiv 1$ or 3 $(mod\ 4)$ and $k \equiv 4$ $(mod\ 16)$, set $\epsilon_1 = 1$ or 0 and $\epsilon_2 = 1$ correspondingly. Then, the sequence can be obtained by letting

$S_1 = \{(3p-1-\epsilon_1-2j-2ir, 3p+3-\epsilon_1+2j+2ir), (3p-\epsilon_1-2j-2ir, 3p+4-\epsilon_1+2j+2ir) \mid i = 0, \ldots, (k-20-8\epsilon_2)/16$ and $j = 0, \ldots, r-2\} \cup \{(q+1-2\epsilon_1+2i, 2q-2-2i), (q+2-2\epsilon_1+2i, 2q-1-2i) \mid i = 0, \ldots, ((1+2\epsilon_2)r-7+2\epsilon_1)/4\}$,

$S_2 = \{(p-1-\epsilon_1-2i, p+1-\epsilon_1+2i), (p-\epsilon_1-2i, p+2-\epsilon_1+2i) \mid i = 0, \ldots, (r-3)/2\} \cup \{(p-r-2-\epsilon_1-2j-2ir, p+r+2-\epsilon_1+2j+ir), (p-r-1-\epsilon_1-2j-2ir, p+r+3-\epsilon_1+2j+2ir) \mid i = 0, \ldots, (k-36+8\epsilon_2)/16$ and $j = 0, \ldots, r-2\} \cup \{(2i, q-1-2\epsilon_1-2i), (2i+1, q-2\epsilon_1-2i) \mid i = 0, \ldots, ((3-2\epsilon_2)r-5-2\epsilon_1)/4\}$, and

$S_3 = \{(3p+1-\epsilon_1, 3p+2-\epsilon_1)\} \cup \{(3p+2-2r-\epsilon_1-2ir, 3p+1+2r-\epsilon_1+2ir), (3p+1-2r-\epsilon_1-2ir, 3p+2+2r-\epsilon_1+2ir) \mid i = 0, \ldots, (k-20-8\epsilon_2)/16\} \cup \{(p-r+1-\epsilon_1-2ir, p+r-\epsilon_1+2ir), (p-r-\epsilon_1-2ir, p+r+1-\epsilon_1+2ir) \mid i = 0, \ldots, (k-20+8\epsilon_2)/16\}$.

Case 7. $r \equiv 1$ $(mod\ 2)$ and $k \equiv 0$ $(mod\ 8)$

Let $\epsilon = 0$ or 1 according to whether $k \equiv 8$ or 0 $(mod\ 16)$. Now, we defined the sequences S ~~we need.~~ as follows: for $k \equiv 8$ (mod 16), $S = S_{1,2} \cup S_{2,1} \cup S_3$ and $S = S_{1,1} \cup S_{2,1} \cup S_{2,2} \cup S_3$ for $k \equiv 0$ (mod 16).

~~If $k \equiv 0$ $(mod\ 16)$, then~~

$S_{1,1} = \{(3q/2-3+\epsilon-2j-2ir, 3q/2+1+\epsilon+2j+2ir), (3q/2-2+\epsilon-2j-$

$2ir, 3q/2+2+\epsilon+2j+2ir)|\ i=0,\ldots,(k-24-8\epsilon)/16$ and $j=0,\ldots,r-2\}$.

~~If $k \equiv 8\ (mod\ 16)$, then~~

$S_{1,2}=\{(q+2i, 2q-2-2i),(q+1+2i, 2q-1-2i)|\ i=0,\ldots,(r-3)/2\}$.

$S_{2,1}=\{(q/2-2+\epsilon-2i, q/2+\epsilon+2i),(q/2-1+\epsilon-2i, q/2+1+\epsilon+2i)|i=0,\ldots,(r-3)/2\}\cup\{(q/2-r-3+\epsilon-2j-2ir, q/2+r+1+\epsilon+2j+2ir),(q/2-r-2+\epsilon-2j-2ir, q/2+r+2+\epsilon+2j+2ir)|\ i=0,\ldots,(k-24-8\epsilon)/16$ and $j=0,\ldots,r-2\}$.

~~If $k \equiv 0\ (mod\ 16)$, then~~

$S_{2,2}=\{(r-2-2i, q-r+2+2i),(r-1-2i, q-r+3+2i)|\ i=0,\ldots,(r-3)/2\}$.

~~Finally, let~~

$S_3=\{(3q/2-1+\epsilon, 3q/2+\epsilon)\}\cup\{(0, q-1+2\epsilon)\}\cup\{(3q/2-2r+\epsilon-2ir, 3q/2+2r-1+\epsilon+2ir),(3q/2-2r-1+\epsilon-2ir, 3q/2+2r+\epsilon+2ir)|\ i=0,\ldots,(k-24-8\epsilon)/16\}\cup\{(q/2-r+\epsilon-2ir, q/2+r-1+\epsilon+2ir),(q/2-r-1+\epsilon-2ir, q/2+r+\epsilon+2ir)|\ i=0,\ldots,(k-24+8\epsilon)/16\}$.

This concludes the proof of the main theorem. ■

4. Concluding remark

In this paper, we manage to construct a 2-cyclically resolvable 4-cycle system of $K_{r(k)}$ for each admissible pair (r,k). It is then natural to ask if a cyclically resolvable m-cycle system of $K_{r(k)}$ can be constructed for each admissible pair (r,k). Unfortunately, not every m-sufficient $K_{r(k)}$ admits a cyclic m-cycle system even $m|rk$. It has been proved by Wu and Fu in [15] that if there exists a cyclic m-cycle system of $K_{r(m)}$ with m even and $m>4$, then $r \not\equiv 3 \ (mod\ 4)$ and $m \not\equiv 2 \ (mod\ 4)$ or $r, m \not\equiv 2 \ (mod\ 4)$. Therefore, finding a necessary and sufficient condition for the existence of a cyclically resolvable m-cycle system of $K_{r(k)}$ is going to be ~~very~~ more difficult.

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